For all problems, *SHOW ALL OF YOUR WORK*. While partial credit will be given, partial solutions that could be obtained directly from a calculator or a guess are worth no points. Continue your work on the back of the page or extra sheet at the end of the exam if you need additional space. *You do not need but may use the normal graphing calculator functions of any graphing calculator, but NOT any differential equations functionality it may have.* If you need to borrow a graphing calculator, ask me.

1. Find real-valued solutions to the following, as indicated. Where possible, give an explicit answer.

   a. Find the solution to $\frac{dy}{dx} = 3y - e^{2x}$, $y(0) = 4$. (8 points)

   b. Find the solution to $y'' + 2y' + y = 0$, $y(0) = 1$, $y'(0) = 2$. (8 points)

   c. Find the general solution to $(x^2 + 1)y' = 3xy$. (5 points)

   d. Find the general solution to $y''' + 9y' = 0$. (5 points)

2. Four solutions of the differential equation $x^2y'' + 4xy' + 2y = 0$ $(x > 0)$ are $y_1 = \frac{1}{x^2}$, $y_2 = 4x^{-2}$, $y_3 = \frac{2+x}{x^2}$ and $y_4 = x^{-1}$. Find a general solution to the problem. Explain why you arrive at the solution you do. (8 points)

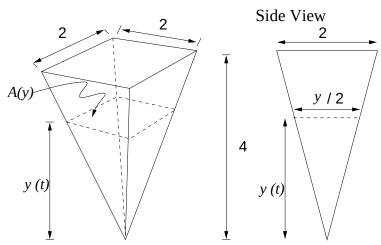
   a. What does "general solution" mean, anyway? (4 points)

3. Suppose that the population of blue ruffle-headed sneetches is modeled by the differential equation $\frac{dP}{dt} = 2(1 - P)(P^2 - 4)$. Use *qualitative methods* (that is, you don't have to—and probably don't want to—solve the equation) to predict what you expect the population of sneetches is likely to be in the long-term. Does your answer depend on the initial number of sneetches? Explain. (15 points)

4. Find each of the following.
   a. $(1 + i\sqrt{3})^{10}$ (5 points)

   b. The real and imaginary parts of $z = \frac{1+i}{1+2i}$. (5 points)

5. Let $y(t)$ be the height of water in a tank and $A(y)$ be the area of a horizontal cross-section of the tank at the height y, as shown in the figure to the right. Then as water drains from the tank, the time rate of change of y is proportional to the square root of y divided by the area of the cross-section at that height.



   a. Write a differential equation for $y(t)$ in this case. You should be able to write $A(y)$ in terms of y. (6 points)

   b. If the height of water in the tank is initially 3 and at $t = 1$ the height has dropped to 2, find when the tank will be empty. (15 points)

6. Recall that the Euler and Improved Euler methods for the first-order differential equation $\frac{dy}{dx} = f(x, y)$ have the iteration formulae

$$y_{n+1} = y_n + hf(x_n, y_n) \quad \text{and}$$
$$y_{n+1} = y_n + \tfrac{1}{2}h(k_1 + k_2), \quad \text{where} \quad k_1 = f(x_n, y_n) \quad \text{and} \quad k_2 = f(x_{n+1}, y_n + hk_1),$$

respectively. If we are approximating solutions to the initial value problem $y' + \frac{2}{x}y = x$, $y(1) = 1$, fill in the missing boxes in the following table. Be sure it is clear how you obtain your results. (8 points)

Method	$x =$	1	1.025	1.05	1.075	1.1	1.125	1.15	1.175	1.2
1. Euler, $h = 0.1$	$y \approx$	1	X	X	X	0.9	X	X	X	0.8464
2. Euler, $h = 0.025$	$y \approx$	1	0.975	0.9531	0.9339		0.9032	0.8912		0.8730
3. Impr Euler, $h = 0.1$	$y \approx$	1	X	X	X	0.9232	X	X	X	0.8821
4. Impr Euler, $h = 0.025$	$y \approx$	1	0.9765		0.9379	0.9224	0.9091	0.8978	0.8885	0.8809

a. The exact solution to this IVP is $y = \frac{1}{4}x^2 + \frac{3}{4}x^{-2}$. Find the *cumulative error* for each of the methods. Explain why (or why not) it changes as you would expect as the step size decreases. (8 points)