Chapter 2
Solution Manual

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2.1:

For a wavelength of $\lambda = 1\mu m$, an energy density of $2 \times 10^{-9} \frac{J}{cm^3}$ corresponds to a photon density:

$$N_p = \frac{2 \times 10^{-9}}{\left(\frac{hc}{1\mu m}\right)} = 1.01 \times 10^{10} \frac{\text{photons}}{cm^3} \tag{1}$$

The rate of absorption events per atom with its lower state occupied ($f_1 = 1, f_2 = 0$) for a rate constant $C = 1.5 \times 10^{-7} \frac{cm^3}{s}$ is then:

$$\begin{aligned} \frac{1}{N}\frac{dN_p}{dt} &= -CN_p \\ &= 1508.296 \frac{\text{transitions}}{s} \end{aligned} \tag{2}$$

which corresponds to a time of $0.663ms$ between transitions.

2.2:

We first note that two levels corresponding to light emitted at $\lambda = 880nm$ corresponds to an energy difference of:

$$\Delta E = \frac{hc}{\lambda} = 2.26 \times 10^{-19} J \tag{3}$$

If the Fermi level is midway between the levels, it is separated from each level by $1.13 \times 10^{-19} J$. At a temperature of $T = 300K$, the occupation probability for each level is given by:

$$f(E_{upper}) = \frac{1}{1 + exp\left(\frac{1.13 \times 10^{-19}}{k_B(300)}\right)} = 1.4 \times 10^{-12} \tag{4}$$

$$f(E_{lower}) = \frac{1}{1 + exp\left(\frac{-1.13 \times 10^{-19}}{k_B(300)}\right)} = 1 - 1.4 \times 10^{-12} \tag{5}$$

For a atomic density of $N_a = 10^{18} cm^{-3}$, the electron densities in each level are given by:

$$N_{upper} = 10^{18}(1.4 \times 10^{-12}) = 1.4 \times 10^6 cm^{-3} \tag{6}$$

$$N_{lower} = 10^{18} - N_{upper} \tag{7}$$

For a photon density of $N_{ph} = 10^{10} cm^{-3}$, the net transition rate in a volume V is given by:

$$\begin{aligned} \frac{dN_{ph}}{dt} &= C\left(N_a\right)\left(f_2 - f_1\right)N_{ph} \\ &\approx -CN_aN_{ph} \\ &= -1.5 \times 10^{21} \end{aligned} \tag{8}$$

where we ignore the downward transitions rate since it is 12 orders of magnitude smaller than the upward transition rate. Decreasing photon population corresponds to upward transitions.

2.3:

We first note that at equilibrium (when the Fermi level is in the middle of the bandgap), the difference between the two levels and the Fermi energy is $E_2 - E_F = E_F - E_1 = 0.7eV$. Out of equilibrium, the energy differences become:

$$E_2 - E_{Fc} = 0.7 - \frac{1}{2}(E_{Fc} - E_{Fv}) \tag{9}$$

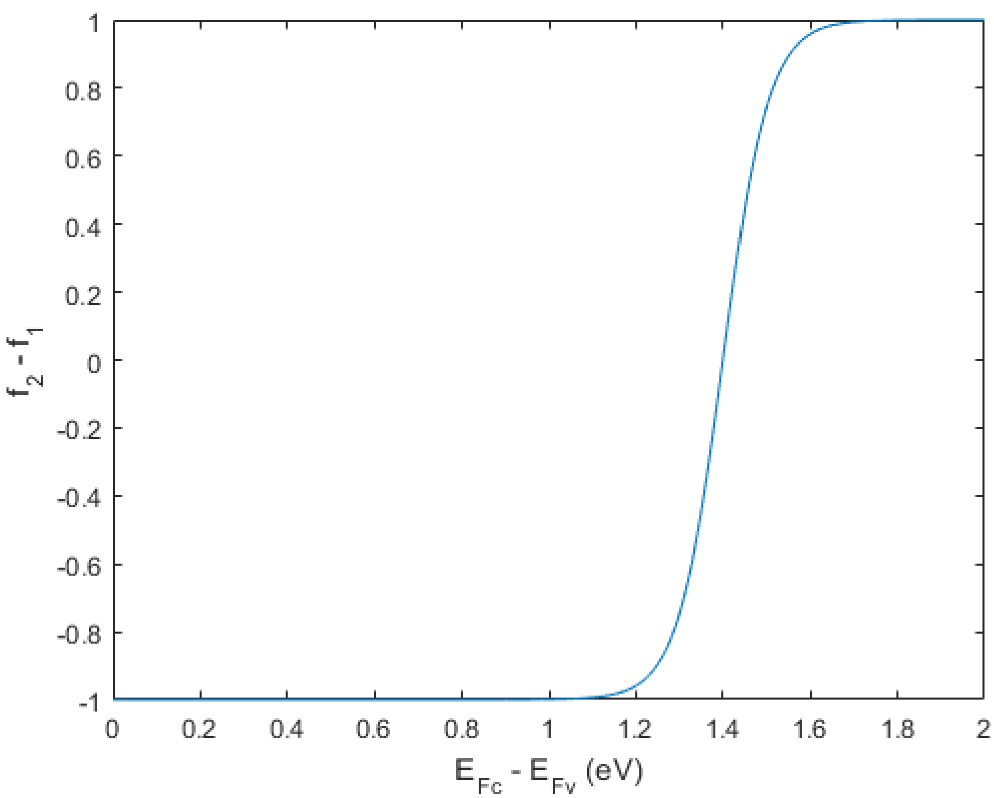
$$E_{Fv} - E_1 = 0.7 - \frac{1}{2}(E_{Fc} - E_{Fv}) \tag{10}$$

The occupation probabilities are then:

$$f(E_2) = \frac{1}{1 + exp\left(\frac{0.7 - \frac{1}{2}(E_{Fc} - E_{Fv})}{k_B T}\right)} \tag{11}$$

$$f(E_1) = \frac{1}{1 + exp\left(\frac{\frac{1}{2}(E_{Fc} - E_{Fv}) - 0.7}{k_B T}\right)} \tag{12}$$

Plotting the difference:



The point at which the curve crosses $f_2 - f_1$ represents the onset of population inversion, the critical condition necessary for optical gain.

2.4:

The condition for gain is for the occupation probability of the conduction band to be greater than that of the valence band:

$$f(E_{cb}) > f(E_{vb}) \quad \leftrightarrow \quad \frac{1}{1+exp\left(\frac{E_{cb}-E_{Fc}}{k_BT}\right)} > \frac{1}{1+exp\left(\frac{E_{vb}-E_{Fv}}{k_BT}\right)} \tag{13}$$

This condition can be rewritten as follows:

$$\begin{aligned} \frac{1+exp\left(\frac{E_{vb}-E_{Fv}}{k_BT}\right)}{\left[1+exp\left(\frac{E_{vb}-E_{Fv}}{k_BT}\right)\right]\left[1+exp\left(\frac{E_{cb}-E_{Fc}}{k_BT}\right)\right]} &> \frac{1+exp\left(\frac{E_{cb}-E_{Fc}}{k_BT}\right)}{\left[1+exp\left(\frac{E_{vb}-E_{Fv}}{k_BT}\right)\right]\left[1+exp\left(\frac{E_{cb}-E_{Fc}}{k_BT}\right)\right]} \\ 1+exp\left(\frac{E_{vb}-E_{Fv}}{k_BT}\right) &> 1+exp\left(\frac{E_{cb}-E_{Fc}}{k_BT}\right) \\ E_{vb}-E_{Fv} &> E_{cb}-E_{Fc} \\ E_{Fc}-E_{Fv} &> E_{cb}-E_{vb} \end{aligned} \tag{14}$$

We have thus derived the Bernard-Duraffourg condition.

2.5:

We first note that from the curve in problem (2.3), the occupation probability difference at $\Delta E_F = 1.45eV$ is found to be:

$$f_2 - f_1 = 0.4487 \quad (\text{at } \Delta E_F = 1.45eV) \tag{15}$$

For the parameters given, the gain is thus:

$$G = \frac{CNn}{c}(f_2 - f_1) = 897.4 \tag{16}$$

which is also the maximum gain since only one photon energy can induce transitions between the two-level system.

2.6:

The intensity of the light as a function of distance traveled L is given by:

$$I(L) = I_0 e^{GL} \tag{17}$$

For the intensity to be tripled, we have the condition:

$$\begin{aligned} 3I_0 &= I_0 e^{GL} \\ \to L &= \frac{ln(3)}{G} = 0.055cm \end{aligned} \tag{18}$$