For all problems, *SHOW ALL OF YOUR WORK*. While partial credit will be given, partial solutions that could be obtained directly from a calculator or a guess are worth no points. Continue your work on the back of the page or extra sheet at the end of the exam if you need additional space. *You do not need but may use the normal graphing calculator functions of any graphing calculator, but NOT any differential equations functionality it may have.* If you need to borrow a graphing calculator, ask me.

1. Solve each of the following differential equations to obtain a general solution or, where possible, a particular one. If possible, give an explicit solution.
   a. $2y' = \frac{1}{x}y - 4x^2$, $y(1) = -1$ (8 points)

   b. $y'' + 2y' + 5y = 0$ (8 points)

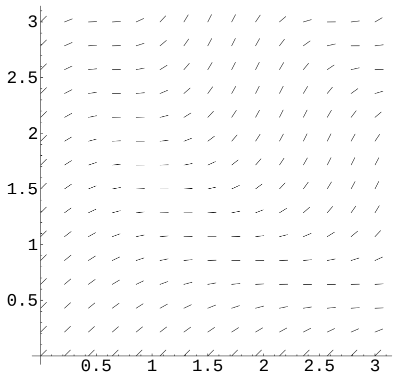
   c. $xy\frac{dy}{dx} = x^2 - x^2y^2$ (8 points)

   d. $2y'' + 8y' + 8y = 0$, $y(0) = 0$, $y'(0) = 3$ (8 points)

2. A passing tortoise is heard to assert that $y_1 = x^2(1-x)^{-1}$ and $y_2 \equiv 0$ are both solutions to the differential equation
$$\frac{1}{y}\frac{dy}{dx} = \frac{y+2x}{x^2}$$
with initial condition $y(1) = 0$.
   a. Is the tortoise correct? (6 points)

   b. How is this related to the Existence and Uniqueness Theorem for first-order ordinary differential equations? (8 points)

3. The direction field for a differential equation $\frac{dy}{dx} = f(x,y)$ is shown to the right.
   a. Sketch a solution that goes through $y(0.5) = 1.0$. (3 points)
   b. How many solution curves go through the point $(0.5, 1.0)$? How do you know? (3 points)



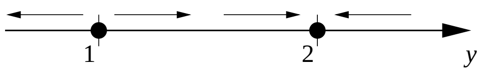
   c. For the solution you drew, what are the approximate values of $y(0)$ and $y(3)$? (3 points)

4. An alert student notes that her initially stationary professor has an initial acceleration of 10 m/s^2. Air resistance, however, results in a retardation of this acceleration which is proportional to the professor's speed.
   a. Write an initial value problem modeling this. (6 points)

   b. What can you say about the professor's motion without solving the differential equation? (3 points)

   c. Solve your initial value problem. What additional data do you need to be able to obtain an explicit solution? (6 points)

5. Write a differential equation that could result in the following phase diagram. Explain briefly why you choose the equation you do. (8 points)



   a. What are the equilibrium solutions of this differential equation? Which are stable? (4 points)

6. Euler's method, Improved Euler's method, and Runge-Kutta are used to solve a differential equation $y'(x) = f(x, y(x))$, $y(0) = 1$. Some of the resulting points are given in the following table. The cumulative error for each method at $x = 0.4$ is then given.

Method	$x =$	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4
1.	$y \approx$	1	1.0487		1.1377	1.1776	1.2143	1.2476		1.3044
2.	$y \approx$	1		1.0974	1.1419		1.2217	1.2566	1.2882	1.3164
3.	$y \approx$	1	1.0487	1.0947			1.2144	1.2477	1.2778	1.3045

Method	1	2	3
Error	0.0001260	0.01192	1.700×10^{-6}

a. What is the step size h? (2 points)

b. Identify which method is which by filling in the first column of the table. (How do you know?) (4 points)

c. Fill in the missing values in the Euler's method row, if the differential equation being solved is $y' = 1 - \sin(xy)$. Be sure it is clear how you obtain your result. (6 points)

d. Suppose that we recalculated the values in the table with $h = 0.2$. What would you expect the error at $x = 0.4$ to be for the Euler's method calculation? For the Improved Euler's method? Why? (6 points)