Formation of Rainbows

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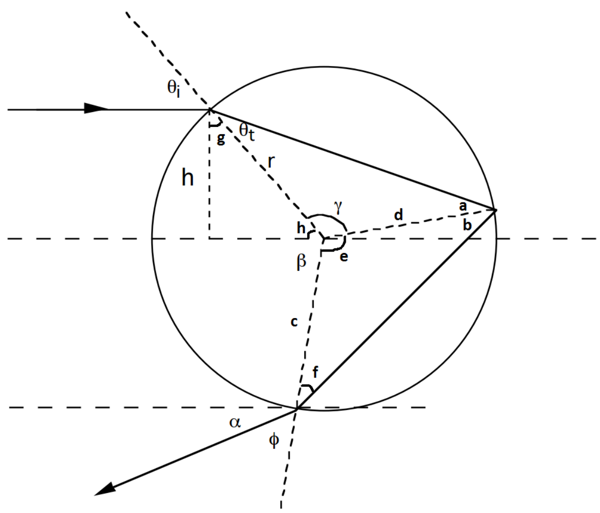
We have the following diagram of light entering a raindrop:



We first write Snell's law for the incident beam:

$$sin(\theta_i) = nsin(\theta_t) \tag{1}$$

We then note that the length d is simply the radius of the drop r: $d = r$. The law of sines thus gives:

$$\frac{r}{sin(a)} = \frac{r}{sin(\theta_t)} \rightarrow a = \theta_t \tag{2}$$

From $a = \theta_t$ we get $\gamma = 180 - 2\theta_t$. This also gives $b = \theta_t$ from the law of reflection. From inspection, $c = r$ so the law of sines in the bottom triangle gives:

$$\frac{r}{sin(\theta_t)} = \frac{r}{sin(f)} \rightarrow f = \theta_t \tag{3}$$

Since the sum of all angles in a triangle equals 180°, we get $e = 180° - 2\theta_t$. Also, $g = 90° - \theta_i$ so $h = 180 - 90° - g = \theta_i$. We thus get β:

$$\begin{aligned}\beta = 360° - h - \gamma - e &= 360° - \theta_i - 180° + 2\theta_t - 180° + 2\theta_t \\ &= 4\theta_t - \theta_i \end{aligned} \tag{4}$$

To find ϕ, we write Snell's law:

$$\begin{aligned}
&nsin(f = \theta_t) = sin(\phi) \\
&\rightarrow sin(\theta_i) = sin(\phi) \\
&\rightarrow \phi = \theta_i
\end{aligned} \tag{5}$$

From inspection we note that $\beta = \alpha + \phi$. We thus find:

$$\begin{aligned}
\alpha &= \beta - \phi \\
&= 4\theta_t - 2\theta_i
\end{aligned} \tag{6}$$

We now note:

$$\begin{aligned}
&h = rsin(\theta_i) \\
&\rightarrow \theta_i = sin^{-1}\left(\frac{h}{r}\right)
\end{aligned} \tag{7}$$

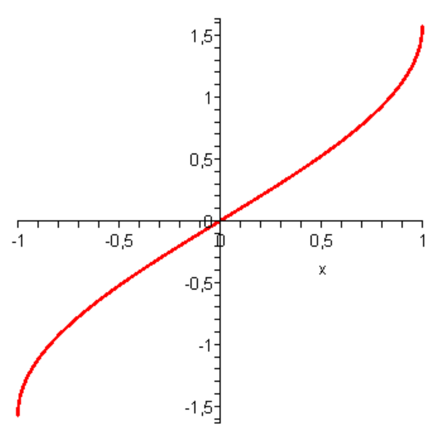
Then using $\theta_i = sin^{-1}\left(nsin(\theta_t)\right)$:

$$\begin{aligned}
&h = rsin(\theta_i) \\
&\rightarrow h = rnsin(\theta_t) \\
&\rightarrow \theta_t = sin^{-1}\left(\frac{h}{rn}\right)
\end{aligned} \tag{8}$$

Plugging in these expressions for θ_i and θ_t gives:

$$\alpha = 4sin^{-1}\left(\frac{h}{nr}\right) - 2sin^{-1}\left(\frac{h}{r}\right) \tag{9}$$

From the graph of $sin^{-1}(x)$:



we can see that increasing n results in decreasing α, and vice versa.

Examining the index of refraction for the common colors:

- Red: n = 1.325
- Orange: n = 1.330
- Green: n = 1.335
- Blue: n = 1.340

we see that the exit angle of red light is the greatest, while the exit angle of blue light is the smallest. Because red light reaches our eyes at a steeper angle than blue light, we will see the red light from higher raindrops and the blue light from lower raindrops. A pictorial representation is as follows: