Normal Modes of a 2D Lattice

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We now extend the method developed for a 1D lattice to two dimensions. A common system to analyze is a 2D monatomic lattice with lattice constant a in both dimensions (essentially a square Bravais lattice). For an atom of mass M at position (x_0, y_0), a displacement in the $\hat{x}$ direction will result in a restoring force with spring constant K_1 due to its nearest neighbors at positions $(x_0 \pm a, y_0)$ and another restoring force with spring constant K_2 due to the other two nearest neighbors at positions $(x_0, y_0 \pm a)$.

For an atom at position (na, ma) where n and m are integers, we denote its displacement from equilibrium as $u(na, ma)$. Newton's third law gives:

$$\begin{aligned} M\frac{\partial^2 u(na,ma)}{\partial t^2} =& -K_1\left[2u(na,ma) - u([n+1]a,ma) - u([n-1]a,ma)\right] \\ & - K_2\left[2u(na,ma) - u(na,[m+1]a) - u(na,[m-1]a)\right] \end{aligned} \tag{1}$$

The Born-von Karman (periodic) boundary conditions now imply a plane wave solution:

$$u(na, ma) = Ae^{i(k_x na + k_y ma - \omega t)} \tag{2}$$

Plugging the assumed solution into the equation of motion:

$$\omega^2 = \frac{4K_1}{M} sin^2\left(\frac{k_x a}{2}\right) + \frac{4K_2}{M} sin^2\left(\frac{k_y a}{2}\right) \tag{3}$$

As a quick sanity check, we calculate the dispersion relation for waves propagating in the $\hat{x}$ direction ($k_y = 0$):

$$\omega^2 = \sqrt{\frac{4K_1}{M}} sin^2\left(\frac{k_x a}{2}\right) \tag{4}$$

which is the same result we obtained for a 1D monatomic lattice, as expected.

An interesting quantity is the velocity:

$$\begin{aligned} \mathbf{v} &= \frac{\partial \omega}{\partial k_x}\hat{x} + \frac{\partial \omega}{\partial k_y}\hat{y} & (5) \\ &= \frac{a}{M\omega}\left[K_1 sin(k_x a)\hat{x} + K_2 sin(k_y a)\hat{y}\right] & (6) \end{aligned}$$

with magnitude:

$$v = \frac{a}{M\omega}\sqrt{K_1^2 sin^2(k_x a) + K_2^2 sin^2(k_y a)} \tag{7}$$