Electron Gas Density of States

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Recall that in a 3D electron gas, there are $2\left(\frac{L}{2\pi}\right)^3$ modes per unit k-space volume. The number of modes N that a sphere of radius k in k-space encloses is thus:

$$N = 2\left(\frac{L}{2\pi}\right)^3 \frac{4}{3}\pi k^3 = \frac{V}{3\pi^2}k^3 \tag{1}$$

A useful quantity is the derivative with respect to k:

$$\frac{dN}{dk} = \frac{V}{\pi^2}k^2 \tag{2}$$

We also recall the relation between energy E and k:

$$E = \frac{\hbar^2 k^2}{2m} \tag{3}$$

$$k = \sqrt{\frac{2mE}{\hbar^2}} \tag{4}$$

Taking the derivative of k with respect to E:

$$\frac{dk}{dE} = \sqrt{\frac{m}{2\hbar^2 E}} \tag{5}$$

The density of states $D(E)$ in 3D, which is the number of states per unit energy, can now be calculated as follows:

$$D_{3D}(E) = \frac{dN}{dE} = \frac{dN}{dk}\frac{dk}{dE} = \frac{V}{\pi^2}k^2\sqrt{\frac{m}{2\hbar^2 E}} = \frac{V}{\pi^2}\sqrt{2E}\left(\frac{m}{\hbar^2}\right)^{\frac{3}{2}} \tag{6}$$

In 2D, the number of modes that a circle of radius k encloses is:

$$N = 2\left(\frac{L}{2\pi}\right)^2 \pi k^2 = \frac{A}{2\pi}k^2 \tag{7}$$

which gives:

$$\frac{dN}{dk} = \frac{A}{\pi}k \tag{8}$$

The density of states in 2D is thus:

$$D_{2D}(E) = \frac{dN}{dk}\frac{dk}{dE} = \frac{A}{\pi}k\sqrt{\frac{m}{2\hbar^2 E}} = \frac{Am}{\pi\hbar^2} \tag{9}$$

In 1D, the number of modes enclosed by the range $(-k, k)$ is:

$$N = 4\left(\frac{L}{2\pi}\right)k \tag{10}$$

which gives:

$$\frac{dN}{dk} = \frac{2L}{\pi} \tag{11}$$

The density of states in 1D is thus:

$$D_{1D}(E) = \frac{dN}{dk}\frac{dk}{dE} = \frac{2L}{\pi}\sqrt{\frac{m}{2\hbar^2 E}} \tag{12}$$

Example: Consider a 1D ballistic wire, meaning that the coherence length of the electrons is longer than the channel (assume electrons do not experience any collisions). What is the conductance of the wire? We begin by rewriting the 1D density of states in terms of the electron velocity v:

$$\begin{aligned} D_{1D}(E) &= \frac{2L}{\pi}\sqrt{\frac{m}{2\hbar^2 E}} \\ &= \frac{2L}{\pi}\sqrt{\frac{m}{2\hbar^2}\frac{2m}{p^2}} \\ &= \frac{4L}{hv} \end{aligned} \tag{13}$$

We now note that only the electrons moving in one direction will contribute to a current. Depending on the direction of the applied voltage, these will be the electrons in either k range $(-k, 0)$ or $(0, k)$, which means the density of states we use will be divided in half:

$$D_{1D}^{curr}(E) = \frac{1}{2}D_{1D}(E) = \frac{2L}{hv} \tag{14}$$

If a potential difference $\Delta V = (V_2 - V_1)$ is applied across the wire, there is a corresponding potential energy difference of $\Delta E = -e\Delta V$. The electron density can thus be expressed as:

$$n = \frac{electrons}{m} = \frac{1}{L}\Delta E\frac{dN}{dE} = -\frac{2e}{hv}\Delta V \tag{15}$$

We now write the current:

$$I = -nev = \frac{2e^2}{h}\Delta V \tag{16}$$

Which gives the conductance:

$$G_0 = \frac{I}{\Delta V} = \frac{2e^2}{h} \tag{17}$$

The quantity G_0 is known as the **conductance quantum**, which will be discussed in detail later.