For all problems, *SHOW ALL OF YOUR WORK*. While partial credit will be given, partial solutions that could be obtained directly from a calculator or a guess are worth no points. Continue your work on the back of the page or extra sheet at the end of the exam if you need additional space. *You do not need but may use the normal graphing calculator functions of any graphing calculator, but NOT any differential equations functionality it may have.* If you need to borrow a graphing calculator, ask me.

1. a. The solution to the complementary homogeneous problem for $y'' + 4y' + 3y = 9x^2$ is $y_c = c_1 e^{-x} + c_2 e^{-3x}$. Find the general solution to this problem. (8 points)

   b. Two linearly independent solutions to $y'' + 4y' + 5y = 0$ are $y_1 = e^{-2x}\cos(x)$ and $y_2 = e^{-2x}\sin(x)$. Find the general solution to $y'' + 4y' + 5y = 4\sin(x)$. (8 points)

   c. $y_1 = x^{1/3}$ and $y_2 = x^{2/3}$ are two linearly independent solutions to $y'' + \frac{2}{x}y' + \frac{1}{x^2}y = 0$. What would be a good guess for y_p for the problem $y'' + \frac{2}{x}y' + \frac{1}{x^2}y = 3x$? Why? *(Don't actually solve for y_p.)* (6 points)

2. Solve the vector equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$ to find real-valued solutions for each of the following matrices $\mathbf{A}$.

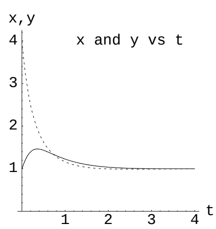
   a. $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ (12 points)

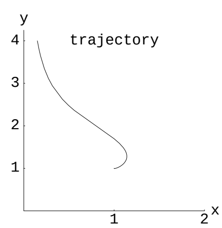
   b. $\mathbf{A} = \begin{pmatrix} 0 & 2 \\ -4 & -4 \end{pmatrix}$ (12 points)

3. To the right is the plot vs. t of a solution $x(t)$ (solid curve) and $y(t)$ (dashed curve) of some system

$$\begin{aligned} x' &= f(t, x, y) \\ y' &= g(t, x, y). \end{aligned}$$

   Next to it is the graph of a trajectory in the phase plane. Do the two graphs describe the the same solution? Explain. (6 points)





4. The tortoise passing by the first midterm has finally made it to the second one. After very careful thought, it says the following: "Consider the system

$$x' = 3x(1-2x) - 2xy$$
$$y' = 2y(1-y) + xy$$

Letting $\mathbf{z} = (x\ y)^T$, I know a general solution to this will be

$$\mathbf{z} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_3 t},$$

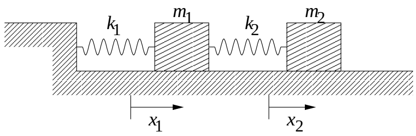
where λ_1, λ_2, and λ_3 are the eigenvalues of the coefficient matrix $\mathbf{A}$ when the system is written in matrix form and $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are the corresponding eigenvectors." Is the tortoise correct? Answer as completely as possible. (10 points)

5. A hyperactive kitten named Newton is batting at a mass attached to a spring. The resulting displacement of the mass is modeled by $mx'' + cx' + kx = F\sin(\omega t)$.
   a. Suppose that the parameters in the problem were picked so that the system was at resonance. Sketch a graph of what you expect the solution for x to look like in this case. Then suppose that we instead were seeing beats. Sketch another graph showing what you expect the solution for x to look like then. (4 points)

   b. What conditions on m, c, k, F and/or ω would ensure that the system was at resonance? What conditions would produce beats? (6 points)

   c. Solve the system with $m = 1$, $c = 0$, $k = 4$, $F = 3$ and $\omega = 2$ and the initial conditions $x(0) = x'(0) = 0$. Does your solution exhibit resonance, beats, practical resonance, or none of these? Why? (8 points)

6. Consider the mass-spring system shown to the right.



a. Explain why the system

$$x_1'' = -k_1 x_1 + k_2(x_2 - x_1)$$
$$x_2'' = \qquad - k_2(x_2 - x_1) + F_1$$

is a good model this. What assumptions are we making for this to be the case? (4 points)

b. Write this as a matrix equation. (6 points)

c. If $F_1 = 0$, what would you guess to solve the matrix equation? *(Don't actually solve it, however!)* (4 points)

d. If $F_1 = 3\sin(2t)$, how does your guess in (c) change? *(Again, don't actually solve the problem.)* (4 points)