For all problems, *SHOW ALL OF YOUR WORK*. While partial credit will be given, partial solutions that could be obtained directly from a calculator or a guess are worth no points. Continue your work on the back of the page or extra sheet at the end of the exam if you need additional space. *You do not need but may use the normal graphing calculator functions of any graphing calculator, but NOT any differential equations functionality it may have.* If you need to borrow a graphing calculator, ask me.

1. a. The solution to the complementary homogeneous problem for $y'' + 4y' + 3y = 9x^2$ is $y_c = c_1e^{-x} + c_2e^{-3x}$. Find the general solution to this problem. (8 points)

Solution: To find a particular solution, use the Method of Undetermined Coefficients and guess $y_p = Ax^2 + Bx + C$. Plugging in, $2A + 4(2Ax + B) + 3(Ax^2 + Bx + C) = 9x^2$, so, collecting powers of x, $3Ax^2 = 9x^2$, $8Ax + 3Bx = 0$, and $2A + 4B + 3C = 0$. Thus $A = 3$, $B = -8$, and $C = \frac{26}{3}$. A general solution is therefore

$$y = c_1e^{-x} + c_2e^{-3x} + 3x^2 - 8x + \frac{26}{3}.$$

b. Two linearly independent solutions to $y'' + 4y' + 5y = 0$ are $y_1 = e^{-2x}\cos(x)$ and $y_2 = e^{-2x}\sin(x)$. Find the general solution to $y'' + 4y' + 5y = 4\sin(x)$. (8 points)

Solution: As before, we use the Method of Undetermined Coefficients and guess $y_p = A\cos(x) + B\sin(x)$. Plugging in and collecting the $\cos(x)$ and $\sin(x)$ terms, we get the two equations $-A + 4B + 5A = 0$ and $-B - 4A + 5B = 4$. The first is $A = -B$, so that the second gives $-8A = 4$, or $A = -\frac{1}{2}$, so $B = \frac{1}{2}$. Thus the general solution is

$$y = c_1e^{-2x}\cos(x) + c_2e^{-2x}\sin(x) - \frac{1}{2}\cos(x) + \frac{1}{2}\sin(x).$$

c. $y_1 = x^{1/3}$ and $y_2 = x^{2/3}$ are two linearly independent solutions to $y'' + \frac{2}{x}y' + \frac{1}{x^2}y = 0$. What would be a good guess for y_p for the problem $y'' + \frac{2}{x}y' + \frac{1}{x^2}y = 3x$? Why? *(Don't actually solve for y_p.)* (6 points)

Solution: In this case the differential equation is no constant-coefficient, so we don't expect the Method of Undetermined Coefficients to work. A better guess is therefore to use the method of Variation of Parameters, and guess that $y_p = u_1(x)x^{1/3} + u_2(x)x^{2/3}$.

2. Solve the vector equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$ to find real-valued solutions for each of the following matrices $\mathbf{A}$.

   a. $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ (12 points)

   Solution: Let $\mathbf{x} = \mathbf{v}e^{\lambda t}$. Then for non-trivial solutions we must have that the determinant $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$, or $\lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0$. Thus $\lambda = 3$ or $\lambda = -1$. If $\lambda = 3$, then $\mathbf{v}$ must satisfy $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0}$, so that $\mathbf{v} = (\,1 \quad 1\,)^T$. If $\lambda = -1$, $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0}$, so that $\mathbf{v} = (\,-1 \quad 1\,)^T$. Thus the solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}.$$

   b. $\mathbf{A} = \begin{pmatrix} 0 & 2 \\ -4 & -4 \end{pmatrix}$ (12 points)

   Solution: Again let $\mathbf{x} = \mathbf{v}e^{\lambda t}$. Then for non-trivial solutions $\begin{vmatrix} -\lambda & 2 \\ -4 & -4-\lambda \end{vmatrix} = 0$, or $\lambda^2 + 4\lambda + 8 = 0$. This gives (using the quadratic formula) $\lambda = -2 \pm 2i$. If $\lambda = -2 + 2i$, then $\mathbf{v}$ must satisfy $\begin{pmatrix} 2-2i & 2 \\ -4 & -2-2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0}$. The first of these equations is $(2 - 2i)v_1 + 2v_2 = 0$, so that, taking $v_1 = 1$, $v_2 = -1 + i$. A complex-valued solution is therefore

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1+i \end{pmatrix} e^{(-2+2i)t}.$$

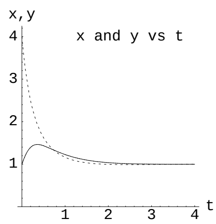
   To get a real-valued solution, we find the real and imaginary parts of this, using those as the two linearly independent solutions that we need to form the general solution:

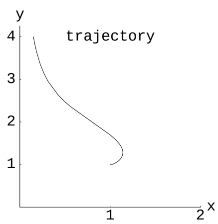
$$\mathbf{x} = c_1 e^{-2t} \begin{pmatrix} \cos(2t) \\ -\cos(2t) - \sin(2t) \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} \sin(2t) \\ \cos(2t) - \sin(2t) \end{pmatrix}.$$

3. To the right is the plot vs. t of a solution $x(t)$ (solid curve) and $y(t)$ (dashed curve) of some system

$$x' = f(t, x, y)$$
$$y' = g(t, x, y).$$

   Next to it is the graph of a trajectory in the phase plane. Do the two graphs describe the the same solution? Explain. (6 points)





   Solution: No. When $t = 0$ in the first figure, $x = 1$ and $y = 4$. Thus the point (4,1) has to be on the trajectory if they represent the same solution, which it isn't. Thus they cannot be the same.

4. The tortoise passing by the first midterm has finally made it to the second one. After very careful thought, it says the following: "Consider the system

$$x' = 3x(1-2x) - 2xy$$
$$y' = 2y(1-y) + xy$$

Letting $\mathbf{z} = (x\ y)^T$, I know a general solution to this will be

$$\mathbf{z} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_3 t},$$

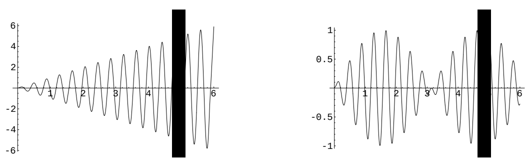
where λ_1, λ_2, and λ_3 are the eigenvalues of the coefficient matrix $\mathbf{A}$ when the system is written in matrix form and $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are the corresponding eigenvectors." Is the tortoise correct? Answer as completely as possible. (10 points)

Solution: The tortoise is again wrong. The system under consideration is nonlinear, so that it doesn't make sense to talk about a general solution in the first place. Further, the nonlinearity disallows the superposition that lets us form a general solution in this way for linear problems. Not only that, but for nonlinear problems we can't use the eigenvalue method to find a solution. And if that weren't enough, even if it were a linear problem and we could use the eigenvalue method, find solutions and add them together to get a general solution there could only be two linearly independent solutions, not three.

5. A hyperactive kitten named Newton is batting at a mass attached to a spring. The resulting displacement of the mass is modeled by $mx'' + cx' + kx = F\sin(\omega t)$.

a. Suppose that the parameters in the problem were picked so that the system was at resonance. Sketch a graph of what you expect the solution for x to look like in this case. Then suppose that we instead were seeing beats. Sketch another graph showing what you expect the solution for x to look like then. (4 points)

Solution:



(Resonance on the left, beats on the right.)

b. What conditions on m, c, k, F and/or ω would ensure that the system was at resonance? What conditions would produce beats? (6 points)

Solution: For resonance, $c = 0$ and $\sqrt{\frac{k}{m}} = \omega$. It doesn't matter what F is. For beats, $c = 0$ and $\sqrt{\frac{k}{m}} \approx \omega$, and again, the value for F doesn't matter.

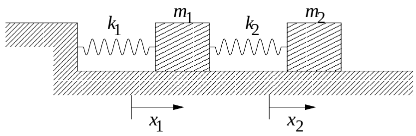
c. Solve the system with $m = 1$, $c = 0$, $k = 4$, $F = 3$ and $\omega = 2$ and the initial conditions $x(0) = x'(0) = 0$. Does your solution exhibit resonance, beats, practical resonance, or none of these? Why? (8 points)

Solution: We're solving $x'' + 4x = 3\sin(2t)$. The complementary homogeneous solution is $x_c = c_1\cos(2t) + c_2\sin(2t)$, so that using the Method of Undetermined Coefficients we guess $x_p = At\cos(2t)$. Then $x_p' = A\cos(2t) - 2At\sin(2t)$ and $x_p'' = -4A\sin(2t) - 4At\cos(2t)$. Plugging these in, we get $-4A\sin(2t) = 3\sin(2t)$, so that $A = -\frac{3}{4}$. Thus $x = c_1\cos(2t) + c_2\sin(2t) - \frac{3}{4}t\cos(2t)$. Applying the initial conditions, we have $x(0) = c_1 = 0$ and $x'(0) = 2c_2 - \frac{3}{4} = 0$. Thus $c_2 = -\frac{3}{8}$. The solution is thus

$$x = -\frac{3}{8}\sin(2t) - \frac{3}{4}t\cos(2t).$$

This is at resonance because our forcing frequency $\omega = 2$ is the same as the natural frequency.

6. Consider the mass-spring system shown to the right.



a. Explain why the system

$$x_1'' = -k_1 x_1 + k_2(x_2 - x_1)$$
$$x_2'' = \qquad - k_2(x_2 - x_1) + F_1$$

is a good model this. What assumptions are we making for this to be the case? (4 points)

Solution: We assume that $m_1 = m_2 = 1$ and that there is no friction. Then Newton's law ($F = ma$) gives $x_i'' = \Sigma$(spring forces). ($i = 1$ or 2.) For each spring the force is the spring constant k_i times the amount of compression of stretching, with a sign to oppose the compression or stretching. Thus for x_1 we get

$$\begin{aligned} x_1'' &= -k_1(\text{stretch spring 1}) - k_2(\text{stretch spring 2}) \\ &= -k_1 x_1 + -k_2(x_1 - x_2) \end{aligned}$$

and a similar equation follows for x_2. We add the external force F_1 to the second equation to reflect an external force on that mass.

b. Write this as a matrix equation. (6 points)
Solution:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}'' = \begin{pmatrix} -(k_1 + k_2) & k_2 \\ k_2 & -k_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ F_1 \end{pmatrix}.$$

c. If $F_1 = 0$, what would you guess to solve the matrix equation? *(Don't actually solve it, however!)* (4 points)
Solution: This is a linear constant-coefficient problem, so we would guess an exponential form:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{\alpha t},$$

where v_1, v_2, and α are constants.

d. If $F_1 = 3\sin(2t)$, how does your guess in (c) change? *(Again, don't actually solve the problem.)* (4 points)
Solution: Now we also need a guess for a particular solution. Thus we use the previous guess to find x_c and also guess

$$\mathbf{x}_p = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \sin(2t).$$

We might need a term proportional to $\cos(2t)$, except that there are no first-derivative terms in the equation. We're assuming that the system is not at resonance.