For all problems, *SHOW ALL OF YOUR WORK*. While partial credit will be given, partial solutions that could be obtained directly from a calculator or a guess are worth no points. Continue your work on the back of the page or extra sheet at the end of the exam if you need additional space. *You do not need but may use the normal graphing calculator functions of any graphing calculator, but NOT any differential equations functionality it may have.*

1. Find the real-valued solutions to the following, as indicated.
   a. Find the general solution to $x'' + 4x' + 8x = 4t$. The complementary homogeneous solution is $x_c = c_1e^{-2t}\cos(2t) + c_2e^{-2t}\sin(2t)$. (6 points)

Solution: We know that the general solution is $x = x_c + x_p$, where x_p is a particular solution, so we have only to find the particular solution. Using the Method of Undetermined Coefficients, we guess $x_p = At + B$. Then $x_p' = A$ and $x_p'' = 0$, so $4A + 8At + 8B = 4t$. Thus $A = \frac{1}{2}$ and $B = -\frac{1}{4}$, and the general solution is

$$x = c_1e^{-2t}\cos(2t) + c_2e^{-2t}\sin(2t) + \frac{1}{2}t - \frac{1}{4}.$$

■

   b. Find the general solution to $\frac{d^3x}{dt^3} + 8x = 3\sin(2t)$. The complementary homogeneous solution is $x_c = c_1e^{-2t} + c_2e^{t/2}\cos(\sqrt{3}\,t/2) + c_3e^{t/2}\sin(\sqrt{3}\,t/2)$. (8 points)

Solution: Again, we need only find x_p. Using the Method of Undetermined Coefficients again, a good guess is $x_p = A\cos(2t) + B\sin(2t)$. Then $x_p''' = 8A\sin(2t) - 8B\cos(2t)$, so

$$(8A\sin(2t) - 8B\cos(2t)) + 8(A\cos(2t) + B\sin(2t)) = 3\sin(2t).$$

Matching the coefficients of $\sin(2t)$ and $\cos(2t)$, $8A + 8B = 3$ and $8A - 8B = 0$. Thus $A = B = \frac{3}{16}$, and the general solution is

$$x = c_1e^{-2t} + c_2e^{t/2}\cos(\sqrt{3}\,t/2) + c_3e^{t/2}\sin(\sqrt{3}\,t/2) + \frac{3}{16}(\cos(2t) + \sin(2t)).$$

■

   c. Find the solution to $x'' + 3x' + 2x = 3e^{-2t}$, $x(0) = 2$, $x'(0) = 0$. The complementary homogeneous solution is $x_c = c_1e^{-2t} + c_2e^{-t}$. (10 points)

Solution: In this case our first Method of Undetermined Coefficients guess, $x_p = Ae^{-2t}$, appears in the complementary homogeneous solution, so we must multiply it by t and instead guess $x_p = Ate^{-2t}$. Then $x_p' = Ae^{-2t} - 2Ate^{-2t}$ and $x_p'' = -4Ae^{-2t} + 4tAe^{-2t}$. Plugging in,

$$(-4Ae^{-2t} + 4tAe^{-2t}) + 3(Ae^{-2t} - 2Ate^{-2t}) + 2(Ate^{-2t}) = 3e^{-2t},$$

or $-Ae^{-2t} = 3e^{-2t}$, so $A = -3$. Thus a general solution is

$$x = c_1e^{-2t} + c_2e^{-t} - 3te^{-2t}.$$

Applying the initial conditions, $x(0) = 2$ requires $c_1 + c_2 = 2$, and $x'(0) = 0$ requires $-2c_1 - c_2 = 3$. Adding these, $c_1 = -5$, so that $c_2 = 7$. The solution is therefore

$$x = -5e^{-2t} + 7e^{-t} - 3te^{-2t}.$$

■

2. Find the real-valued solutions to the following, as indicated.

a. Find the general solution to $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. (10 points)

Solution: We'll solve this with the eigenvalue method. Letting $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{v}e^{\lambda t}$, we get

$$\begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0}.$$

For non-trivial solutions, the determinant of the matrix on the left-hand side must be zero, so that $(1-\lambda)(2-\lambda) - 6 = \lambda^2 - 3\lambda - 4 = 0$, giving $\lambda = 4$ or $\lambda = -1$. To find $\mathbf{v}$, we plug in these values for λ. If $\lambda = 4$,

$$\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

so that $-3v_1 + 2v_2 = 0$, and $v_1 = 2$, $v_2 = 3$ is a solution. If $\lambda = -1$,

$$\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

so that $v_1 + v_2 = 0$, and $v_1 = 1$, $v_2 = -1$ is a solution. Therefore a general solution to the problem is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}.$$

■

b. If $\mathbf{y} = (x_1\ x_2\ x_3)^{\mathrm{T}}$, use the eigenvalue method to find the general solution to $\mathbf{y}' = \mathbf{A}\mathbf{y}$ if $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -4 \\ 0 & 1 & 0 \end{pmatrix}$. *(For 10 points, you may solve instead for* $\mathbf{A} = \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix}$*).* (15 points)

Solution: Again, let $\mathbf{y} = \mathbf{v}e^{\lambda t}$. Then

$$\begin{pmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & -4 \\ 0 & 1 & -\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \mathbf{0}.$$

The determinant of the matrix must be zero, so that, expanding along the top row and noting that the second two terms in the determinant will vanish, $(1-\lambda)(\lambda^2+4) = 0$. Thus $\lambda = 1$ or $\lambda = \pm 2i$. For $\lambda = 1$, the eigenvector calculation is

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & -4 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \mathbf{0},$$

so that $v_1 - v_2 - 4v_3 = 0$ and $v_2 = v_3$. Combining these, $v_1 = 5v_3$ and $v_2 = v_3$. Thus if we take $v_3 = 1$, we get $v_2 = 1$ and $v_1 = 5$. For $\lambda = 2i$, we get

$$\begin{pmatrix} 1-2i & 0 & 0 \\ 1 & -2i & -4 \\ 0 & 1 & -2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \mathbf{0},$$

so that $(1-2i)v_1 = 0$, or $v_1 = 0$. Then $v_2 - 2iv_3 = 0$, so that picking $v_3 = 1$ we get $v_2 = 2i$. To find real-valued solutions we take the complex valued solution $\mathbf{x} = \begin{pmatrix} 0 \\ 2i \\ 1 \end{pmatrix} e^{2it}$ and break it into its real and imaginary parts:

$$\begin{aligned}
\mathbf{x} &= \begin{pmatrix} 0 \\ 2i \\ 1 \end{pmatrix} e^{2it} = \begin{pmatrix} 0 \\ 2i \\ 1 \end{pmatrix} (\cos(2t) + i\sin(2t)) \\
&= \begin{pmatrix} 0 \\ 2i\cos(2t) - 2\sin(2t) \\ \cos(2t) + i\sin(2t) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ -2\sin(2t) \\ \cos(2t) \end{pmatrix} + i \begin{pmatrix} 0 \\ 2\cos(2t) \\ \sin(2t) \end{pmatrix}.
\end{aligned}$$

Then our general solution is the combination of the real solution ($\lambda = 1$) and the real and imaginary parts we have here, or

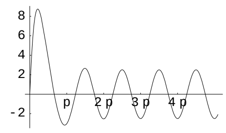
$$\mathbf{x} = c_1 \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ -2\sin(2t) \\ \cos(2t) \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 2\cos(2t) \\ \sin(2t) \end{pmatrix}.$$

■

3. For the following,
   i. write down a differential equation that could give the behavior described *(Note that there might be a number of reasonable answers to this!)*, and
   ii. **without** solving the equation, write down what functions you expect to see in the solution to the problem (e.g., e^{-t^2}, $12\ln(\arctan(t))$, or t^{762}).

   a. Resonance, with the resonant solution having frequency $\omega = 7$. (8 points)

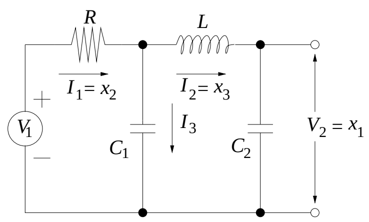
   Solution:
   i. Resonance implies that we are forcing the system at its natural frequency. The equation $x'' + 49x = 0$ has solution $x = c_1\cos(7t) + c_2\sin(7t)$, so it has the desired natural frequency $\omega_0 = 7$. To get resonance, we force it at this frequency: $x'' + 49x = \cos(7t)$.
   ii. The functions in the solution are $\cos(7t)$, $\sin(7t)$, and $t\sin(7t)$. (Technically we'd expect $t\cos(7t)$ as well, but it will turn out that this drops out.) ∎

   b. The behavior shown to the right. (8 points)

   Solution:
   i. This has a transient and steady state solution. The steady state is the long-term sinusoidal solution, which we see has period approximately π. Thus we must be forcing the problem with a frequency of 2. The transient indicates a decaying complementary homogeneous solution, so we expect something like $x'' + cx' + kx = \cos(2t)$. c and k can be any positive numbers; for ease of calculation, let's assume that they are 2 and 1, respectively.
   ii. Given our choice of c and k, the homogeneous solution will have terms like e^{-t} and te^{-t}, while from the forcing we'll get terms like $\cos(2t)$ and $\sin(2t)$. ∎

4. A circuit that is commonly found in electronic equipment is a "two-loop low-pass filter," which regulates an input voltage by filtering out high frequency oscillations in that voltage. For the interested, the circuit is shown in the figure to the right. A model for the circuit is the following. Write it in the matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$. (All capital letters and ω in the equations are constants, and primes denote derivatives with respect to t.)



$$x_1' = \frac{1}{C_2}x_3$$
$$x_2' = -\frac{1}{RC_1}x_2 + \frac{1}{RC_1}x_3 - \frac{\omega}{R}\cos(\omega t)$$
$$x_3' = -\frac{1}{L}x_1 - \frac{R}{L}x_2 + \frac{1}{L}\cos(\omega t)$$

Solution:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 0 & 0 & 1/C_2 \\ 0 & -1/RC_1 & 1/RC_1 \\ -1/L & -R/L & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ -\omega\cos(\omega t)/R \\ \cos(\omega t)/L \end{pmatrix}$$

∎

5. Let $\mathbf{M} = \begin{pmatrix} -1/2 & -2 \\ 1/3 & 4/3 \end{pmatrix}$ and $\mathbf{P} = \begin{pmatrix} -6 & 12 \\ 2 & -3 \end{pmatrix}$.

   a. Is $\mathbf{P}$ the inverse matrix for $\mathbf{M}$? (Why or why not?) (5 points)

   Solution: To check, let's multiply the two matrices:

   $$\mathbf{PM} = \begin{pmatrix} 7 & 28 \\ -2 & -8 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

   so no, it isn't. ■

   b. If $\mathbf{x} = (x_1\ x_2)^T$ and $\mathbf{b} = (1\ 3)^T$, how many solutions does $\mathbf{M}\,\mathbf{x} = \mathbf{0}$ have? How many solutions are there to $\mathbf{M}\,\mathbf{x} = \mathbf{b}$? Be sure to indicate why you give the answers you do. *Note that you do not have to solve these problems to answer this question!* (6 points)

   Solution: We can test this by looking at $\det(\mathbf{M})$ to determine if $\mathbf{M}$ is singular or not. $\det(\mathbf{M}) = -\frac{4}{6} + \frac{2}{3} = 0$, so $\mathbf{M}$ is singular. Therefore we know that $\mathbf{Mx} = \mathbf{0}$ has an infinite number of solutions, and $\mathbf{Mx} = \mathbf{b}$ will have either zero or an infinite number. Note that the second row in $\mathbf{M}$ is $-2/3$ times the first. The second row of $\mathbf{b}$ is not $-2/3$ times the first, so $\mathbf{Mx} = \mathbf{b}$ will have no solutions. ■

6. Consider the problem $\mathbf{y}' = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{y}$, where $\mathbf{y} = (x_1\ x_2)^T$.

   a. Verify that the vector functions $\mathbf{y}_1 = \begin{pmatrix} e^t \\ 0 \end{pmatrix}$ and $\mathbf{y}_2 = \begin{pmatrix} te^t \\ \frac{1}{2}e^t \end{pmatrix}$ are solutions to this problem. (6 points)

   Solution: To verify that these are solutions, we can just plug them in: $\mathbf{y}_1' = \begin{pmatrix} e^t \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{y}_1 = \begin{pmatrix} e^t + 2(0) \\ 0 + 0 \end{pmatrix}$, so it is clear that $\mathbf{y}_1' = \mathbf{A}\mathbf{y}_1$ for this matrix $\mathbf{A}$. Similarly, for $\mathbf{y}_2$, $\mathbf{y}_2' = \begin{pmatrix} te^t + e^t \\ \frac{1}{2}e^t \end{pmatrix}$ and $\mathbf{A}\mathbf{y}_2 = \begin{pmatrix} te^t + e^t \\ 0 + \frac{1}{2}e^t \end{pmatrix}$, so that $\mathbf{y}_2$ is also a solution. ■

   b. Write a general solution to the problem. What is true that allows you to construct the general solution in the manner you do? (6 points)

   Solution: The general solution is $\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2$, provided $\mathbf{y}_1$ and $\mathbf{y}_2$ are linearly independent (we need two such solutions because it's a 2×2 system). We can verify linear independence with the Wronskian:

   $$\mathrm{W}(\mathbf{y}_1, \mathbf{y}_2) = \begin{vmatrix} e^t & te^t \\ 0 & \frac{1}{2}e^t \end{vmatrix} = \frac{1}{2}e^{2t} \neq 0,$$

   so the two are linearly independent. ■

   c. Use your general solution to find the particular solution to this problem satisfying the initial condition $\mathbf{y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. (6 points)

   Solution: If $\mathbf{y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, we have from our general solution above that $c_1 + 0 = 1$, so $c_1 = 1$, and $\frac{1}{2}c_2 = 1$, so $c_2 = 2$. The particular solution is therefore

   $$\mathbf{y} = \begin{pmatrix} e^t \\ 0 \end{pmatrix} + \begin{pmatrix} 2te^t \\ e^t \end{pmatrix} = \begin{pmatrix} 1 + 2t \\ 1 \end{pmatrix} e^t.$$

   ■