

# Chapter 2

## Solution Manual

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### 2.1:

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Given the Hamiltonian:

$$\hat{H} = \Delta (|\phi_1\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_1|) \quad (1)$$

The time-dependent Schrodinger equation reads:

$$\frac{\partial}{\partial t} |\Psi(t)\rangle = -\frac{i}{\hbar} \hat{H} |\Psi(t)\rangle \quad (2)$$

which, for a time-independent Hamiltonian yields the solution:

$$|\Psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t} |\Psi(0)\rangle \quad (3)$$

Plugging in the initial state given, we find:

$$\begin{aligned} |\Psi(t)\rangle &= \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar}\Delta(|\phi_1\rangle\langle\phi_2|+|\phi_2\rangle\langle\phi_1|)t} (|\phi_1\rangle + |\phi_2\rangle) \\ &= \frac{1}{\sqrt{2}} \left[ 1 - \frac{i\Delta}{\hbar} (|\phi_1\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_1|)t + \frac{1}{2!} \left(\frac{i\Delta}{\hbar}\right)^2 (|\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|)t^2 + \dots \right] (|\phi_1\rangle + |\phi_2\rangle) \end{aligned} \quad (4)$$

$$= \frac{1}{\sqrt{2}} \left[ 1 - \frac{i\Delta}{\hbar}t + \frac{1}{2!} \left(\frac{i\Delta}{\hbar}t\right)^2 + \dots \right] (|1\rangle + |2\rangle) \quad (5)$$

$$= \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar}\Delta t} (|1\rangle + |2\rangle) \quad (5)$$

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### 2.2:

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We have the forward and backward time evolution operators:

$$u(t, t_0) = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)} \quad (6)$$

$$u^\dagger(t, t_0) = e^{\frac{i}{\hbar}\hat{H}(t-t_0)} \quad (7)$$

Using the Baker-Hausdorff lemma, we find:

$$\begin{aligned} u(t, t_0)u^\dagger(t, t_0) &= e^{-\frac{i}{\hbar}\hat{H}(t-t_0)} e^{\frac{i}{\hbar}\hat{H}(t-t_0)} \\ &= e^{\frac{i}{\hbar}(-\hat{H}+\hat{H})(t-t_0)} e^{\frac{i}{2\hbar}[-\hat{H}, \hat{H}](t-t_0)} \\ &= 1 \end{aligned} \quad (8)$$

One may easily show  $u^\dagger(t, t_0)u(t, t_0) = 1$  by the same procedure.

**2.3:**

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The Heisenberg picture operator is defined in (2.12) by:

$$\hat{A}_H(t) = e^{\frac{i}{\hbar} \hat{H}(t-t_0)} \hat{A} e^{-\frac{i}{\hbar} \hat{H}(t-t_0)} \quad (9)$$

Taking the time derivative:

$$\begin{aligned} \frac{\partial \hat{A}_H(t)}{\partial t} &= \frac{i}{\hbar} \hat{H} e^{\frac{i}{\hbar} \hat{H}(t-t_0)} \hat{A} e^{-\frac{i}{\hbar} \hat{H}(t-t_0)} - e^{\frac{i}{\hbar} \hat{H}(t-t_0)} \hat{A} \left( \frac{i}{\hbar} \hat{H} \right) e^{-\frac{i}{\hbar} \hat{H}(t-t_0)} \\ &= \frac{i}{\hbar} \hat{H} e^{\frac{i}{\hbar} \hat{H}(t-t_0)} \hat{A} e^{-\frac{i}{\hbar} \hat{H}(t-t_0)} - \frac{i}{\hbar} e^{\frac{i}{\hbar} (t-t_0)} \hat{A} e^{-\frac{i}{\hbar} \hat{H}(t-t_0)} \hat{H} \\ &= \frac{i}{\hbar} \hat{H} \hat{A}_H - \frac{i}{\hbar} \hat{A}_H \hat{H} \\ &= \frac{i}{\hbar} [\hat{H}, \hat{A}_H] \end{aligned} \quad (10)$$

which is the Heisenberg equation.

**2.4:**

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The forward time evolution operator for a time-dependent Hamiltonian is defined as:

$$U(t, t_0) = 1 + \sum_{n=1}^{\infty} \left( -\frac{i}{\hbar} \right)^n \int_{t_0}^t \int_{t_0}^{\tau_n} \dots \int_{t_0}^{\tau_2} \hat{H}(\tau_n) \hat{H}(\tau_{n-1}) \dots \hat{H}(\tau_1) d\tau_1 \dots d\tau_{n-1} d\tau_n \quad (11)$$

If the Hamiltonian is in fact time-independent, the above operator reduces as follows:

$$\begin{aligned} U(t, t_0) &= 1 + \sum_{n=1}^{\infty} \left( -\frac{i}{\hbar} \right)^n \int_{t_0}^t \int_{t_0}^{\tau_n} \dots \int_{t_0}^{\tau_2} \hat{H}^n d\tau_1 \dots d\tau_{n-1} d\tau_n \\ &= 1 + \sum_{n=1}^{\infty} \left( -\frac{i}{\hbar} \right)^n \hat{H}^n (t - t_0)^n \\ &= e^{-\frac{i}{\hbar} \hat{H}(t-t_0)} \\ &= u(t, t_0) \end{aligned} \quad (12)$$

**2.5:**

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Note that we can split the time-evolution operator  $U(t, t_0)$  into an arbitrary number  $n$  of time-evolution operators spanning  $t_0 \rightarrow t$ :

$$\begin{aligned} U(t, t_0) &= U(t, t_n) U(t_n, t_{n-1}) \dots U(t_1, t_0) \\ &= \left[ 1 - \frac{i}{\hbar} \int_{t_n}^t H(\tau) U(\tau, t_n) d\tau \right] \left[ 1 - \frac{i}{\hbar} \int_{t_{n-1}}^{t_n} H(\tau) U(\tau, t_{n-1}) d\tau \right] \dots \left[ 1 - \frac{i}{\hbar} \int_{t_0}^{t_1} H(\tau) U(\tau, t_0) d\tau \right] \end{aligned} \quad (13)$$

Noting:

$$\lim_{a \rightarrow b} \left[ 1 - \frac{i}{\hbar} \int_a^b H(\tau) U(\tau, a) d\tau \right] = 1 - \frac{i}{\hbar} H(b)(b-a) \quad (14)$$

If split the evolution operator into an infinite number of components:

$$\begin{aligned} U(t, t_0) &= \lim_{n \rightarrow \infty} [U(t, t_n) U(t_n, t_{n-1}) \dots U(t_1, t_0)] \\ &= \left[ 1 - \frac{i}{\hbar} H(t)(t - t_n) \right] \left[ 1 - \frac{i}{\hbar} H(t_n)(t_n - t_{n-1}) \right] \dots \left[ 1 - \frac{i}{\hbar} H(t_1)(t_1 - t_0) \right] \end{aligned} \quad (15)$$

and doing the same to the backwards evolution operator:

$$\begin{aligned} U^\dagger(t, t_0) &= \lim_{n \rightarrow \infty} [U^\dagger(t_1, t_0) \dots U^\dagger(t_n, t_{n-1}) U^\dagger(t, t_n)] \\ &= \left[ 1 + \frac{i}{\hbar} H(t_1)(t_1 - t_0) \right] \dots \left[ 1 + \frac{i}{\hbar} H(t_n)(t_n - t_{n-1}) \right] \left[ 1 + \frac{i}{\hbar} H(t)(t - t_n) \right] \end{aligned} \quad (16)$$

We can multiply the two operators together to find:

$$\begin{aligned} U^\dagger(t, t_0) U(t, t_0) &= U(t, t_0) U^\dagger(t, t_0) \\ &= \left[ 1 + \left( \frac{H(t_1)(t_1 - t_0)}{\hbar} \right)^2 \right] \left[ 1 + \left( \frac{H(t_2)(t_2 - t_1)}{\hbar} \right)^2 \right] \dots \left[ 1 + \left( \frac{H(t)(t - t_n)}{\hbar} \right)^2 \right] \end{aligned} \quad (17)$$

Since we take  $n \rightarrow \infty$ , the time differences  $(t_1 - t_0), (t_2 - t_1), \dots, (t - t_n)$  go to zero. Those terms proportional to the time-differences squared can certainly be neglected, and we thus find that  $U(t, t_0)$  is Hermitian:

$$U^\dagger(t, t_0) U(t, t_0) = U(t, t_0) U^\dagger(t, t_0) = 1 \quad (18)$$

## 2.6:

We have the wavefunction written in terms of  $U_I(t, t_0)$ :

$$|\psi(t)\rangle = U(t, t_0) |\psi(t_0)\rangle = U_0(t, t_0) U_I(t, t_0) |\psi(t_0)\rangle \quad (19)$$

We can work backwards from the time-ordered exponential definition of the evolution operators and arrive at the alternative definition:

$$U_0(t, t_0) = 1 - \frac{i}{\hbar} \int_{t_0}^t \hat{H}_0(\tau) U_0(\tau, t_0) d\tau \quad (20)$$

$$U_I(t, t_0) = 1 - \frac{i}{\hbar} \int_{t_0}^t \hat{H}'_I(\tau) U_I(\tau, t_0) d\tau \quad (21)$$

Using these representations and taking the time derivative:

$$\begin{aligned} \frac{\partial}{\partial t} |\psi(t)\rangle &= \left[ \frac{\partial U_0(t, t_0)}{\partial t} U_I(t, t_0) + U_0(t, t_0) \frac{\partial U_I(t, t_0)}{\partial t} \right] |\psi(t_0)\rangle \\ &= \left[ \left( -\frac{i}{\hbar} \hat{H}_0(t) U_0(t, t_0) \right) U_I(t, t_0) + U_0(t, t_0) \left( -\frac{i}{\hbar} \hat{H}'_I(t) U_I(t, t_0) \right) \right] |\psi(t_0)\rangle \\ &= -\frac{i}{\hbar} \left[ \hat{H}_0(t) U_0(t, t_0) U_I(t, t_0) + U_0(t, t_0) U_0^\dagger(t, t_0) \hat{H}'_I(t) U_0(t, t_0) U_I(t, t_0) \right] |\psi(t_0)\rangle \\ &= -\frac{i}{\hbar} [\hat{H}_0(t) + \hat{H}'_I(t)] |\psi(t_0)\rangle \end{aligned} \quad (22)$$

**2.7:**

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There are two situations that may occur:

- $t > \tau$ :

We expand  $U_0(t, t_0)$  as:

$$U_0(t, t_0) = U_0(t, \tau)U_0(\tau, t_0) \quad (23)$$

Multiplying the two together:

$$\begin{aligned} U_0(t, t_0)U_0^\dagger(\tau, t_0) &= U_0(t, \tau)U_0(\tau, t_0)U_0^\dagger(\tau, t_0) \\ &= U_0(t, \tau) \end{aligned} \quad (24)$$

- $t < \tau$ :

We expand  $U_0(\tau, t_0)$  as:

$$U_0(\tau, t_0) = U_0(\tau, t)U_0(t, t_0) \quad (25)$$

Multiplying the two together:

$$\begin{aligned} U_0(t, t_0)U_0^\dagger(\tau, t_0) &= U_0(t, t_0)U_0^\dagger(t, t_0)U_0^\dagger(\tau, t) \\ &= U_0^\dagger(\tau, t) \end{aligned} \quad (26)$$


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**2.8:**

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The perturbative terms of the Heisenberg operator  $\hat{A}_H(t)$  are given in equation form by:

$$\begin{aligned} \hat{A}_H^{(0)}(t) &= U_0^\dagger(t, t_0)\hat{A}(t)U_0(t, t_0) \\ \hat{A}_H^{(1)}(t) &= \left(-\frac{i}{\hbar}\right) \left\{ \left[ \int_{t_0}^t U_0^\dagger(t_0, \tau_1) \hat{H}'(\tau_1) U_0^\dagger(\tau_1, t) d\tau_1 \right] \hat{A}(t) U_0(t, t_0) \right. \\ &\quad \left. + U_0^\dagger(t, t_0) \hat{A}(t) \left[ \int_{t_0}^t U_0(t, \tau_1) \hat{H}'(\tau_1) U_0(\tau_1, t_0) d\tau_1 \right] \right\} \\ \hat{A}_H^{(2)}(t) &= \left(-\frac{i}{\hbar}\right)^2 \left\{ \left[ \int_{t_0}^t \int_{t_0}^{\tau_2} U_0^\dagger(t_0, \tau_1) \hat{H}'(\tau_1) U_0(\tau_1, \tau_2) \hat{H}'(\tau_2) U_0^\dagger(\tau_2, t) d\tau_1 d\tau_2 \right] \hat{A}(t) U_0(t, t_0) \right. \\ &\quad + \left[ \int_{t_0}^t U_0^\dagger(t_0, \tau_1) \hat{H}'(\tau_1) U_0^\dagger(\tau_1, t) d\tau_1 \right] \hat{A}(t) \left[ \int_{t_0}^t U_0(t, \tau_1) \hat{H}'(\tau_1) U_0(\tau_1, t_0) d\tau_1 \right] \\ &\quad \left. + U_0^\dagger(t, t_0) \hat{A}(t) \left[ \int_{t_0}^t \int_{t_0}^{\tau_2} U_0(t, \tau_2) \hat{H}'(\tau_2) U_0(\tau_2, \tau_1) \hat{H}'(\tau_1) U_0(\tau_1, t_0) d\tau_1 d\tau_2 \right] \right\} \\ &\quad \vdots \end{aligned} \quad (27)$$

After factorizing, the total Heisenberg operator is clearly:

$$\begin{aligned} \hat{A}_H(t) &= \hat{A}_H^{(0)}(t) + \hat{A}_H^{(1)}(t) + \hat{A}_H^{(2)}(t) + \dots \\ &= \left[ \sum_{n=0}^{\infty} U_n^\dagger(t, t_0) \right] \hat{A}(t) \left[ \sum_{n=0}^{\infty} U_n(t, t_0) \right] \\ &= U^\dagger(t, t_0) \hat{A}(t) U(t, t_0) \end{aligned} \quad (28)$$

Taking the time derivative:

$$\begin{aligned}
 \frac{\partial}{\partial t} \left[ U^\dagger(t, t_0) \hat{A}(t) U(t, t_0) \right] &= \frac{\partial U^\dagger(t, t_0)}{\partial t} \hat{A}(t) U(t, t_0) + U^\dagger(t, t_0) \hat{A}(t) \frac{\partial U(t, t_0)}{\partial t} \\
 &= \left( U^\dagger(t, t_0) \frac{i}{\hbar} \hat{H}(t) \right) \hat{A}(t) U(t, t_0) + U^\dagger(t, t_0) \hat{A}(t) \left( -\frac{i}{\hbar} \hat{H}(t) U(t, t_0) \right) \\
 &= \frac{i}{\hbar} \hat{H}(t) \hat{A}_H(t) - \frac{i}{\hbar} \hat{A}_H(t) \hat{H}(t) \\
 &= \frac{i}{\hbar} [\hat{H}(t), \hat{A}_H(t)]
 \end{aligned} \tag{29}$$

We thus see this form of  $\hat{A}_H(t)$  indeed obeys the Heisenberg equation as well.

## 2.9:

We have the expression derived for a time-independent zero-order Hamiltonian:

$$TP_{nm}^{(2)}(t = \infty) = \frac{1}{\hbar^2} \left| \int_{t_0}^t e^{i\omega_{nm}\tau} H'_{nm}(\tau) d\tau \right|^2 \tag{30}$$

In the case that the perturbation is time-independent:

$$\begin{aligned}
 TP_{nm}^{(2)}(t) &= \frac{1}{\hbar^2} |H'_{nm}|^2 \left| \int_{t_0}^t e^{i\omega_{nm}\tau} d\tau \right|^2 \\
 &= \frac{1}{\hbar^2} |H'_{nm}|^2 \left| \frac{1}{i\omega_{nm}} [e^{i\omega_{nm}t} - e^{i\omega_{nm}t_0}] \right|^2 \\
 &= \frac{1}{\hbar^2} |H'_{nm}|^2 \left| \frac{e^{i\omega_{nm}t_0}}{i\omega_{nm}} [e^{i\omega_{nm}(t-t_0)} - 1] \right|^2 \\
 &= \frac{1}{\hbar^2} \frac{|H'_{nm}|^2}{\omega_{nm}^2} \left| [e^{i\omega_{nm}(t-t_0)} - 1] \right|^2 \\
 &= \frac{1}{\hbar^2} \frac{|H'_{nm}|^2}{\omega_{nm}^2} [2 - 2\cos(\omega_{nm}(t - t_0))] \\
 &= \frac{4}{\hbar^2} \frac{|H'_{nm}|^2}{\omega_{nm}^2} \sin^2 \left[ \frac{\omega_{nm}}{2} (t - t_0) \right] \\
 &= \frac{4|H'_{nm}|^2}{\hbar^2} \frac{\sin^2 \left[ \frac{\omega_{nm}}{2} (t - t_0) \right]}{\omega_{nm}^2}
 \end{aligned} \tag{31}$$

Noting the following integral identity:

$$\int_{-\infty}^{\infty} \frac{\sin^2(\alpha x)}{x^2} dx = \pi \alpha \tag{32}$$

Instead of a single final state  $|n\rangle$ , if we have a quasi-continuum of states centered around an energy  $E_n$  with a density of states  $\rho(E)$ , the total transition probability is found by integrating over all the possible transition frequencies  $\omega_{nm}$  in the above expression:

$$\begin{aligned}
 TP^{(2)}(t) &= \frac{4|H'_{nm}|^2 \hbar^2}{\hbar^2} \int_{-\infty}^{\infty} \rho(E) \frac{\sin^2 \left[ \frac{E_{nm}}{2\hbar} (t - t_0) \right]}{E_{nm}^2} dE_{nm} \\
 &= \frac{2\pi |H'_{nm}|^2}{\hbar} \rho(E_n)(t - t_0)
 \end{aligned} \tag{33}$$

To find the rate of transitions, we simply take the time derivative:

$$w = \frac{\partial TP^{(2)}(t)}{\partial t} = \frac{2\pi}{\hbar} |H'_{nm}|^2 \rho(E_n) \quad (34)$$

which is Fermi's golden rule.

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## 2.10:

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We first evaluate the conjugate transpose of the  $\overleftarrow{\overline{\xi}}$  diagram:

$$\begin{aligned} \left[ \overleftarrow{\overline{\xi}} |m\rangle \langle n| \right]^\dagger &= \left[ \overleftarrow{\overline{\xi}} \right]^* |n\rangle \langle m| \\ &= \left[ -\left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^t \int_{t_0}^{\tau_2} U_0(t, \tau_1) \hat{H}'(\tau_1) U_0(\tau_1, t_0) |m\rangle \langle n| U_0^\dagger(\tau_2, t_0) \hat{H}'(\tau_2) U_0^\dagger(t, \tau_2) d\tau_1 d\tau_2 \right]^\dagger \\ &= -\left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^t \int_{t_0}^{\tau_2} U_0(t, \tau_2) \hat{H}'(\tau_2) U_0(\tau_2, t_0) |n\rangle \langle m| U_0^\dagger(\tau_1, t_0) \hat{H}'(\tau_1) U_0^\dagger(t, \tau_1) d\tau_1 d\tau_2 \\ &= \overleftarrow{\overline{\xi}} |n\rangle \langle m| \end{aligned} \quad (35)$$

Doing the same for  $\overleftarrow{\overleftarrow{\xi\xi}}$ :

$$\begin{aligned} \left[ \overleftarrow{\overleftarrow{\xi\xi}} |m\rangle \langle n| \right]^\dagger &= \left[ \overleftarrow{\overleftarrow{\xi\xi}} \right]^* |n\rangle \langle m| \\ &= \left[ \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^t \int_{t_0}^{\tau_2} U_0(t, \tau_2) \hat{H}'(\tau_2) U_0(\tau_2, \tau_1) \hat{H}'(\tau_1) U_0(\tau_1, t_0) |m\rangle \langle n| U_0^\dagger(t, t_0) d\tau_1 d\tau_2 \right]^\dagger \\ &= \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^t \int_{t_0}^{\tau_2} U_0(t, t_0) |n\rangle \langle m| U_0^\dagger(\tau_1, t_0) \hat{H}'(\tau_1) U_0^\dagger(\tau_2, \tau_1) \hat{H}'(\tau_2) U_0^\dagger(t, \tau_2) d\tau_1 d\tau_2 \\ &= \overleftarrow{\overleftarrow{\xi\xi}} |n\rangle \langle m| \end{aligned} \quad (36)$$

Note that since the Liouville evolution operator operates on both the Ket and Bra, whether its on the right or left of the density matrix it acts on does not matter as long as we are consistent in writing down the equations.