Sample Solution for a Forced System

Consider the system

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -5 & -4 \end{pmatrix} + \begin{pmatrix} 0 \\ 7 \end{pmatrix} \sin(3t).$$

You can verify that a complementary homogeneous solution is given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_c = c_1 \begin{pmatrix} e^{-2t}\cos(t) \\ e^{-2t}(-2\cos(t) - \sin(t)) \end{pmatrix} + c_2 \begin{pmatrix} e^{-2t}\sin(t) \\ e^{-2t}(\cos(t) - 2\sin(t)) \end{pmatrix}.$$

To find the particular solution, we want to use the *Method of Undetermined Coefficients*, which says "guess a particular solution that looks like the forcing term and derivatives of the forcing term."

Here, the forcing term looks like $\sin(3t)$, and derivatives of this look like $\cos(3t)$ or $\sin(3t)$, so we want to guess $\mathbf{x}_p = \mathbf{a}\cos(3t) + \mathbf{b}\sin(3t)$. This is exactly the same as what we did in §3.5, except that the constants in our guess are now vectors. To emphasize this, we might write the guess out showing the vector components, viz.,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_p = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cos(3t) + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \sin(3t) = \begin{pmatrix} a_1\cos(3t) + b_1\sin(3t) \\ a_2\cos(3t) + b_2\sin(3t) \end{pmatrix}. \tag{1}$$

Then to find the values of a_1, a_2, b_1 and b_2, we just plug this into the system of differential equations. Note that if we had been solving a problem with second derivatives we could have discarded the term in $\cos(3t)$. In this case, however, we cannot do that.

Let's do this two ways: first, in vector notation. We need the derivative of $\mathbf{x}_p$, which is $-3\mathbf{a}\sin(3t) + 3\mathbf{b}\cos(3t)$. Plugging in, then, we have

$$-3\mathbf{a}\sin(3t) + 3\mathbf{b}\cos(3t) = \mathbf{A}(\mathbf{a}\cos(3t) + \mathbf{b}\sin(3t)) + \mathbf{F}\sin(3t),$$

(where $\mathbf{A}$ is the coefficient matrix of the problem and $\mathbf{F} = (0\ 7)^T$) so that we have the two equations

$$-3\mathbf{a} = \mathbf{A}\mathbf{b} + \mathbf{F} \qquad \text{and} \qquad 3\mathbf{b} = \mathbf{A}\mathbf{a}.$$

Rearranging these as usual, we get

$$-\mathbf{F} = \mathbf{A}\mathbf{b} + 3\mathbf{a} \qquad \text{and} \qquad 0 = \mathbf{A}\mathbf{a} - 3\mathbf{b}.$$

It's easiest to next rewrite these in terms of a_1, a_2, b_1 and b_2 by multiplying out the matrix equations. This gives us

$$\begin{aligned} 0 &= b_2 + 3a_1 \\ -7 &= -5b_1 - 4b_2 + 3a_2 \\ 0 &= a_2 - 3b_1 \\ 0 &= -5a_1 - 4a_2 - 3b_2. \end{aligned} \tag{2}$$

Four equations in four unknowns. No problem... I get $a_1 = -\frac{21}{40}$, $a_2 = -\frac{21}{40}$, $b_1 = -\frac{7}{40}$ and $b_2 = \frac{63}{40}$.

Next, suppose we wanted to go straight to a set of (non-vector) equations. From the component form shown in equation (1), we can calculate the derivative and plug into the equation to get

$$\begin{pmatrix} -3a_1\sin(3t) + 3b_1\cos(3t) \\ -3a_2\sin(3t) + 3b_2\cos(3t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -5 & -4 \end{pmatrix} \begin{pmatrix} a_1\cos(3t) + b_1\sin(3t) \\ a_2\cos(3t) + b_2\sin(3t) \end{pmatrix} + \begin{pmatrix} 0 \\ 7\sin(3t) \end{pmatrix},$$

so that

$$\begin{aligned} \begin{pmatrix} -3a_1\sin(3t) + 3b_1\cos(3t) \\ -3a_2\sin(3t) + 3b_2\cos(3t) \end{pmatrix} &= \begin{pmatrix} a_2\cos(3t) + b_2\sin(3t) \\ -5a_1\cos(3t) - 5b_1\sin(3t) - 4a_2\cos(3t) - 4b_2\sin(3t) \end{pmatrix} + \begin{pmatrix} 0 \\ 7\sin(3t) \end{pmatrix} \\ &= \begin{pmatrix} a_2\cos(3t) + b_2\sin(3t) \\ -5a_1\cos(3t) - 5b_1\sin(3t) - 4a_2\cos(3t) - 4b_2\sin(3t) + 7\sin(3t) \end{pmatrix}. \end{aligned}$$

In both the top and bottom of these equations, the coefficients of the cosine terms must match up, as must the coefficients of the sine terms. This gives the set of equations in (2).

Thus a general solution to the problem is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} e^{-2t}\cos(t) \\ e^{-2t}(-2\cos(t) - \sin(t)) \end{pmatrix} + c_2 \begin{pmatrix} e^{-2t}\sin(t) \\ e^{-2t}(\cos(t) - 2\sin(t)) \end{pmatrix} + \begin{pmatrix} -\frac{21}{40} \\ -\frac{21}{40} \end{pmatrix} \cos(3t) + \begin{pmatrix} -\frac{7}{40} \\ \frac{63}{40} \end{pmatrix} \sin(3t).$$