Web Appendix A.3

A.3 Solutions to Differential Equations

A.3.A First-order Ordinary Differential Equations

Also see http://www.ucl.ac.uk/Mathematics/geomath/level2/deqn/de8.html

$$\frac{dy}{dt} + f(t)y = g(t) \qquad \text{(WA.3.A.1)}$$

Integrating factor = $\exp\left(\int f dt\right)$

Multiply through the integrating

$$e^{\int f(y)dt}\frac{dy}{dt} + e^{\int f dt}y = e^{\int f dt}g(t)$$

Collect term

$$d\left(\frac{ye^{\int f dt}}{dt}\right) = e^{\int f dt}g(t)$$

$$ye^{\int f dt} = \int g(t)\, e^{\int f dt}dt + K_1$$

and divide by $e^{\int f dt}$

$$y = e^{-\int f dt}\int g(t)\, e^{\int f dt}dt + K_1 e^{-\int f dt} \qquad \text{(WA.3.A.2)}$$

Example A.3–1 Integrating Factor for Series Reactions

$$\frac{dy}{dt} + k_2 y = k_1 e^{-k_1 t}$$

$$\text{Integration factor} = \exp \int k_2 \, dt = e^{-k_2 t}$$

$$\frac{dye^{k_2 t}}{dt} = e^{k_2 t} k_1 e^{-k_1 t} = k_1 e^{(k_2 - k_1)t}$$

$$e^{k_2 t} y = k_1 \int e^{(k_2 - k_1)t} dt = \frac{k_1}{k_2 - k_1} e^{(k_2 - k_1)t} + K_1$$

$$y = \frac{k_1}{k_2 - k_1} e^{-k_1 t} + K_1 e^{-k_2 t}$$

$$t = 0 \quad y = 0$$

$$y = \frac{k_1}{k_2 - k_1} \left[e^{-k_1 t} - e^{-k_2 t} \right]$$

A.3.B Coupled First-order Linear Ordinary Differential Equations with Constant Coefficients

Also see http://www.mathsci.appstate.edu/~sjg/class/2240/finalss04/Alicia.html.

Consider the following coupled set of linear first order ODE with constant coefficients.

(1) $$\frac{dx}{dt} = ax + by \qquad t = 0 \qquad x = x_0$$

(2) $$\frac{dy}{dt} = cx + dy \qquad t = 0 \qquad y = y_0$$

Or in Matrix Notation

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The solution to these coupled equations is

$$x = Ae^{-\alpha t} + Be^{-\beta t} \tag{3}$$

and

$$y = K_1 e^{-\alpha t} + K_2 e^{-\beta t} \tag{4}$$

where

$$\alpha, \beta = -\left[\frac{TrA \pm \sqrt{TrA^2 - 4DetA}}{2} \right], \; TrA = a + d, \quad DetA = ad - bc$$

From the initial conditions

$$t = 0 \quad x = x_0 \text{ and } y = y_0 \text{ we get}$$

$$x_0 = A + B$$

and

$$y_0 = K_1 + K_2$$

differentiating equations (3) and (4) and evaluating the derivative at t = 0

$$\left.\frac{dx}{dt}\right)_{t=0} = -\alpha A - \beta B = a x_0 + b y_0$$

$$\left.\frac{dy}{dt}\right)_{t=0} = -\alpha K_1 - \beta K_2 = c x_0 + d y_0$$

We have four equations and four unknowns (The arbitrary constants of integration A, B, K_1 and K_2) so we can eliminate these arbitrary constants of integration.

If $y_0 = 0$ then the solution takes the form

$$x = x_0 \left[\frac{(a+\alpha)e^{-\beta t} - (a+\beta)e^{-\alpha t}}{\alpha - \beta} \right]$$

$$y = \frac{c x_0}{\alpha + \beta} e^{-\beta t} - \frac{c x_0}{\alpha + \beta} e^{-\alpha t}$$