For all problems, *SHOW ALL OF YOUR WORK*. While partial credit will be given, partial solutions that could be obtained directly from a calculator or a guess are worth no points. Continue your work on the back of the page or extra sheet at the end of the exam if you need additional space. *You do not need but may use the normal graphing calculator functions of any graphing calculator, but NOT any differential equations functionality it may have.* If you need to borrow a graphing calculator, ask me.

1. Solve each of the following to find, where possible, explicit real-valued solutions.

a. $y' + 2xy = 3x^2 y$, $y(1) = 5$. (6 points)

Solution: This is linear and separable. We'll separate variables to solve it. The equation is the same as $y' = (3x^2 - 2x)y$, so that

$$\frac{dy}{y} = (3x^2 - 2x)dx.$$

Integrating both sides, $\ln|y| = x^3 - x^2 + \hat{C}$, and, exponentiating to find y, $y = Ce^{x^3 - x^2}$. (Where $C = \pm e^{\hat{C}}$.) The initial condition requires that $C = 5$, so $y = 5e^{x^3 - x^2}$. ■

b. $y''' + 4y'' + 5y' = 0$. (5 points)

Solution: This is a linear, constant-coefficient homogeneous problem, so we guess $y = e^{rx}$. Then $r(r^2 + 4r + 5) = 0$, or $r((r+2)^2 + 1) = 0$, so $r = 0$ or $r = -2 \pm i$. A general solution is therefore $y = c_1 + c_2 e^{-2x} \cos(x) + c_3 e^{-2x} \sin(x)$. ■

c. $y'' + 6y' + 9y = 0$, $y(0) = 0$, $y'(0) = 4$. (6 points)

Solution: Another linear, constant-coefficient homogeneous problem. Guess $y = e^{rx}$ to get $(r+3)^2 = 0$. Thus $r = -3$ twice, and a general solution is $y = c_1 e^{-3x} + c_2 x e^{-3x}$. If $y(0) = 0$, $c_1 = 0$, and $y'(0) = c_2 = 4$. Thus $y = 4xe^{-3x}$. ■

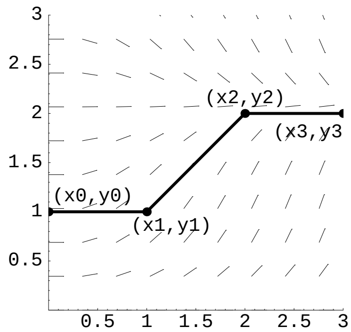
d. $y' = \tan(x)y - 3$. (5 points)

Solution: This is a linear, not-separable, first-order equation. We solve it with an integrating factor. First, rewrite the equation as $y' - \tan(x)y = -3$. The integrating factor is $\rho(x) = \exp(-\int \tan(x)dx) = \exp(\ln|\cos(x)|) = \cos(x)$. Thus, multiplying both sides of the equation by the integrating factor,

$$\cos(x)(y' - \tan(x)y) = (\cos(x)y)' = -3\cos(x).$$

Integrating both sides, $\cos(x)y = -3\sin(x) + C$, and $y = -3\tan(x) + C\sec(x)$ $(= (-3\sin(x) + C)/\cos(x))$. ■

2. The figure to the right shows the direction field for some equation
$$\frac{dy}{dx} = f(x,y)$$
Use this and *Euler's Method* with a step size of $h = 1.0$ to estimate $y(3)$ if $y(0) = 1$. Be sure that it is clear from your work how you obtain your answer, and how it is related to the numerical method you are using. (10 points)



Solution: Euler's method calculates the next point in the approximation with the formula $y_{n+1} = y_n + hf(x_n, y_n)$. $f(x_n, y_n)$ is just the slope at the previous point, which is represented on the graph as the slope of the direction field line at that point.

The easiest way of estimating with Euler's method is therefore to draw on the direction field the lines starting at each point and extending a distance $h = 1$ with the correct slope, as shown. This gives $y(1) \approx 1$, $y(2) \approx 2$, and $y(3) \approx 2$.

(We could also estimate the slope from the graph and determine successive points using this in Euler's method. At $x = 0$, the slope $f(0, y) = 0$, so $y(1) \approx y(0) + hf(0, y(0)) = 1 + (1)(0) = 1$. At $(1, 1)$, we estimate the slope to be $f(1, 1) \approx 1$, so that $y(2) \approx y(1) + hf(1, y(1)) = 1 + (1)(1) = 2$, etc.) ∎

3. Consider the differential equation $y'' = -2yy'$, $x \neq -1$.

a. Verify that $y_1 = \frac{1}{2}$ and $y_2 = \frac{1}{x+1}$ are both solutions to this. (6 points)

Solution: We can do this by plugging in to the differential equation. It is clear that y_1 is a solution: $y_1' = 0 = y_1''$, so we get $0 = -2(\frac{1}{2})(0)$, which is obviously true.
For y_2, note that $y_2' = -(x+1)^{-2}$, and $y_2'' = 2(x+1)^{-3}$. The right hand side of the equation therefore becomes $(-2)((x+1)^{-1})(-(x+1)^{-2}) = 2(x+1)^{-3}$, which is the same as y_2'', and this is therefore also a solution. ∎

b. Are these two functions linearly independent? Why or why not? (5 points)

Solution: They are linearly independent, as they are clearly not constant multiples. We can verify this with the Wronskian:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & (x+1)^{-1} \\ 0 & -(x+1)^{-2} \end{vmatrix} = -\frac{1}{2}(x+1)^{-2} \neq 0,$$

so the two functions are linearly independent. ∎

c. Can you write a general solution to this problem? Explain. (4 points)

Solution: We cannot write a general solution (using y_1 and y_2) to this problem because it is a nonlinear ODE. To find a general solution we would have to have some method of solving the problem "en masse," which we don't have. (Note that while y_1 and y_2 are independently solutions to the problem, the sum $y_1 + y_2 = \frac{1}{2} + \frac{1}{x+1}$ is not a solution! Try it and see!) ∎

4. A not-so-cool-as-all-that cat lounges languidly in a room $9\times10\times5$ meters large. The air flow in the room introduces clean air at a rate of $1\text{m}^3/\text{min}$, and the cat's smoky cigarette converts 0.002m^3 of air per minute into a mixture containing 5% carbon monoxide. This is well mixed in the room, and the ventilation system removes 1m^3 of smoky air from the room every minute. Find the amount of carbon monoxide in the room as a function of time. (16 points)

Solution: This is a mixture problem. Let $A(t) =$ the amount of carbon monoxide in the room. Then

$$\frac{dA}{dt} = (\text{rate of addition of CO}) - (\text{rate of loss of CO}).$$

The only input of CO that we know of is the CO created by the cat's cigarette, which produces $(0.002\text{m}^3\ \text{air/min})(0.05\text{m}^3\ \text{CO}/\text{m}^3\ \text{air}) = 0.0001\text{m}^3\ \text{CO/min}$. (The clean air coming into the room doesn't change the amount of CO there.) The CO leaves the room at a rate of $(1\text{m}^3\ \text{air/min})(\text{concentration of CO in room})$. The concentration of CO in the room is just the amount present divided by the volume of the room, or $A/450$, so we get the equation

$$\frac{dA}{dt} = 0.0001 - \frac{A}{450}.$$

We can solve this with an integrating factor. After moving the term in A to the left-hand side of the equation, the integrating factor is $e^{t/450}$, so $(Ae^{t/450})' = 0.0001e^{t/450}$. Integrating both sides, $Ae^{t/450} = 0.045e^{t/450} + C$. Divide through by the exponential to get $A(t) = 0.045 + Ce^{-t/450}$. We may assume that $A(0) = 0$ (there is initially no CO in the room), so that $A(t) = 0.045(1 - e^{-t/450})$. ■

5. Below are three phase diagrams for differential equations of the form $\frac{dP}{dt} = f(P)$.

i. 

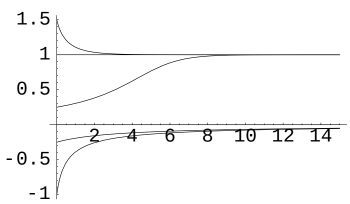
ii. 

iii. 

a. Which, if any, of these phase diagrams could correspond to the differential equation $\frac{dP}{dt} = P(1-P^2)$? Why? (6 points)

Solution: We note that this differential equation has three equilibrium solution, $P = 0$, $P = 1$ and $P = -1$. Therefore none of the phase diagrams (all of which have only two equilibria) can correspond to this differential equation. ■

b. Which, if any, of these phase diagrams could correspond to the plot of solution curves to a problem of this type shown to the right? (6 points)



Solution: This figure shows solutions approaching the equilibrium solution $P = 0$ from below (giving an arrow to the right), and all solutions above $P = 0$ approaching the equilibrium solution $P = 1$ (giving an arrow to the right for $0 < P < 1$ and to the left for $P > 1$). This behavior is shown in phase diagram (ii). ■

6. Rewrite each of the following as indicated. Be sure to show all of your work. Yes, every little step.
   a. $z = \frac{2i}{(1-3i)}$ as $z = x + iy$. (4 points)

   Solution: To rewrite this, multiply the numerator and denominator by the complex conjugate of the denominator:

$$z = \frac{2i}{(1-3i)} = \frac{2i}{(1-3i)} \cdot \frac{1+3i}{1+3i} = \frac{-6+2i}{10} = -\frac{3}{5} + \frac{1}{5}\, i.$$

■

   b. $z = -4 + i$ as $z = re^{i\theta}$. (4 points)

   Solution: The modulus $r = |z| = \sqrt{(-4)^2 + 1^2} = \sqrt{17}$. The angle θ is the angle from the x (Re(z))-axis to the point z in the complex plane. Here z is in the second quadrant, so we need the angle $\theta = \pi - \arctan(1/4)$. Thus $z = \sqrt{17}\, e^{i(\pi - \arctan(1/4))}$. ■

7. Archimedes, brilliant man that he was, told us that the buoyancy force on an object is equal to the weight of the water it displaces. For a cylindrical buoy suspended with its axis perpendicular to the surface of the water, this means that $F_B \propto y$, where y is the amount the buoy is submerged beyond its equilibrium position. Suppose that we have a buoy with mass m for which $F_B = 576y$ (in English units).
   a. Explain why the equation $y'' + \frac{576}{m} y = 0$ is a good model for this problem. (5 points)

   Solution: This is a good model, because if we start with Newton's assertion that $ma = \Sigma\,$(forces), we have $ma = my'' = -F_B$. We assume no frictional forces, and are able to ignore the gravitional force because we're measuring the displacement of the buoy from the equilibrium position. The negative sign arises because the buoyant force opposes the submersion of the buoy. Thus $my'' + 576y = 0$, or $y'' + \frac{576}{m} y = 0$. (Alternately, we know that the motion of the buoy will be oscillatory in the same manner as a mass-spring system, with the buoyant force playing the role of the restoring force. Thus we can start with the equation $my'' + cy' + F_B = 0$, or, with $c = 0$, $y'' + \frac{576}{m} y = 0$.) ■

   b. Suppose that a playful meremaid pulls the buoy down slightly to start it oscillating. What is the mass of the buoy to be if its period of oscillation is π seconds?[1] (12 points)

   Solution: Solving the differential equation above, we let $y = e^{rt}$, getting $r = \pm i\sqrt{\frac{576}{m}} = \pm i\frac{24}{\sqrt{m}}$. Thus the solution for y is $y = c_1 \cos(\frac{24}{\sqrt{m}} t) + c_2 \sin(\frac{24}{\sqrt{m}} t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$. The period of oscillation for y is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{24/\sqrt{m}} = \frac{\sqrt{m}\,\pi}{12}.$$

   This must equal π, so $\sqrt{m} = 12$, or $m = 144$. If we're in English units, this is likely to be 144 slugs, or $144 \cdot 32.2 = 4637$ lbs. That's a pretty strong meremaid. ■

[1] : *could the meremaid beat you in an arm-wrestling contest? (No, you don't have to answer this question to get full credit on the problem. . .)*