Module 5

Solving Systems of Equations

EXECUTIVE MBA

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MICHIGAN ROSS SCHOOL OF BUSINESS | Quantitative Skills Refresher

AGENDA

- Introduction
- Sketching Multiple Lines & Curves
- Solving Systems of Equations
- Calculating Areas
- Application: Evaluating Profit Opportunities

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Introduction

Solving Systems of Equations

Let's start with an example. Recall the application of sales forecasting from the "Working with Linear Equations" module. We used historical data to derive the equation of a line relating software sales at a particular store to time:

$$Sales = 247{,}500t + 715{,}000$$

where t = time (1999 = 0, 2000 = 1, etc.).

The manager of this software store has also estimated a line relating total costs to time:

$$Costs = 320{,}000t + 352{,}500$$

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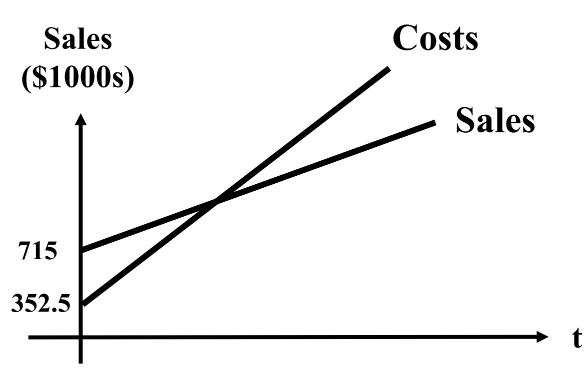
Introduction

Solving Systems of Equations

- The manager is worried that if sales and costs continue on this track, the business is headed for losses. When will costs overtake sales revenues?



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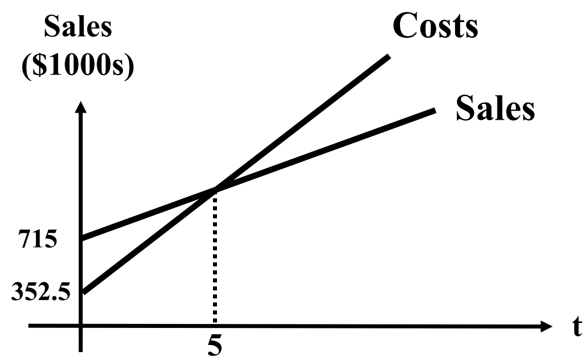
Introduction

Solving Systems of Equations

- **Solve the equation: Costs = Sales**

$320{,}000t + 352{,}500 = 247{,}500t + 715{,}000$

$\Rightarrow 72{,}500t = 362{,}500 \Rightarrow t = 5$ (i.e., in 2004)



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Introduction

Solving Systems of Equations

- **The example we just worked through is not typical of the types of systems of equations relevant for business. We had two equations, but only one unknown, time.**
- **Saying that costs and sales depend on time alone is unrealistic. In reality it not only takes many equations, but also many variables to describe how a business or market works.**
- **In this module we will be working with two equations and two unknowns (two variables). The solution to a system equations is a set of values for the variables that simultaneously satisfies all the equations.**

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Sketching Multiple Lines & Curves

- Before we begin to solve systems of equations, it helps to have a visual depiction of what we are doing.
- Use the same graphing skills developed earlier.
- The only difference is that we have to draw the curves accurately enough so that we can see how they might intersect (or not intersect).

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Sketching Multiple Lines & Curves

Examples: (restricted to $x > 0$ and $y > 0$)

1. A downward sloping line & horizontal line:
   $y = 20 - 0.01x$ $\quad$ $y = 5$
2. A downward & upward sloping line:
   $y = 20 - 0.01x$ $\quad$ $y = 10 + 0.07x$
3. A downward & upward sloping line:
   $y = 20 - 0.01x$ $\quad$ $y = 22 + x$
4. A horizontal line & parabola:
   $y = 24$ $\quad$ $y = 24 - 15x + 3x^2$
5. Two parabolas:
   $y = 24 - 8x + x^2$ $\quad$ $y = 24 - 16x + 3x^2$

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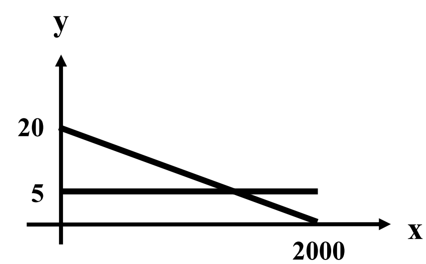
Sketching Multiple Lines & Curves

Examples:

1. A downward sloping line & horizontal line:

$y = 20 - 0.01x$ $\qquad$ $y = 5$



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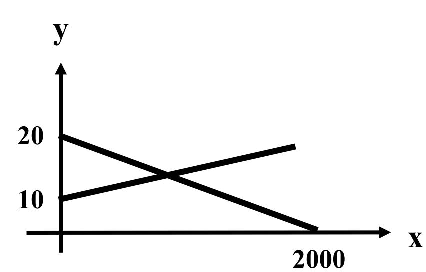
Sketching Multiple Lines & Curves

Examples:

2. A downward & upward sloping line:

$y = 20 - 0.01x$ $\qquad$ $y = 10 + 0.07x$



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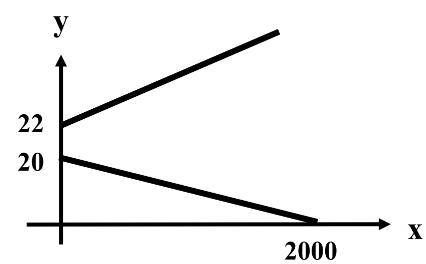
Sketching Multiple Lines & Curves

Examples:

3. **A downward & upward sloping line:**

$y = 20 - 0.01x$ $y = 22 + x$



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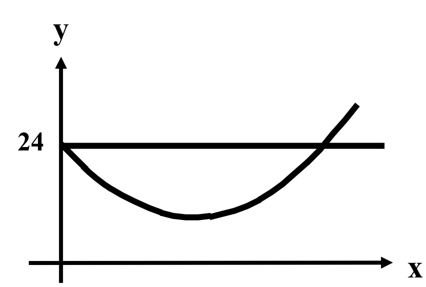
Sketching Multiple Lines & Curves

Examples:

4. **A horizontal line & parabola:**

$y = 24$ $y = 24 - 15x + 3x^2$



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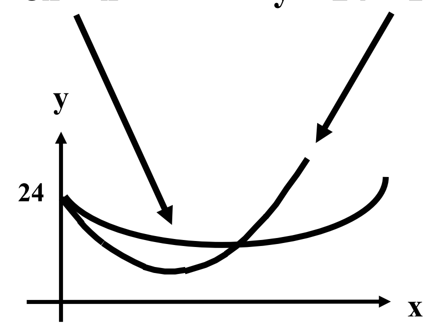
Sketching Multiple Lines & Curves

Examples:

5. Two parabolas:

$y = 24 - 8x + x^2$ $\quad\quad$ $y = 24 - 16x + 3x^2$



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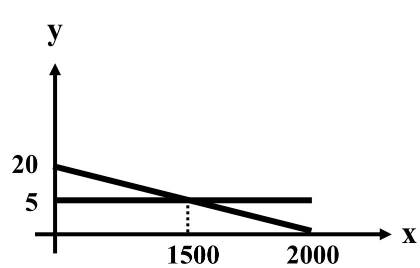
Solving Systems of Equations

(Solving For Intersections)

Using the Same Examples:

1. A downward sloping line & horizontal line:

$y = 20 - 0.01x$ $\quad\quad$ $y = 5$



Set the equations equal and solve:

$5 = 20 - 0.01x$

$0.01x = 15 \Rightarrow x = 1500$

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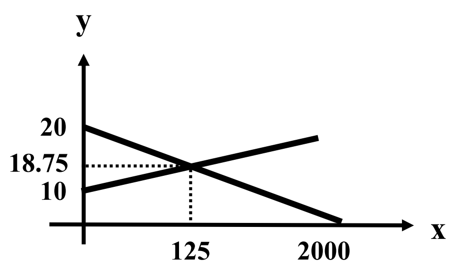
Solving Systems of Equations

Using the Same Examples:

2. A downward & upward sloping line:

y = 20 – 0.01x   y = 10 + 0.07x



Set the equations equal and solve:

20 – 0.01x = 10 + 0.07x

0.08x = 10 ⇒ x = 125

⇒ y = 20 – 0.01(125) = 18.75

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Solving Systems of Equations

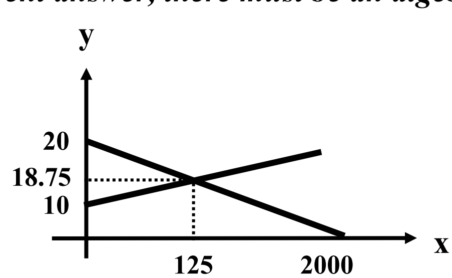
*****Note: You Can Check Your Answer*****

When solving systems of equations, you can check your answer. In the last slide we found x = 125 and then plugged that answer into the equation y = 20 – 0.1x to find y = 18.75.

It does not matter which equation you choose. Plug x = 125 into the other equation:

y = 10 + 0.07(125) = 18.75; you get the *same answer*

If you get a different answer, there must be an algebraic error.



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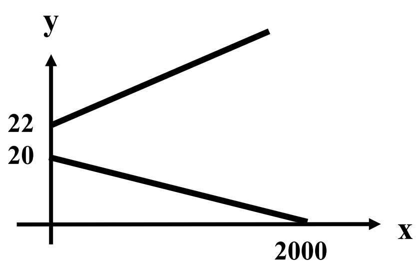
Solving Systems of Equations

Using the Same Examples:

3. A downward & upward sloping line:

$y = 20 - 0.01x$ $\qquad$ $y = 22 + x$



Graph these equations and you will see that there is no intersection (assuming x > 0 and y > 0). Or you can try to solve for the intersection:

$$20 - 0.01x = 22 + x$$

$$1.01x = -2 \Rightarrow x = -1.98$$

The lines do intersect, but not for x's and y's of interest to us.

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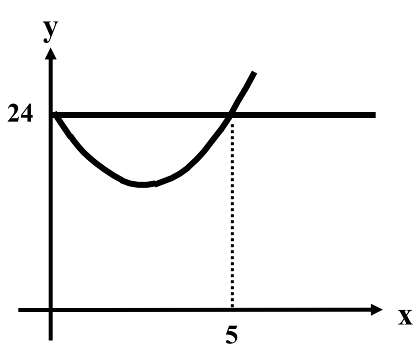
Solving Systems of Equations

Using the Same Examples:

4. A horizontal line & quadratic curve:

$y = 24$ $\qquad$ $y = 24 - 15x + 3x^2$



The graph shows two intersections; ignore the intersection at x = 0 and solve:

$$24 - 15x + 3x^2 = 24$$

$$\Rightarrow -15x + 3x^2 = 0$$

$$\Rightarrow 3x^2 = 15x \Rightarrow 3x = 15$$

(you can divide both sides by x as long as $x \neq 0$)

$$\Rightarrow x = 5$$

We already know that y = 24.

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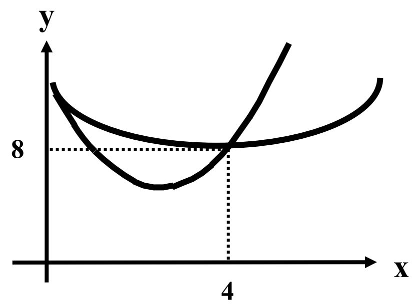
Solving Systems of Equations

Using the Same Examples:

5. Two parabolas:

$y = 24 - 8x + x^2$ $y = 24 - 16x + 3x^2$



Solve for the intersection:

$24 - 16x + 3x^2 = 24 - 8x + x^2$

$\Rightarrow 2x^2 = 8x \Rightarrow 2x = 8$

(you can divide both sides by x as long as $x \neq 0$)

$\Rightarrow x = 4$

$\Rightarrow y = 24 - 8(4) + 4^2 = 8$

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Solving Systems of Equations

Exercises (5 questions)

Solve these systems of equations and sketch the curves. Restrict all solutions to $x > 0$ and $y > 0$.

1. $x = 72 - 4y$ and $y = 3 + 0.5x$.
2. $P = 10 - 0.2Q$ and $P = 1.8Q$. For the drawing, put P on the y-axis and Q on the x-axis.
3. $2y - x = 24$ and $y = 8$
4. $y = 10 + 2x^2$ and $y = 135 - 3x^2$

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Solving Systems of Equations

Exercises (5 questions)

5. **Cube, Inc. makes two types of home office desks, standard and deluxe. The table below shows how many machine hours it takes to make each type of desk, the manufacturing cost of each desk, the constraints on the number of machine hours per year, and the annual budget. (The hidden costs are labor and materials.) You would like to know how many desks of each type to make, hitting the constraints exactly. It is not necessary to draw a graph.**

	Standard	Deluxe	Constraints
Machine Hours	5	1	8,000
Production Cost	$4,000	$8,000	$10,000,000

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Solving Systems of Equations

*Exercises: **Answers***

1. $x = 72 - 4y$ *and* $y = 3 + 0.5x$

Begin with the first equation and substitute in for y:

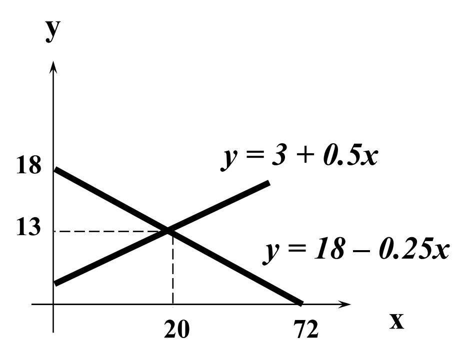
$x = 72 - 4y$

$\Rightarrow x = 72 - 4(3 + 0.5x)$

$\Rightarrow x = 20$

Now use one of the equations to find y:

$y = 3 + 0.5(20) \Rightarrow y = 13$



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Solving Systems of Equations

*Exercises: **Answers***

2. ***P = 10 – 0.2Q and P = 1.8Q***

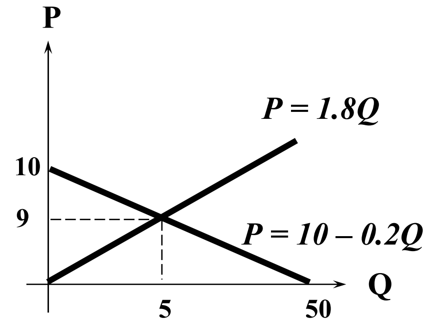
Set the equations equal:

10 – 0.2Q = 1.8Q

⇒ Q = 5

Now use one of the equations to find P:

P = 1.8(5) = 9



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Solving Systems of Equations

*Exercises: **Answers***

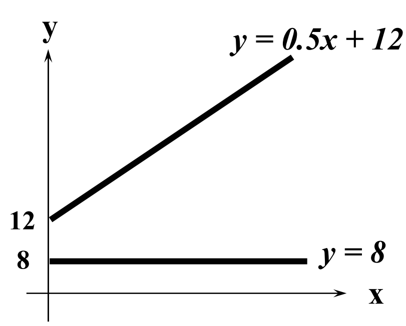
3. ***2y – x = 24 and y = 8***

Substitute y = 8 and solve:

2(8) – x = 24

⇒ x = -8

An intersection for a negative value of x is not relevant for us. There is no solution to this system of equations.



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Solving Systems of Equations

*Exercises: **Answers***

4. $y = 10 + 2x^2$ *and* $y = 135 - 3x^2$

Set the equations equal and solve:

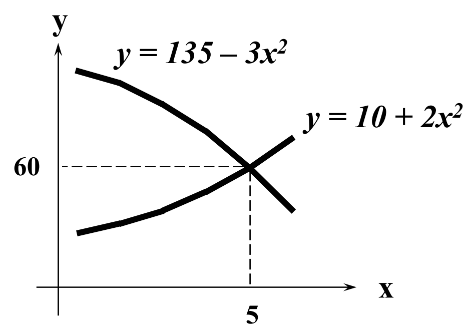
$10 + 2x^2 = 135 - 3x^2$

$\Rightarrow 5x^2 = 125 \Rightarrow x^2 = 25$

$\Rightarrow x = 5$

Solve for y:

$y = 10 + 2(5^2) = 60$



The Graph: These are two quadratic equations. One is hill-shaped, and the other is U-shaped. Here we show only the part of each curve that has positive x and y. There is actually more than one intersection in this case, and you would see it if you drew the complete curves. The other solution to $x^2 = 25$ is $x = -5$, but it is not relevant for us.

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Solving Systems of Equations

*Exercises: **Answers***

5.

	Standard	Deluxe	Constraints
Machine Hours	5	1	8,000
Production Cost	$4,000	$8,000	$10,000,000

Setup:

Let x = #standard desks and y = #deluxe desks

(1) $5x + 1y = 8{,}000$ (machine hours constraint)

(2) $4{,}000x + 8{,}000y = 10{,}000{,}000$ (budget constraint)

Solution:

Solve equation (1) for y and substitute into (2)

$4{,}000x + 8{,}000(8{,}000 - 5x) = 10{,}000{,}000$

$\Rightarrow 36{,}000x = 54{,}000{,}000 \Rightarrow$ *$x = 1{,}500$*

$\Rightarrow y = 8000 - 5(1500) \Rightarrow$ *$y = 500$*

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Calculating Areas

Introduction

- Sometimes, particularly in the field of economics, we find it valuable to draw and calculate certain areas defined by intersecting equations.
- In this module we will focus on visually depicting and calculating two concepts that will come up in the Economics of Business course:
  - The Firm's Total Revenue
  - The Consumer's Total Willingness To Pay

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Calculating Areas

Introduction

- It will be helpful to talk about price and quantity as variables, rather than x and y.
- For each of the examples that follow, we will use the following setup:
  - Price (P) is on the vertical axis and Quantity (Q) is on the horizontal axis.
  - P is measured in dollars and Q is measured in units per month.
  - The demand equation (and the corresponding line we will draw) represents one individual consumer's willingness to pay. Anna is the name of our consumer. Anna's willingness to pay is determined by her tastes and her income.

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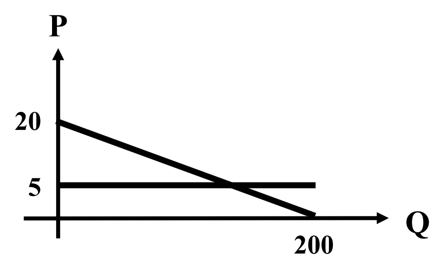
Calculating Areas

Example #1

- **1. P = 5 and P = 20 – 0.1Q**
  - **The price line at P = 5 shows that the company is charging a single price of $5 per unit, regardless of how many units are purchased. The demand line shows that the maximum price Anna is willing to pay is $20. As the price falls, she is willing to buy more. The maximum amount of this product that she will buy is 200 units per month.**



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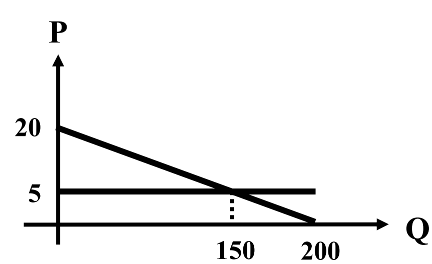
Calculating Areas

Example #1

- **1. P = 5 and P = 20 – 0.1Q**
  - **We will use this example for <u>three different calculations</u>.**
  - **First, we solve for the intersection of these two lines to find out how many units Anna will purchase at the price of $5.**

$5 = 20 - 0.1Q \Rightarrow Q = 150$



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Calculating Areas

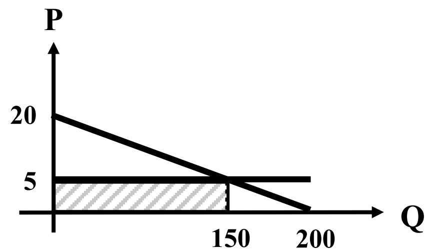
Example #1

- Second, we find total revenue earned by the company:

Total Revenue = Price × Quantity

= 5 × 150 = $750 per month

Total Revenue (TR) is the area of the rectangle shaded below:



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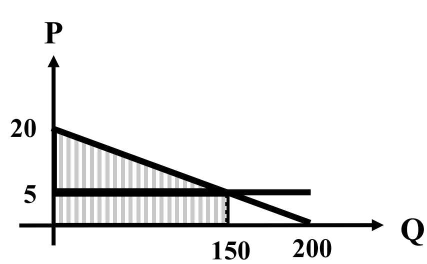
Calculating Areas

Example #1

- Third, the company would like to know how much Anna is actually willing to pay for 150 units.
  - *Willingness To Pay* (WTP) for all the units consumed is the willingness to pay for each unit, from the first unit up to the last unit.
  - WTP = shaded area under the demand line, up to 150.



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Calculating Areas

Example #1

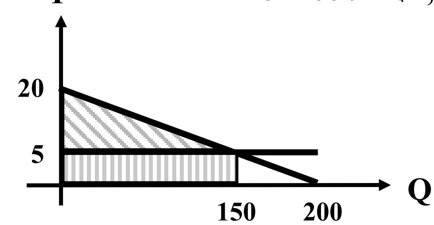
- **In order to calculate this area, we break it up into a triangle and a rectangle:**
  - **WTP = area of the triangle and rectangle shaded below**

    Area of Triangle **= ½(base × height)**

    Area of Rectangle **= base × height**

    In this case: WTP = ½[150(20 – 5)] + (150 × 5)

    = 1125 + 750 = $1,875 per month



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Calculating Areas

Example #2

- **2. Anna's Demand: P = 20 – 0.1Q (same as before)**

  Company's Price Schedule:

  P = $10 for first 50 units

  P = $5 for all remaining units

  - **The company is trying to take advantage of the fact that Anna is willing to pay more for the first block of units consumed.**

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Calculating Areas

Example #2

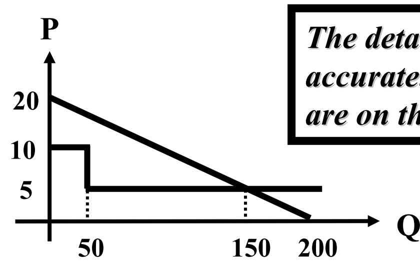
➢ **2. Anna's Demand: P = 20 – 0.1Q (same as before)**

Company's Price Schedule:

P = \$10 for first 50 units

P = \$5 for all remaining units

- **This price schedule is graphed below:**



The details on how to accurately sketch this are on the next slide…

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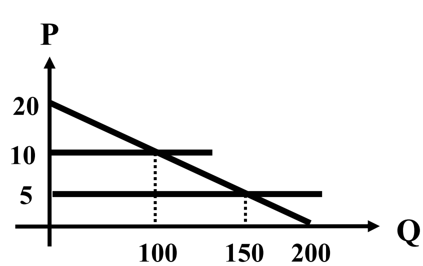
Calculating Areas

Aside: Graphing Price Schedule

➢ **It is critical to know how the company's price schedule overlaps with Anna's demand line. In order to find this out, you have to plug the company's prices into the demand equation:**

$$P = 10 \Rightarrow 10 = 20 - 0.1Q \Rightarrow Q = 100$$

$$P = 5 \Rightarrow 5 = 20 - 0.1Q \Rightarrow Q = 150$$



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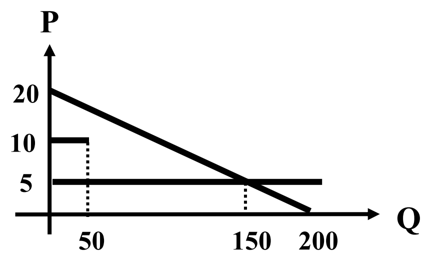
Calculating Areas

Aside: Graphing Price Schedule

- The P = 10 line intersects the demand line at Q = 100; *but the company stops charging P = 10 once Q = 50.* Therefore, the price schedule at P = 10 lies within (to the left of) the demand line. It does not intersect the demand line.



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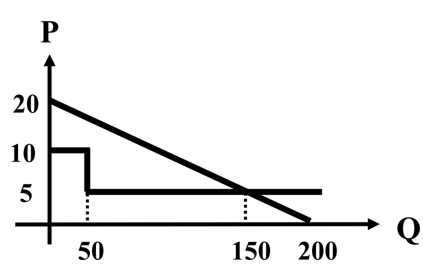
Calculating Areas

Aside: Graphing Price Schedule

- The P = 10 line intersects the demand line at Q = 100; *but the company stops charging P = 10 once Q = 50.* Therefore, the price schedule at P = 10 lies within (to the left of) the demand line. It does not intersect the demand line.

- The only intersection is at P = 5, where Q = 150, as before.



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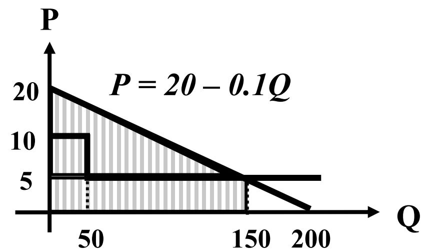
Calculating Areas

Example #2, conclusion

- **2. WTP is $1,875, the same as in Example #1 since we are working with the same demand line in both examples and total quantity consumed in both examples is Q = 150.**



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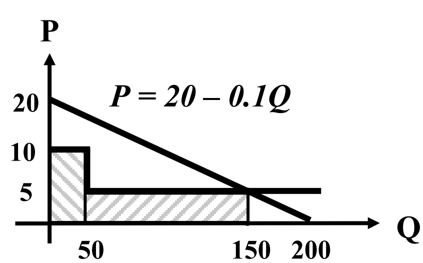
Calculating Areas

Example #2, conclusion

- **2. But the Total Revenue calculation is different.**

TR = two shaded rectangles
= 50(10) + (150 – 50)5 = $1000, compared to $750 before



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Calculating Areas

Exercises

In each case, calculate the quantity demanded, total revenue, and total willingness to pay. Graph your answer.

1. **Demand: P = 100 – 5Q**
   Company's Price: P = 40

2. **Demand: P = 100 – 5Q**
   Company's Price Schedule:
   P = $40 for first 10 units
   P = $30 for next 4 units
   P = $20 for all remaining units

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Calculating Areas

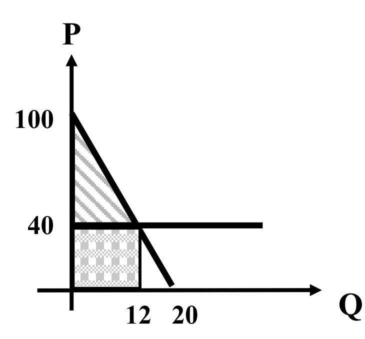
*Exercises: **Answers***

1. Demand: P = 100 – 5Q
Company's Price: P = 40

P = 40 ⇒ 40 = 100 – 5Q ⇒ Q = 12

TR = 12(40) = $480

WTP = triangle + rectangle
= ½(12)(100 – 40) + 480
= 360 + 480
= $840



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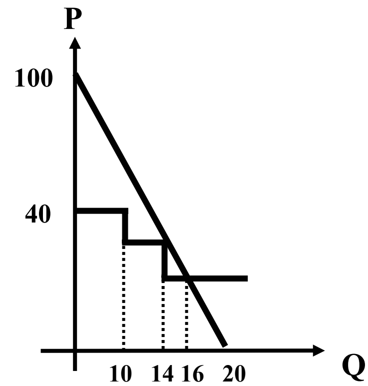
Calculating Areas

*Exercises: **Answers***

2. Demand: P = 100 – 5Q
Company's Price Schedule:
P = $40 for first 10 units
P = $30 for next 4 units
P = $20 for all remaining units



First, find the intersections:
P = 40 ⇒ 40 = 100 – 5Q ⇒ Q = 12
P = 30 ⇒ 30 = 100 – 5Q ⇒ Q = 14
P = 20 ⇒ 20 = 100 – 5Q ⇒ Q = 16

Therefore, the first price ($40) does not intersect the demand line; the second intersection exactly touches the demand line; and the third intersection crosses through the demand line.

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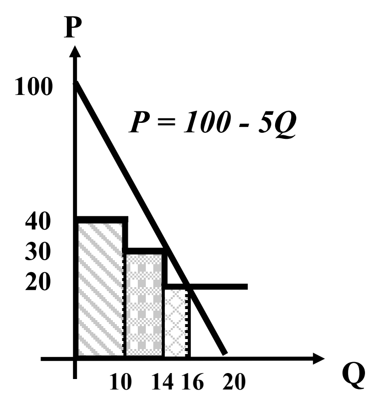
Calculating Areas

*Exercises: **Answers***

2. Now calculate the areas:



TR = shaded rectangles
= 10(40) + (14 – 10)30
+ (16 – 14)20
= 400 + 120 + 40
= $560

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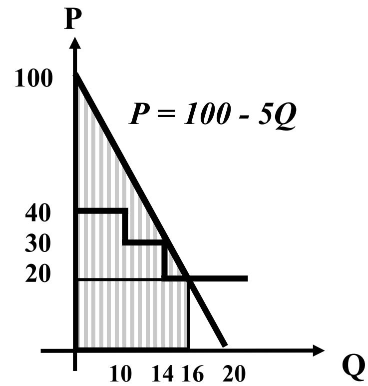
Calculating Areas

*Exercises: **Answers***

2. Now calculate the areas:

WTP = entire shaded area under the demand curve up to Q = 16.



Calculating Areas

*Exercises: **Answers***

2. Now calculate the areas:

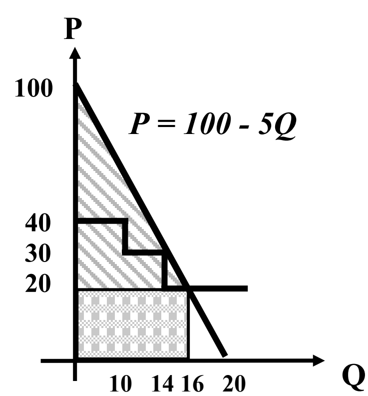
WTP = entire shaded area under the demand curve up to Q = 16.

WTP = triangle + rectangle

= ½(16)(100 – 20) + 16(20)

= 640 + 320

= $960



Application

Evaluating Profit Opportunities

- You have estimated your average (unit) cost structure for a new product and you are wondering if there is a profit opportunity in this new market. We can illustrate how you might think about this in terms of solving a system of equations.
- There are two relevant equations (at the broadest level) – an average cost equation and a market demand equation.

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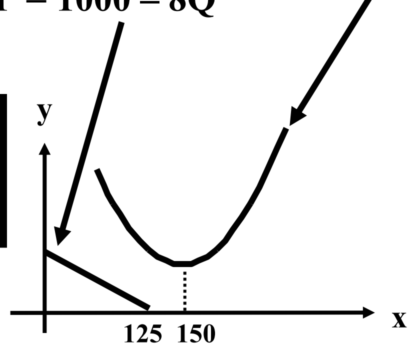
Application

Evaluating Profit Opportunities

- Average Cost: $AC = 100 + 0.25(Q - 150)^2$
- Demand: $P = 1000 - 8Q$

As usual, this is a rough sketch of what the graph looks like, but it is not to scale.



Bottom Line: There is no intersection (no solution to the system of equations), which means there is no profit opportunity.

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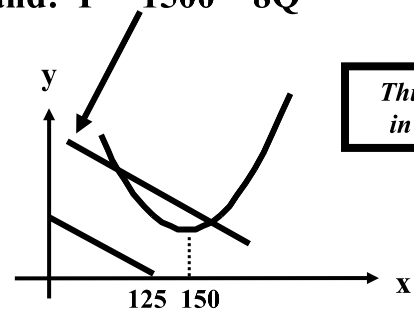
Application

Evaluating Profit Opportunities

- ***What if the market were bigger?***
- **Old Demand: P = 1000 – 8Q**
- **New Demand: P = 1500 – 8Q**



This is a parallel shift in the demand line.

Now there is a profit opportunity. That is, consumers are willing to pay prices that, at least for some volumes of production, more than cover cost.

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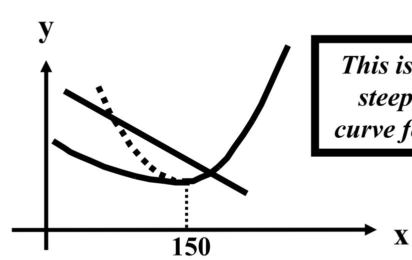
Application

Evaluating Profit Opportunities

- ***What if it weren't so expensive to be a small firm?***
  - **What if you could decrease the fixed costs of production, distribution, or product development? This lowers startup costs.**



This is a decrease in the steepness of the cost curve for small volumes.

Example: Private-label breakfast cereals can lower their up-front development costs by manufacturing only the simplest types of cereals. This is one of the reasons why, despite charging a lower price, they can have a positive profit margin (depending, of course, on the extent of demand for their product).

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