For all problems, *SHOW ALL OF YOUR WORK*. While partial credit will be given, partial solutions that could be obtained directly from a calculator or a guess are worth no points. Continue your work on the back of the page or extra sheet at the end of the exam if you need additional space. *You do not need but may use the normal graphing calculator functions of any graphing calculator, but NOT any differential equations functionality it may have.* If you need to borrow a graphing calculator, ask me.

1. Find real-valued solutions to the following, as indicated. Where possible, give an explicit answer.

a. Find the solution to $\frac{dy}{dx} = 3y - e^{2x}$, $y(0) = 4$. (8 points)

Solution: This is a first-order linear problem, and is not separable, so we use an integrating factor to solve it. First, move the $3y$ to the LHS: $\frac{dy}{dx} - 3y = -e^{2x}$, so the integrating factor is $\rho(x) = \exp(\int -3\,dx) = e^{-3x}$. Thus, multiplying both sides of the equation by $\rho(x)$,

$$e^{-3x}(y' - 3y) = (e^{-3x}y)' = -e^{-x}.$$

(Where the collapse of the LHS is because of how we picked the integrating factor.) Integrating both sides, $e^{-3x}y = e^{-x} + C$, so $y = e^{2x} + Ce^{3x}$. Applying the initial condition, $4 = 1 + C$, so $C = 3$, and $y = e^{2x} + 3e^{3x}$. ■

b. Find the solution to $y'' + 2y' + y = 0$, $y(0) = 1$, $y'(0) = 2$. (8 points)

Solution: For higher-order, linear constant-coefficient problems, we guess $y = e^{rt}$. Then $r^2 + 2r + 1 = 0$, so $r = -1$ (twice). Thus a general solution is $y = c_1e^{-t} + c_2te^{-t}$. Applying the initial conditions, $1 = c_1$, and $2 = -c_1 + c_2$, so $c_1 = 1$ and $c_2 = 3$. The solution is $y = e^{-t} + 3te^{-t}$. ■

c. Find the general solution to $(x^2 + 1)y' = 3xy$. (5 points)

Solution: This is linear and separable, so we could solve it either by separation or with an integrating factor. Let's do the former, in that the two in this case degenerate to essentially the same calculation. Separating,

$$\frac{dy}{y} = \frac{3x}{x^2 + 1}$$

so that, integrating both sides, $\ln|y| = \frac{3}{2}\ln|x^2 + 1| + \hat{C}$. Moving the $\frac{3}{2}$ into the exponent on the right-hand side and exponentiating both sides, we get $y = C(x^2 + 1)^{3/2}$. Note that $y = 0$ is also a solution (by inspection), which our general solution captures if we drop the restriction $C \neq 0$ implied by its derivation as $C = \pm e^{\hat{C}}$. ■

d. Find the general solution to $y''' + 9y' = 0$. (5 points)

Solution: Again, we use $y = e^{rt}$, getting the characteristic equation $r^3 + 9r = 0$, which factors as $r(r^2 + 9) = 0$. Thus $r = 0$ or $r = \pm 3i$, and the general solution is $y = c_1 + c_2\cos(3t) + c_3\sin(3t)$. ■

2. Four solutions of the differential equation $x^2y''+4xy'+2y = 0$ $(x > 0)$ are $y_1 = \frac{1}{x^2}$, $y_2 = 4x^{-2}$, $y_3 = \frac{2+x}{x^2}$ and $y_4 = x^{-1}$. Find a general solution to the problem. Explain why you arrive at the solution you do. (8 points)

Solution: A general solution is $y = c_1Y_1+c_2Y_2$, where Y_1 and Y_2 are two linearly independent solutions to the differential equation. (Because it's a second-order equation we know that we need only two such solutions to make the general solution.) We're given that all four of y_1, y_2, y_3 and y_4 are solutions, so the question is which two we should pick. Let's use y_1 and y_4, because they are "different" and easy to work with. Their Wronskian is

$$W(y_1, y_4) = \begin{vmatrix} x^{-2} & x^{-1} \\ -2x^{-3} & -x^{-2} \end{vmatrix} = x^{-4}.$$

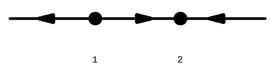
We know $x > 0$, so this is nonzero (and defined), and y_1 and y_4 are therefore linearly independent. Thus a general solution is $y = c_1x^{-2} + c_2x^{-1}$. (It would be possible to make a general solution with other pairs—in particular, y_1 and y_3, or y_3 and y_4, but not with y_1 and y_2.) ■

a. What does "general solution" mean, anyway? (4 points)

Solution: By "general solution," we mean a solution that captures all possible solutions to the differential equation. That is, any solution to the differential equation may be written as $y = c_1x^{-2} + c_2x^{-1}$ for some choice of c_1 and c_2. Another way of thinking about this is that the different solutions to the differential equation will be distinguished by different pairs of initial conditions $x(0) = a$ and $x'(0) = b$. By picking c_1 and c_2 in our general solution, we are able to satisfy initial conditions for any such (a, b). ■

3. Suppose that the population of blue ruffle-headed sneetches is modeled by the differential equation $\frac{dP}{dt} = 2(1 - P)(P^2 - 4)$. Use *qualitative methods* (that is, you don't have to—and probably don't want to—solve the equation) to predict what you expect the population of sneetches is likely to be in the long-term. Does your answer depend on the initial number of sneetches? Explain. (15 points)

Solution: For a qualitative analysis we construct a phase diagram. Equilibrium populations are when $\frac{dP}{dt} = 0$, which is when $P = 1$, $P = 2$ or $P = -2$. We can probably disregard the negative population of sneetches. Then we note that if $P > 2$, $2(1 - P)(P^2 - 4) < 0$, so $\frac{dP}{dt} < 0$, and the population is decreasing. Similarly, if $1 < P < 2$, $\frac{dP}{dt} > 0$, and if $0 < P < 1$, $\frac{dP}{dt} < 0$. Thus we have the phase diagram



This tells us that if we start with a population of sneetches greater than one (population unit), the population will end up (in the long-term) at a population of $P = 2$, while if we start less than $P = 1$, the population will go to zero. Thus the answer does depend on the initial number of sneetches. ■

4. Find each of the following.
   a. $(1+i\sqrt{3})^{10}$ (5 points)

   Solution: It is easiest to do this by first rewriting the complex number $z = 1 + i\sqrt{3}$ in exponential form, $z = re^{i\theta}$. $r = |z| = (1^2 + (\sqrt{3})^2)^{1/2} = 2$, and $\theta = \arctan(\sqrt{3}) = \pi/3$. Thus we want $(2e^{i\pi/3})^{10} = 2^{10}e^{10i\pi/3} = 2^{10}e^{i(2\pi+4\pi/3)}$. $e^{2i\pi} = 1$, so this is the same as $2^{10}e^{4i\pi/3} = 2^{10}(\cos(4\pi/3) + i\sin(4\pi/3))$. (Or, $-2^9 - \sqrt{3}\cdot 2^9 i$.) ∎

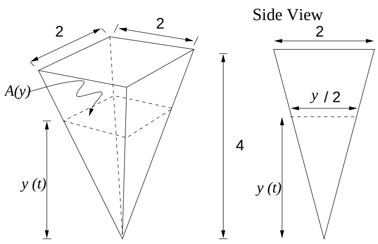
   b. The real and imaginary parts of $z = \frac{1+i}{1+2i}$. (5 points)

   Solution: To do this, we multiply the numerator and denominator by the complex conjugate of the denominator:

$$z = \frac{1+i}{1+2i}\cdot\frac{1-2i}{1-2i} = \frac{3-i}{5}.$$

   Thus $\mathrm{Re}(z) = 3/5$ and $\mathrm{Im}(z) = -1/5$. ∎

5. Let $y(t)$ be the height of water in a tank and $A(y)$ be the area of a horizontal cross-section of the tank at the height y, as shown in the figure to the right. Then as water drains from the tank, the time rate of change of y is proportional to the square root of y divided by the area of the cross-section at that height.



   a. Write a differential equation for $y(t)$ in this case. You should be able to write $A(y)$ in terms of y. (6 points)

   Solution: From the description, we have that

$$\frac{dy}{dt} = \frac{-\hat{k}\sqrt{y}}{A(y)},$$

   where we have picked $\hat{k} > 0$ and written in the negative sign to explicitly indicate that the tank is draining. Here $A(y) = y^2/4$, so that this becomes $y' = -ky^{-3/2}$ (with $k = 4\hat{k}$). ∎

   b. If the height of water in the tank is initially 3 and at $t = 1$ the height has dropped to 2, find when the tank will be empty. (15 points)

   Solution: We can solve our differential equation from above using separation of variables. Separating, we get $y^{3/2}dy = -kdt$. Integrating both sides, $\frac{2}{5}y^{5/2} = -kt + C$. Applying the initial condition and condition at $t = 1$, we have

$$\frac{2}{5}(3^{5/2}) = 0 + C$$
$$\frac{2}{5}(2^{5/2}) = -k\cdot 1 + C$$

   The first gives that $C = \frac{2}{5}(3^{5/2})$, and the second that $k = C - \frac{2}{5}(2^{5/2})$. The tank is empty when $y = 0$, which is when $0 = -kt + C$, so $t = C/k$, or

$$t = \frac{3^{5/2}}{3^{5/2} - 2^{5/2}} \approx 1.57.$$

∎

6. Recall that the Euler and Improved Euler methods for the first-order differential equation $\frac{dy}{dx} = f(x, y)$ have the iteration formulae

$$y_{n+1} = y_n + hf(x_n, y_n) \quad \text{and}$$
$$y_{n+1} = y_n + \tfrac{1}{2}h(k_1 + k_2), \quad \text{where} \quad k_1 = f(x_n, y_n) \quad \text{and} \quad k_2 = f(x_{n+1}, y_n + hk_1),$$

respectively. If we are approximating solutions to the initial value problem $y' + \frac{2}{x}y = x$, $y(1) = 1$, fill in the missing boxes in the following table. Be sure it is clear how you obtain your results. (8 points)

Method	$x =$	1	1.025	1.05	1.075	1.1	1.125	1.15	1.175	1.2
1. Euler, $h = 0.1$	$y \approx$	1	X	X	X	0.9	X	X	X	0.8464
2. Euler, $h = 0.025$	$y \approx$	1	0.975	0.9531	0.9339		0.9032	0.8912		0.8730
3. Impr Euler, $h = 0.1$	$y \approx$	1	X	X	X	0.9232	X	X	X	0.8821
4. Impr Euler, $h = 0.02$	$y \approx$	1	0.9765		0.9379	0.9224	0.9091	0.8978	0.8885	0.8809

Solution: For the missing points from the Euler row, we apply the formula given with $f(x, y) = x - 2y/x$.

$$y(1.1) \approx 0.9339 + 0.025(1.075 - 2(\frac{0.9339}{1.075})) = 0.9173, \quad \text{and}$$
$$y(1.175) \approx 0.8912 + 0.025(1.15 - 2(\frac{0.8912}{1.15})) = 0.8812.$$

For the missing point in the improved Euler row, $k_1 = f(1.025, 0.9765) = 1.025 - 2(0.9765/1.025) = -0.8804$. Thus

$$k_2 = f(1.05, 0.9765 + .025(-0.8804)) = f(1.05, 0.9542)$$
$$= 1.05 - 2(\frac{0.9542}{1.05}) = -0.7673.$$

Then

$$y(1.05) \approx 0.9765 + 0.0125(-0.8804 - 0.7673) = 0.9559.$$

∎

a. The exact solution to this IVP is $y = \frac{1}{4}x^2 + \frac{3}{4}x^{-2}$. Find the *cumulative error* for each of the methods. Explain why (or why not) it changes as you would expect as the step size decreases. (8 points)

Solution: The cumulative error is the error at the end of the approximations. Thus we need to know $y(1.2) = \frac{1}{4}(1.2)^2 + \frac{3}{4}(1.2)^{-2} = 0.8808$. The errors for the different methods and step sizes are therefore

Euler	$h = 0.1$	$\lvert 0.8808 - 0.8464\rvert = 0.0344$
	$h = 0.025$	$\lvert 0.8808 - 0.8730\rvert = 0.0078$
Impr Euler	$h = 0.1$	$\lvert 0.8808 - 0.8821\rvert = 0.0013$
	$h = 0.025$	$\lvert 0.8808 - 0.8809\rvert = 0.0001$

We know that Euler is a first-order method, so that if decrease the step size by a factor of four we should similarly divide the error by four. Here $0.0344/4 = 0.0086 \approx 0.0078$, so this looks reasonable. Similarly, Improved Euler is a second-order method, so the error should decrease by a factor of $4^2 = 16$ when we decrease the step size by a factor of four. Here $0.0013/16 = 0.0000825 \approx 0.0001$, so this again looks reasonable. ∎