For all problems, *SHOW ALL OF YOUR WORK*. Partial solutions and problems with missing steps will be marked wrong. Continue your work on the back of the page or extra sheet at the end of the exam if you need additional space. *You do not need but may use the normal graphing calculator functions of any graphing calculator, but not any differential equations functionality it may have.*

1. Solve each of the following, explicitly if possible and implicitly otherwise. (33 points)

   a. $5t\frac{dz}{dt} = -10z + \dfrac{1}{t}\cos(4t)$, $z(\pi) = 2$.

   b. $5z\frac{dz}{dt} = (z^2+1)^{1/2}\cos(4t)$.

   c. $y''' + 4y' = 0$, with $y(0) = 0$, $y'(0) = 1$ and $y''(0) = 0$.

2. Consider the differential equation $4t^2y''(t) - ty'(t) + y(t) = 0$ (which we don't know how to solve). (15 points)

   a. Show that $y_1 = t^{1/4}$ and $y_2 = t$ are solutions to this differential equation (for $t > 0$).

   b. Write the general solution to the differential equation. What is true about y_1 and y_2 that allows you to do this? How do you know?

   c. Find the particular solution for the differential equation if $y(1) = 3$ and $y'(1) = 2$.

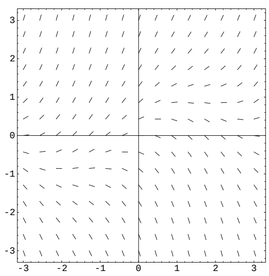
3. Find $|z|$, $Arg(z)$ and the exponential (polar) form of $z = \left(\frac{1}{2-2i}\right)^8$. (6 points)

4. An alert turtle named Yert observes that tin organ pipes decay with age as a result of a chemical reaction which is catalyzed by the decayed tin. As a result, the rate at which the tin decays is proportional to the product of the amount of tin left and the amount that has already decayed. Let M be the total amount of tin before any has decayed. (14 points)

   a. Write a differential equation for the amount of decayed tin, $p(t)$. Be sure it is clear why your equation has the form it does.

   b. Draw a phase diagram for your differential equation (take $M = 10$ and your constant of proportionality, $k = 2$, if you like) and explain what this tells you about the decay of the tin if initially none of it is decayed. How does this change if there is a very small amount decayed initially?

5. Sketch solutions to the differential equation whose direction field is shown to the right through the initial conditions $y(0) = 1$ and $y(-2) = -1$. Then carefully explain how you know that the direction field is *not* the the direction field for the differential equation $\frac{dy}{dx} = x - \sin(y)$. (8 points)



6. The initial value problem $yy' = y - y^2$, $y(0) = 0$ has two solutions. (14 points)
   a. Does this conform to or contradict the existence and uniqueness theorem for first-order ordinary differential equations? (Recall that the theorem starts "*For* $y'(x) = f(x, y)$*, if the function* $f(x, y)$ *is. . .*")

   b. Find the two solutions to $yy' = y - y^2$, $y(0) = 0$.

7. A not entirely defensible numerical method for approximating the solution to the differential equation $\frac{dy}{dx} = f(x, y)$ might use the iteration formula $y_{n+1} = y_n + hf(x_n, z)$, where $z = y_n + \frac{1}{2}hf(x_n, y_n)$. Approximate $y(0.2)$ using $h = 0.1$ and this numerical method (tentatively called "Yert's method") if $\frac{dy}{dx} = 2 - xy^2$ and $y(0) = 0.5$. (10 points)