Chapter 2
Solution Manual

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2.1:

We have the general eigenstate expansion of an arbitrary wavefunction:

$$|\Psi\rangle = \sum_n c_n e^{-i\frac{E_n}{\hbar}t} |n\rangle \tag{1}$$

Substituting into the time-dependent Schrodinger equation:

$$\sum_n i\hbar \left[\frac{\partial}{\partial t} c_n(t) - \frac{i}{\hbar} E_n c_n(t)\right] e^{-i\frac{E_n}{\hbar}t} |n\rangle = \sum_n c_n(t) e^{-i\frac{E_n}{\hbar}t} \left(\hat{H}_0 + \hat{W}(t)\right) |n\rangle$$

Operating from the left by $\langle m|$, we find:

$$\frac{\partial}{\partial t} c_m(t) = -\frac{i}{\hbar} \sum_n c_n(t) e^{-i\frac{E_n - E_m}{\hbar}t} \left\langle m \middle| \hat{W}(t) \middle| n \right\rangle \tag{2}$$

2.2:

We have the coherent and incoherent states:

$$\rho_c = \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix} \quad \text{(coherent state)} \tag{3}$$

$$\rho_{ic} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad \text{(incoherent state)} \tag{4}$$

We then note that the dipole operator has only off-diagonal elements:

$$\mu = \begin{pmatrix} 0 & \mu_{01} \\ \mu_{10} & 0 \end{pmatrix} \tag{5}$$

Then calculating the expectation value of the two states by (2.37):

$$\begin{aligned} \langle \mu \rangle_c &= \mathrm{Tr}\left[\rho_c \mu\right] \\ &= \mathrm{Tr}\left[\begin{pmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & \mu_{01} \\ \mu_{10} & 0 \end{pmatrix}\right] \\ &= \mathrm{Tr}\left[\begin{pmatrix} -\frac{i\mu_{10}}{2} & \frac{\mu_{01}}{2} \\ \frac{\mu_{10}}{2} & \frac{i\mu_{01}}{2} \end{pmatrix}\right] \\ &= \frac{i}{2}\left(\mu_{01} - \mu_{10}\right) \end{aligned} \tag{6}$$

$$\begin{aligned}\langle\mu\rangle_{ic} &= \mathrm{Tr}\left[\rho_{ic}\mu\right] \\ &= \mathrm{Tr}\left[\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}0 & \mu_{01}\\ \mu_{10} & 0\end{pmatrix}\right] \\ &= \mathrm{Tr}\left[\begin{pmatrix}0 & \frac{\mu_{01}}{2}\\ \frac{\mu_{10}}{2} & 0\end{pmatrix}\right] \\ &= 0 \end{aligned} \tag{7}$$

2.3:

We again begin with a ground state population, which evolves along the pathway R_2 according to:

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \xrightarrow{i\rho\mu_0} \begin{pmatrix}0 & i\\ 0 & 0\end{pmatrix} \xrightarrow{t_1} \begin{pmatrix}0 & ie^{i\omega_{01}t_1}\\ 0 & 0\end{pmatrix} \xrightarrow{i\rho\mu_0\mu_1} \begin{pmatrix}ie^{i\omega_{01}t_1} & 0\\ 0 & 0\end{pmatrix} \xrightarrow{i\mu_2\rho\mu_0\mu_1} \begin{pmatrix}0 & 0\\ ie^{i\omega_{01}t_1} & 0\end{pmatrix} \xrightarrow{t_3}$$
$$\begin{pmatrix}0 & 0\\ ie^{-i\omega_{01}(t_3-t_1)} & 0\end{pmatrix} \xrightarrow{i\mathrm{Tr}[\mu_3\mu_2\rho\mu_0\mu_1]} ie^{-i\omega_{01}(t_3-t_1)} \tag{8}$$

and along the R_5 pathway:

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \xrightarrow{i\mu_0\rho} \begin{pmatrix}0 & 0\\ i & 0\end{pmatrix} \xrightarrow{t_1} \begin{pmatrix}0 & 0\\ ie^{-i\omega_{01}t_1} & 0\end{pmatrix} \xrightarrow{i\mu_1\mu_0\rho} \begin{pmatrix}ie^{-i\omega_{01}t_1} & 0\\ 0 & 0\end{pmatrix} \xrightarrow{i\mu_2\mu_1\mu_0\rho} \begin{pmatrix}0 & 0\\ ie^{-i\omega_{01}t_1} & 0\end{pmatrix} \xrightarrow{t_3}$$
$$\begin{pmatrix}0 & 0\\ ie^{-i\omega_{01}(t_3+t_1)} & 0\end{pmatrix} \xrightarrow{i\mathrm{Tr}[\mu_3\mu_2\mu_1\mu_0\rho]} ie^{-i\omega_{01}(t_3+t_1)} \tag{9}$$

We see that the R_2 and R_5 pathways both pass through the ground state after the second interaction, rather than through the excited state as in the R_1 and R_4 pathways.

2.4:

We now consider an initial population in the excited state:

$$\rho(-\infty) = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \tag{10}$$

The first interaction at $t = 0$ gives:

$$i\mu(0)\rho(-\infty) = i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0 & i\\ 0 & 0\end{pmatrix} \tag{11}$$

After propagating to $t = t_1$, the second interaction gives:

$$\begin{aligned} i\mathrm{Tr}\left[\mu(t_1)\mu(0)\rho(-\infty)\right] &= i\mathrm{Tr}\left[\mu(0)\rho(-\infty)\mu(t_1)\right] \\ &= \mathrm{Tr}\left[\begin{pmatrix}0 & ie^{+i\omega_{01}t_1}\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right] \\ &= \mathrm{Tr}\left[\begin{pmatrix}ie^{+i\omega_{01}t_1} & 0\\ 0 & 0\end{pmatrix}\right] \\ &= ie^{+i\omega_{01}t_1} \end{aligned} \tag{12}$$

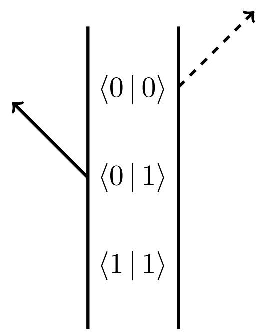
Note that we cycled the arguments in the trace to ensure that the system ended in the ground state to ensure energy conservation. The first order polarization correction for $\omega = \omega_{01}$ and without dephasing is thus given by:

$$\begin{aligned} P^{(1)}(t) &= i\int_0^\infty E'(t-t_1)\left[e^{-i\omega_{01}(t-t_1)} + e^{+i\omega_{01}(t-t_1)}\right]e^{+i\omega_{01}t_1}dt_1 + c.c. \\ &= i\int_0^\infty E'(t-t_1)\left[e^{-i\omega_{01}t}e^{2i\omega_{01}t_1} + e^{+i\omega_{01}t}\right]dt_1 + c.c. \end{aligned} \tag{13}$$

Under the rotating wave approximation, the first oscillating term vanishes and we are left with:

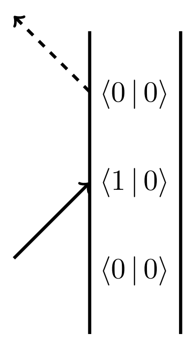
$$P^{(1)}(t) = i\int_0^\infty E'(t-t_1)e^{+i\omega_{01}t}dt_1 + c.c. \tag{14}$$

The Feynman diagram of $i\mathrm{Tr}[\mu(t_1)\mu(0)\rho(-\infty)]$ is given by:



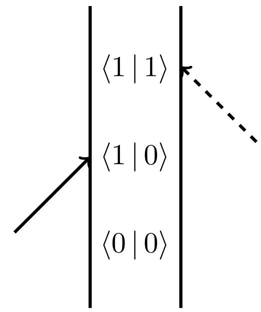
2.5:

The left term's Feynman diagram is given by:



Since the left and right terms are mathematically the same, we must maintain the directions of the arrows after cycling the terms.

The right term's Feynman diagram is thus:

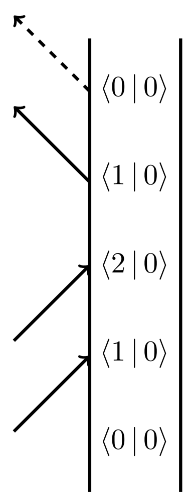


2.6:

The fields have the form:

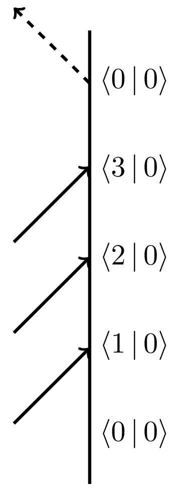
$$\vec{E}_j(\vec{r},t) = \vec{E}_j e^{i(\vec{r}\cdot\vec{k}_j - \omega t)} \tag{15}$$

For an emitted field to have a wavevector $\vec{k}_1 + \vec{k}_2 - \vec{k}_3$, the input fields must be $\vec{E}_1$, $\vec{E}_2$, and $\vec{E}_3^*$ (with arrows pointing to the right, right, and left respectively). This corresponds to the Feynman diagram:



2.7:

For an emitted field to have a wavevector $\vec{k}_1 + \vec{k}_2 + \vec{k}_3$, the input fields must be $\vec{E}_1$, $\vec{E}_2$, and $\vec{E}_3$ (with all arrows pointing to the right). This corresponds to the Feynman diagram:

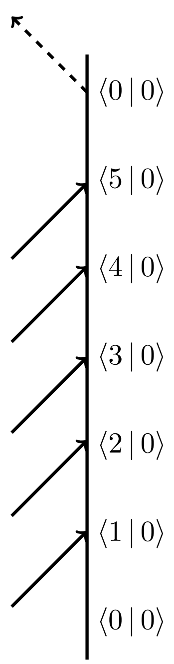


Since the emission is due to the transition between the virtual state $|3\rangle$ and the ground state $|0\rangle$, the emitted field's frequency must be the sum of all the input field's. We can also deduce this from the input fields $\vec{E}_1$, $\vec{E}_2$, and $\vec{E}_3$, since they all have positive frequencies.

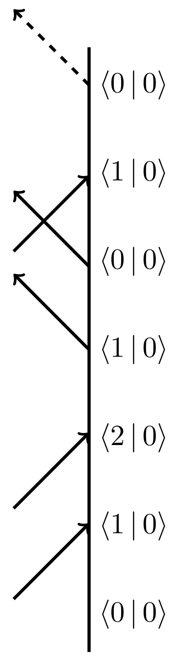
2.8:

For three input beams, some possible phase matching conditions for fifth-order processes are:

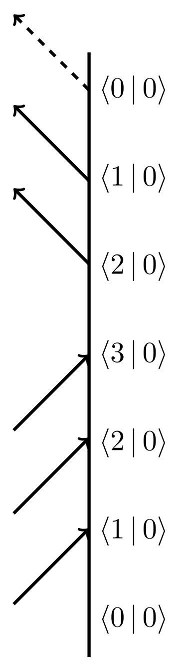
1. $\vec{k} = 2\vec{k}_1 + 2\vec{k}_2 + \vec{k}_3$, $\vec{k} = 2\vec{k}_1 + \vec{k}_2 + 2\vec{k}_3$, $\vec{k} = \vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3$:



2. $\vec{k} = 2\vec{k}_1 - 2\vec{k}_2 + \vec{k}_3$:



3. $\vec{k} = 3\vec{k}_2 - 2\vec{k}_3$:



$\vdots$