For all problems, *SHOW ALL OF YOUR WORK*. While partial credit will be given, partial solutions that could be obtained directly from a calculator or a guess are worth no points. Continue your work on the back of the page or extra sheet at the end of the exam if you need additional space. *You do not need but may use the normal graphing calculator functions of any graphing calculator, but NOT any differential equations functionality it may have.* If you need to borrow a graphing calculator, ask me.

1. Solve each of the following differential equations to obtain a general solution or, where possible, a particular one. If possible, give an explicit solution.

   a. $2y' = \frac{1}{x}y - 4x^2$, $y(1) = -1$ (8 points)
   Solution: This is a first-order linear problem, so we first put it in standard form, $y' - \frac{1}{2x}y = -2x^2$ and then find an integrating factor. The integrating factor is $u(x) = e^{\int -1/(2x)dx} = x^{-1/2}$, so

$$(yx^{-1/2})' = -2x^{3/2}.$$

   Integrating both sides, $yx^{-1/2} = -\frac{4}{5}x^{5/2} + C$, so that

$$y = -\frac{4}{5}x^3 + Cx^{1/2}.$$

   To satisfy the initial condition $y(1) = -1$, $C = -\frac{1}{5}$ and $y = -\frac{4}{5}x^3 - \frac{1}{5}\sqrt{x}$.

   b. $y'' + 2y' + 5y = 0$ (8 points)
   Solution: This is a homogeneous constant coefficient linear problem, so we guess the solution $y = e^{rt}$. Plugging in, $r^2 + 2r + 5 = 0$, so that $r = -1 \pm 2i$. Thus the solution is

$$y = C_1e^{-t}\cos(2t) + C_2e^{-t}\sin(2t).$$

   c. $xy\frac{dy}{dx} = x^2 - x^2y^2$ (8 points)
   Solution: This is first-order and nonlinear, but separable. Separating, we get

$$\frac{ydy}{1-y^2} = x,$$

   so that, integrating both sides, $-\frac{1}{2}\ln(|1-y^2|) = \frac{1}{2}x^2 + \hat{C}$, or

$$y = \sqrt{1 - Ce^{-x^2}}.$$

   d. $2y'' + 8y' + 8y = 0$, $y(0) = 0$, $y'(0) = 3$ (8 points)
   Solution: This is a homogeneous constant coefficient linear problem, so we divide through by 2 and guess the solution $y = e^{rt}$. Plugging in, $r^2 + 4r + 4 = 0$, so that $r = -2$ twice. Thus the general solution is $y = C_1e^{-2t} + C_2te^{-2t}$. Plugging in the initial conditions, we get $C_1 = 0$ from $y(0) = 0$ and $C_2 = 3$ from $y'(0) = 3$. Thus

$$y = 3te^{-2t}.$$

2. A passing tortoise is heard to assert that $y_1 = x^2(1-x)^{-1}$ and $y_2 \equiv 0$ are both solutions to the differential equation

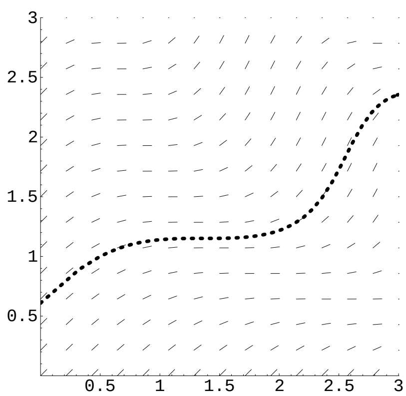
$$\frac{1}{y}\frac{dy}{dx} = \frac{y+2x}{x^2}$$

with initial condition $y(1) = 0$.

   a. Is the tortoise correct? (6 points)
   Solution: The tortoise is incorrect. We note that to satisfy an initial value problem the proposed solution must satisfy both the initial condition and the differential equation. The initial condition $y(1) = 0$ is satisfied by $y_2 \equiv 0$, but not by y_1. Thus they cannot both be solutions. We note that y_2 is in fact a solution (if we multiply through by y to get the standard form $y' = y\frac{y+2x}{x^2}$), and it can be shown that y_1 also satisfies the differential equation (but not the initial condition). *(One can defensibly argue that $\frac{1}{y}\frac{dy}{dx}$ is an indeterminant form if $y = 0$, so that the equation before multiplying by y is more restrictive. For our purposes we will take the two forms to be equivalent, and the tortoise is in any event incorrect.)*

   b. How is this related to the Existence and Uniqueness Theorem for first-order ordinary differential equations? (8 points)
   Solution: The Existence and Uniqueness Theorem says that if $\frac{dy}{dx} = f(x,y)$ and $f(x,y)$ is continuous in a region around the initial condition (here, $(x,y) = (1,0)$), then there is a solution. If $\frac{\partial f}{\partial x}$ is similarly continuous, the solution is unique. Here $f(x,y) = y\frac{y+2x}{x^2}$ satisfies both of these conditions, so the differential equation $\frac{dy}{dx} = y\frac{y+2x}{x^2}$ is guaranteed to have a unique solution (which is $y \equiv 0$).

3. The direction field for a differential equation $\frac{dy}{dx} = f(x,y)$ is shown to the right.



   a. Sketch a solution that goes through $y(0.5) = 1.0$. (3 points)

   b. How many solution curves go through the point $(0.5, 1.0)$? How do you know? (3 points)
   Solution: Only one curve can go through the point (0.5,1.0). We know that for well-behaved ODEs solutions curves cannot cross, and this is evidentially well-behaved. Further, for there to be no solutions or more than one solution the requirements of the Existence and Uniqueness Theorem would have to be violated, which would require that the direction field have kinks or breaks, which it doesn't.

   c. For the solution you drew, what are the approximate values of $y(0)$ and $y(3)$? (3 points)
   Solution: $y(0) \approx 0.61$ and $y \approx 2.36$.

4. An alert student notes that her initially stationary professor has an initial acceleration of 10 m/s^2. Air resistance, however, results in a retardation of this acceleration which is proportional to the professor's speed.

   a. Write an initial value problem modeling this. (6 points)
   Solution:

$$\frac{dv}{dt} = 10 - kv, \qquad v(0) = 0.$$

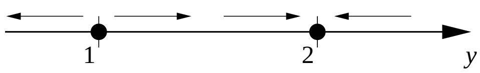
   b. What can you say about the professor's motion without solving the differential equation? (3 points)
   Solution: Note that if $\frac{dv}{dt} = 0$ then $10 - kv = 0$, so $v = 10/k$. This is what we expect terminal velocity to be.

   c. Solve your initial value problem. What additional data do you need to be able to obtain an explicit solution? (6 points)
   Solution: This is a separable first-order linear differential equation, so we can separate variables or use integrating factors. Separating, $dv(10 - kv)^{-1} = dt$, so $-\frac{1}{k}\ln(|10 - kv|) = t + \hat{C}$, and, solving for v, $v = Ce^{-kt} + \frac{10}{k}$. The initial condition is $v(0) = 0$, so

$$v = \frac{10}{k}(1 - e^{-kt}).$$

   To obtain an explicit solution we need to know k. We could get this if we knew the professor's velocity at an additional time.

5. Write a differential equation that could result in the following phase diagram. Explain briefly why you choose the equation you do. (8 points)



   Solution: The derivative has to be zero at the equilibrium points 1 and 2, so a good first guess is $\frac{dy}{dx} = k(y - 1)(y - 2)$. To have $\frac{dy}{dx} < 0$ for $y < 1$ and $y > 2$ (and $\frac{dy}{dx} > 0$ for $1 < y < 2$) we must reverse the sign on this, giving

$$\frac{dy}{dx} = -k(y - 1)(y - 2).$$

   $(k > 0)$.

   a. What are the equilibrium solutions of this differential equation? Which are stable? (4 points)
   Solution: The equilibrium solutions are $y = 1$ and $y = 2$. $y = 2$ is stable.

6. Euler's method, Improved Euler's method, and Runge-Kutta are used to solve a differential equation $y'(x) = f(x, y(x))$, $y(0) = 1$. Some of the resulting points are given in the following table. The cumulative error for each method at $x = 0.4$ is then given.

Method	$x =$	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4
1.	$y \approx$	1	1.0487		1.1377	1.1776	1.2143	1.2476		1.3044
2.	$y \approx$	1		1.0974	1.1419		1.2217	1.2566	1.2882	1.3164
3.	$y \approx$	1	1.0487	1.0947			1.2144	1.2477	1.2778	1.3045

Method	1	2	3
Error	0.0001260	0.01192	1.700×10^{-6}

a. What is the step size h? (2 points)
Solution: $h = 0.05$.

b. Identify which method is which by filling in the first column of the table. (How do you know?) (4 points)
Solution: Methods 1, 2, and 3 are, respectively, Improved Euler, Euler's and the Runge-Kutta Methods. We know this because we expect the errors for Improved Euler to be much less than those for Euler's method, and those for Runge-Kutta to be even smaller.

c. Fill in the missing values in the Euler's method row, if the differential equation being solved is $y' = 1 - \sin(xy)$. Be sure it is clear how you obtain your result. (6 points)
Solution: Euler's method says $y_{n+1} = y_n + hf(x_n, y_n)$, so

$$y(0.05) \approx y_1 = 1 + 0.05(1 - \sin(0 \cdot 1)) = 1.05,$$

and

$$y(0.20) \approx y_4 = 1.1419 + 0.05(1 - \sin(0.15 \cdot 1.1419)) = 1.1834.$$

d. Suppose that we recalculated the values in the table with $h = 0.2$. What would you expect the error at $x = 0.4$ to be for the Euler's method calculation? For the Improved Euler's method? Why? (6 points)
Solution: $h = 0.2$ is 4 times as large as the previous step size. The error in Euler's method is proportional to the step size, so we expect that the error with $h = 0.2$ will be approximately $4(0.01192) = 0.04768$. For Improved Euler, the error is proportional to the step size squared, so we expect the error to increase by a factor of 16 to $16(0.0001260) = 0.002016$.