Rabi Flopping and the Autler-Townes Effect (AC Stark Shift)

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Semiclassical Derivation

Consider a two-level system with the Hamiltonian:

$$\hat{H} = \hat{H}_0 - \vec{\mu} \cdot \vec{E}_0 cos(\omega t) \tag{1}$$

where we've assumed an incident plane wave field and made the dipole approximation for the interaction Hamiltonian. We can also express the two-level system wavefunction as a superposition of the ground and excited states:

$$|\Psi(t)\rangle = C_1(t)e^{-i\omega_1 t}|1\rangle + C_2(t)e^{-i\omega_2 t}|2\rangle \tag{2}$$

Plugging (1) and (2) into the Schrodinger equation, we obtain the rate equations for $C_1(t)$ and $C_2(t)$ (neglecting damping and decay):

$$\dot{C}_1 = i\frac{\Omega^*}{2}e^{-i\Delta t}C_2(t) \tag{3}$$

$$\dot{C}_2 = i\frac{\Omega}{2}e^{i\Delta t}C_1(t) \tag{4}$$

where we've defined the **resonant Rabi frequency** Ω and the **detuning** Δ:

$$\Omega = \frac{\langle 2 \,|\, \vec{\mu} \,|\, 1\rangle \cdot \vec{E}_0}{\hbar} \qquad \Delta = \omega_0 - \omega \tag{5}$$

We now assume an complex exponential solution for $C_1(t)$ of the form:

$$C_1(t) = e^{irt} \tag{6}$$

where r is a purely real number. Plugging this ansatz into (3), we find:

$$C_2(t) = \frac{2r}{\Omega}e^{i(\Delta+r)t} \tag{7}$$

Plugging (6) and (7) into (4), we find:

$$\begin{aligned} &\frac{2r}{\Omega}(\Delta + r) = \frac{\Omega}{2} \\ &\rightarrow r^2 + \Delta r - \frac{\Omega^2}{4} = 0 \\ &\rightarrow r_{\pm} = \frac{-\Delta \pm \sqrt{\Delta^2 + \Omega^2}}{2} \end{aligned} \tag{8}$$

The general solution for $C_1(t)$ is:

$$C_1(t) = Ae^{ir_+t} + Be^{ir_-t} \tag{9}$$

and according to (3), the general solution for $C_2(t)$ is:

$$C_2(t) = \frac{2}{\Omega} e^{i\Delta t} \left[Ar_+ e^{ir_+t} + Br_- e^{ir_-t} \right] \tag{10}$$

For the initial conditions $C_1(0) = 1$ and $C_2(0) = 0$, A and B are then:

$$A = \frac{r_-}{r_- - r_+} = \frac{\Delta + \sqrt{\Delta^2 + \Omega^2}}{2\sqrt{\Delta^2 + \Omega^2}} \qquad B = \frac{r_+}{r_+ - r_-} = \frac{-\Delta + \sqrt{\Delta^2 + \Omega^2}}{2\sqrt{\Delta^2 + \Omega^2}} \tag{11}$$

We thus find $C_1(t)$ and $C_2(t)$:

$$C_1(t) = \frac{e^{-i\frac{\Delta}{2}t}}{2\sqrt{\Delta^2 + \Omega^2}} \left[\left(\Delta + \sqrt{\Delta^2 + \Omega^2} \right) e^{i\frac{\sqrt{\Delta^2+\Omega^2}}{2}t} + \left(-\Delta + \sqrt{\Delta^2 + \Omega^2} \right) e^{-i\frac{\sqrt{\Delta^2+\Omega^2}}{2}t} \right]$$

$$C_2(t) = e^{i\frac{\Delta}{2}t} \frac{\Omega}{2\sqrt{\Delta^2 + \Omega^2}} \left[e^{i\frac{\sqrt{\Delta^2+\Omega^2}}{2}t} - e^{-i\frac{\sqrt{\Delta^2+\Omega^2}}{2}t} \right] \tag{12}$$

We can examine the wavefunction in the two limiting cases:

- Off-resonance Excitation ($\Delta \gg \Omega$)

  We take the limiting case $\Delta \to \infty$, in which the state coefficients become:

  $$C_1(t) \approx 1 \quad \text{and} \quad C_2(t) \approx 0 \tag{13}$$

  which gives the state:

  $$|\Psi(t)\rangle \approx e^{-i\omega_1 t} |1\rangle \tag{14}$$

  As the frequency of the field is detuned from resonance the field becomes less efficient at driving transitions between $|1\rangle$ and $|2\rangle$. Unsurprisingly, as the detuning frequency goes to infinity no transitions are induced.

- Resonant Excitation ($\Delta = 0$)

  For $\Delta = 0$, the state coefficients become:

  $$C_1(t) = \frac{1}{2} \left[e^{i\frac{\Omega}{2}t} + e^{-i\frac{\Omega}{2}t} \right] \quad \text{and} \quad C_2(t) = \frac{1}{2} \left[e^{i\frac{\Omega}{2}t} - e^{-i\frac{\Omega}{2}t} \right] \tag{15}$$

  The probabilities of each state are thus given by:

  $$P_1(t) = |C_1(t)|^2 = cos^2 \left(\frac{\Omega}{2} t \right) \quad \text{and} \quad P_2(t) = |C_2(t)|^2 = sin^2 \left(\frac{\Omega}{2} t \right) \tag{16}$$

  we thus find that the occupation probabilities of the atom oscillates between $|1\rangle$ and $|2\rangle$, a phenomenon known as **Rabi flopping**.

We now explictly write the state $|\Psi\rangle$:

$$
\begin{aligned}
|\Psi(t)\rangle &= C_1(t)e^{-i\omega_1 t}|1\rangle + C_2(t)e^{-i\omega_2 t}|2\rangle \\
&= \frac{1}{2}\left[e^{-i\left(\omega_1+\frac{\Omega}{2}\right)t}|1\rangle + e^{-i\left(\omega_1-\frac{\Omega}{2}\right)t}|1\rangle + e^{-i\left(\omega_2+\frac{\Omega}{2}\right)t}|2\rangle - e^{-i\left(\omega_2-\frac{\Omega}{2}\right)t}|2\rangle\right] \\
&= \frac{1}{2}\left[e^{-i\frac{E_1^+}{\hbar}t}|1\rangle + e^{-i\frac{E_1^-}{\hbar}t}|1\rangle + e^{-i\frac{E_2^+}{\hbar}t}|2\rangle - e^{-i\frac{E_2^-}{\hbar}t}|2\rangle\right] \qquad (17)
\end{aligned}
$$

We have found something interesting, which is that although there are still only two eigenstates of the Hamilonian, it seems that there are now four oscillation frequencies in our state:

$$\omega_1^{\pm} = \omega_1 \pm \frac{\Omega}{2} \quad \text{and} \quad \omega_2^{\pm} = \omega_2 \pm \frac{\Omega}{2} \qquad (18)$$

This suggests that the two level energy diagram is now split into four levels, which is called the **Autler-Townes effect** or **AC Stark splitting**:

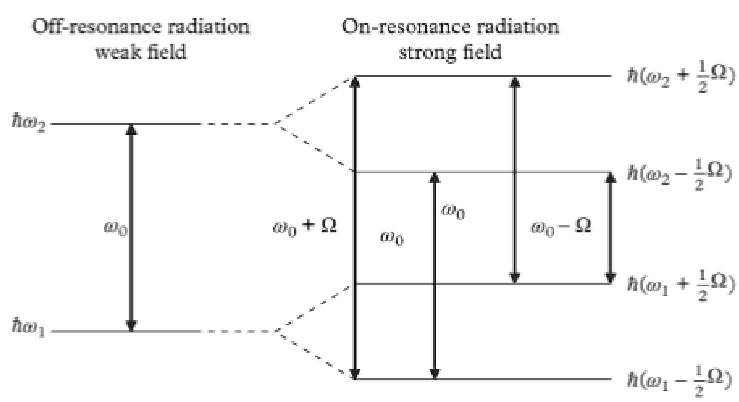

Figure 1: Taken from *Nonlinear and Quantum Optics* - Stephen Rand.

Of course there is a major problem with this derivation, which is that there seems to be four eigenvalues corresponding to only two eigenstates. This prediction of energy splitting is correct however, which we find in a more cautious derivation as well.