Math 216–060 Exam 2, 12 November 2001 Name: Exam Solution

Prof. Gavin LaRose **page 1 of 4** Recitation Section: ________

For all problems, *SHOW ALL OF YOUR WORK*. Partial solutions and problems with missing steps will be marked wrong. Continue your work on the back of the page or extra sheet at the end of the exam if you need additional space. *You do not need but may use the normal graphing calculator functions of any graphing calculator, but not any differential equations functionality it may have.*

1. For each of the following differential equations, two solutions to the complementary homogeneous problem are given. Find the general solution for each.

   a. $y'' + 3y' + 2y = 3x$, $y_1(x) = e^{-2x}, y_2(x) = e^{-x}$. (10 points)
   Solution: The general solution is $y = C_1 y_1 + C_2 y_2 + y_p$. To find y_p we use the method of undetermined coefficients: we guess $y_p = Ax + B$ and plug in to find A and B. $y_p' = A$ and $y_p'' = 0$, so $3A + 2Ax + 2B = 3x$. Matching the coefficients of x and 1, $2A = 3$, so $A = \frac{3}{2}$ and $3A + 2B = 0$, so $B = -\frac{9}{4}$. Thus

   $$y = C_1 e^{-2x} + C_2 e^{-x} + \frac{3}{2}x - \frac{9}{4}.$$

   b. $y'' + 4y' + 4y = 2e^{-2x}$, $y_1(x) = e^{-2x}, y_2(x) = xe^{-2x}$. (10 points)
   Solution: Proceeding as before and using the method of undetermined coefficients, we would guess $y_p = Ae^{-2x}$, but this is a solution to the complementary homogeneous problem, so we multiply by enough factors of x that this is no longer the case and instead guess $y_p = Ax^2e^{-2x}$. Then $y_p' = A(2xe^{-2x} - 2x^2e^{-2x})$ and $y_p'' = A(2e^{-2x} - 8xe^{-2x} + 4x^2e^{-2x}$. Plugging these into the differential equation, we get

   $$A\left((2e^{-2x} - 8xe^{-2x} + 4x^2e^{-2x}) + 4(2xe^{-2x} - 2x^2e^{-2x}) + 4x^2e^{-2x} \right) = 2Ae^{-2x},$$

   so that we must have $A = 1$. The particular solution is therefore

   $$y = C_1 e^{-2x} + C_2 xe^{-2x} + x^2e^{-2x}.$$

2. Find all equilibrium solutions to the system

   $$x' = x(3 - x) - 2xy$$
   $$y' = y(1 - y) + xy.$$

   In what direction are solutions moving when $(x, y) = (1, 2)$? How is this related to a direction field for the system? (6 points)
   Solution: The equilibrium solutions are those for which $x' = y' = 0$, so that

   $$0 = x(3 - x - 2y)$$
   $$0 = y(1 + x - y).$$

   From the first, $x = 0$ or $x = 3-2y$. If $x = 0$ the second equation requires that either $y = 0$ or $y = 1$. Thus two equilibrium solutions are $(0, 0)$ and $(0, 1)$. In the second equation $y = 0$ or $y = x + 1$. If $y = 0$ in the first equation, there is the non-zero solution $x = 3$, so a third equilibrium solution is $(3, 0)$. Finally, if $x = 3 - 2y$ and $y = x + 1$, $x = 3 - 2x - 2$ so $x = \frac{1}{3}$ and $y = \frac{4}{3}$. The last equilibrium point is $(\frac{1}{3}, \frac{4}{3})$.

   When $(x, y) = (1, 2)$, the system gives $(x', y') = (-2, 0)$ (by plugging in $x = 1$ and $y = 2$), so a solution curve at that point must be moving only to the left. This is related to the direction field in that it tells us the slope (in this case, zero) of the line in the direction field that we would draw at that point. We could construct the entire direction field by finding (x', y') at every point, determining the slope $\frac{y'}{x'}$, and drawing a line with that slope at the point.

3. The charge on a capacitor, $Q(t)$, in a simple LRC circuit is given by $LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = E(t)$, where L, R, and C are the inductance, resistance and capacitance of the inductor, resistor and capacitor in the system. Suppose that $R = 1\text{k}\Omega$, $C = 20\mu F$, $L = 0.35\text{h}$, and that the system is forced with an applied charge $E(t) = A\sin(\omega t)$.

   a. For what ω, if any, will this system exhibit resonance? Explain. (8 points)
   Solution: In this case the system is damped, so we will not see any (pure) resonance. We might expect to see practical resonance (though in that the damping is quite large in this case this is unlikely to be obvious), however, when the forcing frequency ω is close to the natural frequency of the undamped system. (This is $\omega_0 = \frac{1}{\sqrt{LC}}$.)

   b. Now suppose $R = 0$ (the resistor is removed). If $\omega = 120\pi$ and $A = 117$, find the charge $Q(t)$ if $Q(0) = Q'(0) = 0$. Be sure that it is clear why you proceed as you do and how you arrive at your answer. Will your solution exhibit resonance, beats, or neither? Why? (10 points)
   Solution: With the resistor removed the problem is $Q'' + \frac{1}{LC}Q = \frac{A}{L}\sin(120\pi t)$. The natural frequency of the system is $\omega_0 = \frac{1}{\sqrt{LC}} \approx 377.964$ (remember that $20\mu\text{F} = 20 \times 10^{-6}\text{F}$). The forcing frequency $\omega \approx 376.991$ is very close to this, so we expect to see beats. If they had been equal we would have resonance, as this is an undamped system. The complementary homogeneous solution for the problem is $Q_c = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$. To find the particular solution, we can use the method of undetermined coefficients. We guess $Q_p = A\sin(\omega t) + B\cos(\omega t)$, but because there is no Q' term on the left-hand side of the equation we note that B will be zero, and so can omit it from the outset. Plugging this Q_p into the equation, we get $-\omega^2 A + \omega_0^2 A = \frac{117}{L}$, so $A = \frac{117}{L(\omega_0^2 - \omega^2)} \approx -0.455$. Our solution is therefore $Q = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t) - 0.455\sin(120\pi t)$. The initial conditions require that $C_1 = 0$ and $\omega_0 C_2 - 120\pi(0.455) = 0$, so $C_2 = 0.454$, and the solution is

$$Q = 0.454 sin(\omega_0 t) - 0.455\sin(120\pi t).$$

4. Suppose that the populations of two interacting species are given by the system

$$x' = x(1-2x) + xy$$
$$y' = y(2-y) + xy$$

Use Euler's method with $h = 0.1$ to approximate $x(0.2)$ and $y(0.2)$ if $x(0) = 1$ and $y(0) = 1.5$. (12 points)
Solution: Euler's method says that $\mathbf{x}_{n+1} = \mathbf{x}_n + h\mathbf{f}(t_n, \mathbf{x}_n)$ (where $\mathbf{x} = (x\ y)^T$ and $\mathbf{f}$ gives the right-hand side of the system). Let's take $f(x, y) = x(1-2x) + xy$ and $g(x, y) = y(2-y) + xy$ and make a table of values showing how this works.

n	t_n	x_n	y_n	$f(x_n, y_n)$	$g(x_n, y_n)$	x_{n+1}	y_{n+1}
0	0	1	1.5	0.5	2.25	$1 + .1 \cdot .5 = 1.05$	$1.5 + .1 \cdot 2.25 = 1.725$
1	0.1	1.05	1.725	0.656	2.286	$1.05 + .1 \cdot .656 = 1.116$	$1.725 + .1 \cdot 2.286 = 1.954$
2	0.2	1.116	1.954				

5. If $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, and $c = 4$, find $\mathbf{Av} - c\mathbf{v}$. (5 points)

   Solution: $\mathbf{Av} = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 13 \\ -2 \end{pmatrix}$ and $c\mathbf{v} = \begin{pmatrix} 4 \\ 16 \end{pmatrix}$, so $\mathbf{Av} - c\mathbf{v} = \begin{pmatrix} 9 \\ -18 \end{pmatrix}$.

   a. Is this an eigenvalue problem? Explain. (5 points)

   Solution: This is not an eigenvalue problem. For an eigenvalue problem we are solving $\mathbf{Av} = \lambda\mathbf{v}$, which is the same as $\mathbf{Av} - \lambda\mathbf{v} = 0$. Thus the problem we have above looks like an eigenvalue problem (with $c = \lambda$), but because the result we obtained isn't zero we know that the c and $\mathbf{v}$ we are using there aren't an eigenvalue and eigenvector of the matrix $\mathbf{A}$.

6. Solve the system $\mathbf{x}' = \mathbf{Ax}$ if $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$. (12 points)

   Solution: We guess $\mathbf{x} = \mathbf{v}e^{\lambda t}$. Plugging this in, we know that λ must be an eigenvalue and $\mathbf{v}$ an eigenvector of the matrix $\mathbf{A}$. This means that λ must satisfy $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, which is

$$\begin{vmatrix} 1-\lambda & -1 \\ 2 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 2 = 0.$$

   Thus $\lambda^2 - 4\lambda + 5 = 0$. This doesn't factor, but an easy quadratic formula or calculator calculation gives $\lambda = 2 \pm i$. The eigenvector corresponding to $\lambda = 2 + i$ must satisfy

$$\begin{pmatrix} -1-i & -1 \\ 2 & 1-i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

   We know that the two equations here are the same because λ is an eigenvalue, so let's use the first. If $v_1 = -1$, $v_2 = 1 + i$, so a solution is

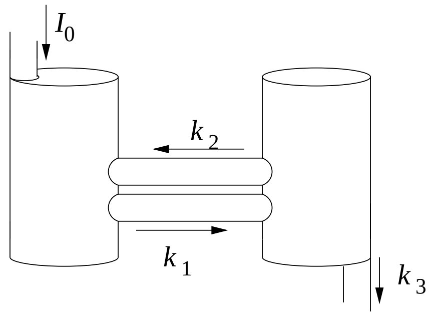
$$\mathbf{x}_a = \begin{pmatrix} -1 \\ 1+i \end{pmatrix} e^{(2+i)t}.$$

   Sadly, this is complex-valued, and we really do like real-valued solutions, so let's find a couple of real-valued ones: we'll use the real and imaginary parts of this solution. Expanding it to find them,

$$\begin{aligned} \mathbf{x}_a &= e^{2t}\begin{pmatrix} -1 \\ 1+i \end{pmatrix}(\cos(t) + i\sin(t)) \\ &= e^{2t}\begin{pmatrix} -\cos(t) - i\sin(t) \\ (\cos(t) - \sin(t)) + i(\cos(t) + \sin(t)) \end{pmatrix}, \end{aligned}$$

   So our general solution is

$$\mathbf{x} = C_1\mathrm{Re}(\mathbf{x}_a) + C_2\mathrm{Im}(\mathbf{x}_a) = C_1 e^{2t}\begin{pmatrix} -\cos(t) \\ \cos(t) - \sin(t) \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} -\sin(t) \\ \cos(t) + \sin(t) \end{pmatrix}.$$



7. Consider the crude model of the lungs and blood shown in the figure to the right. If x_1 is the amount of some toxin in lungs (which comes from inhaling the toxin at a rate I_0), x_2 is the amount of that toxin in the bloodstream, and the constants k_j are the constants of proportionality for the indicated transfers of the toxins, the model predicts that

$$\begin{aligned} x_1' &= -k_1x_1 + k_2x_2 + I_0 \\ x_2' &= \quad k_1x_1 - (k_2+k_3)x_2. \end{aligned}$$

a. Explain why this makes sense. (5 points)
Solution: This makes sense because it says that x_1 is decreased by the factor k_1x_1, reflecting transfer from the lungs to the bloodstream (the lower "pipe" in the figure), and increased by the factors k_2x_2, reflecting transfer back (the upper "pipe" in the figure) and I_0, which is the inhaled amount. Similarly, x_2 is increased by the transfer from the lungs (k_1x_1) and decreased by transfer back (k_2x_2) and removal from the blood (the output "pipe" in the figure), k_3x_2.

b. Rewrite the system in matrix notation. (5 points)
Solution:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -k_1 & k_2 \\ k_1 & -(k_2+k_3) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} I_0 \\ 0 \end{pmatrix}$$

c. If $I_0 = 60$, $k_1 = k_2 = 2$ and $k_3 = 3$, solve it. (12 points)
Solution: With these constants, the system above becomes

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 60 \\ 0 \end{pmatrix}.$$

We solve for the complementary homogeneous solution first (ignoring the forcing term $(60\ 0)^T$). Let $\mathbf{x} = \mathbf{v}e^{\lambda t}$; then, as in problem 6, $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, or $(-2-\lambda)(-5-\lambda) - 4 = \lambda^2 + 7\lambda + 6 = 0$. This factors as $(\lambda+6)(\lambda+1) = 0$, so the eigenvalues are $\lambda = -6$ or $\lambda = -1$. If $\lambda = -6$, the eigenvector must satisfy $\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so $\mathbf{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Similarly if $\lambda = -1$, $\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. The homogeneous solution to the problem is therefore

$$\mathbf{x}_c = C_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-6t} + C_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}.$$

To find the particular solution, use the method of undetermined coefficients. The forcing term is a constant, so let's guess that $\mathbf{x}_p = \mathbf{a}$, a constant vector. Plugging in (and noting that the derivative of a constant vector is zero), we get

$$0 = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} 60 \\ 0 \end{pmatrix}.$$

This is $2a_1 - 2a_2 = 60$ and $2a_1 - 5a_2 = 0$. Subtracting the second from the first, $3a_2 = 60$, so $a_2 = 20$. Then $a_1 = 50$. Our final solution is therefore

$$\mathbf{x} = \mathbf{x}_c + \mathbf{x}_p = C_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-6t} + C_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 50 \\ 20 \end{pmatrix}.$$