

COMMENT

Circles and Squares, Spheres and Cubes: What's the Deal with Circumplex Models?

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A distinction is made between data description and representational space in the context of circumplex models. The representational space provides the language in which data are described, and different languages have their advantages and disadvantages. For instance, points in a two-dimensional Cartesian grid can form a circle. Such a circular pattern corresponds to a description of the data pattern. However, the same data can also be represented in a polar coordinate system, which is a different representational space than the Cartesian grid. I claim that additional theory advancement in applied areas can occur if more attention is given to the particular representational space in which the circumplex is used. I also present three diagnostic properties that all perfect circumplex models must satisfy. © 1996 Academic Press, Inc.

I read the paper by Tracey and Rounds (this issue) with great interest. Fifteen years ago as a confused undergraduate I took a vocational interest inventory. The guidance counselor who interpreted my score suggested I pursue a career in forestry. As with most advice I received during that period in my life, I discarded it and pursued a career in psychology instead. I believe, however, that a scientific understanding of interest measurement and the ways in which vocational interests are interrelated is very important.

Tracey and Rounds (this issue) propose the addition of a third dimension, prestige, to the now standard two-dimensional conceptualization of vocational interests in terms of People/Things and Data/Ideas dimensions. They argue that the three possible two-dimensional subspaces created by all pairwise combinations of the dimensions each conform to a circumplex structure, and that these three circumplexes can be combined to form a sphere. In this comment I focus on the properties of circumplex models and the distinction

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between a description of data and the representational space in which the data are described.

DATA PATTERNS AND REPRESENTATION SPACES

It is essential to distinguish between the *data pattern* and a *representational space* when considering circumplex models. Data pattern refers to the observed configuration of items in a space. For instance, in a scatterplot points can display different patterns such as an oval with a positive slope (indicating a positive correlation), an inverted-U shape (indicating one possible curvilinear form), or a random pattern resembling a blast from a shotgun (indicating a correlation of 0). Note that the underlying representational space (a two-dimensional Cartesian grid) is held constant in these scatterplots; it is the configuration of points, the data pattern, that varies. The typical output from a multidimensional scaling showing the stimulus space is another example of a data pattern. The stimulus space can be plotted and the observed configuration may be interpreted.

A representational space, on the other hand, refers to the particular coordinate system that is used to identify points in that space. For example, the most common representational space used in psychology is the Cartesian coordinate system. Using this system, points in a plane are represented by ordered pairs (e.g., x, y , where x and y are real numbers) and points in three dimensions are represented by ordered triples (e.g., x, y, z , where x, y , and z are real numbers). Polar coordinates offer a different but equally useful representational space for the plane. That is, a point in a plane can be represented in terms of its distance from the origin (denoted r) and its angle from, say, the horizontal axis (denoted θ). As with the Cartesian representation, a point in a polar coordinate system is represented as an ordered pair but the ordered pair consists of a real number r and an angle θ . Alternate representations to the usual three-dimensional (x, y, z) "cube" are spherical spaces and cylindrical spaces. For example, cylindrical coordinates are simply polar coordinates in the (x, y) plane (see Boaz, 1983, for a discussion). The representational space is independent from the data pattern.

I should point out that the cylindrical and spherical representational spaces I referred to above are different from the cylinder and sphere models discussed by Tracey and Rounds (this issue). By cylindrical and spherical spaces I refer to different ways to express coordinates to describe a structure, whereas Tracey and Rounds use cylinder and sphere to refer to the data pattern (i.e., within the particular representational space of a Cartesian grid whether the points are arranged in the form of a cylinder or a sphere).

A representational space is usually chosen because it provides a convenient language that makes particular theoretical expressions easy. For example, sometimes it is convenient to talk about motion in terms of rectangular coordinates; other times, as with the pendulum, it is more convenient to talk about motion in terms of polar coordinates (Boaz, 1983). Sometimes one representa-

tion makes the expression of theory easier than a different representation. In the case of quantum mechanics, wave theories about atoms are relatively easy to express in terms of a spherical representation (Feynman, 1965). The key is to choose a coordinate system that provides a natural “language” for discussing the phenomenon being modeled. The underlying laws and theoretical statements remain the same even though the problems are described using different coordinates. For example, a Pearson correlation can be described in terms of N points in two dimensions where N is the number of subjects (the usual textbook depiction of the scatterplot), or it can be described as an N -dimensional space with two points (Walker, 1940). In the latter representation, the two points are vectors. One point represents the N scores on variable X and the other point represents the N scores on variable Y . The cosine of the angle between the two vectors X and Y is equal to the Pearson correlation between variables X and Y . Thus, different representational spaces can be used to denote the same information about the Pearson correlation. Each representation has its own merits. Note that the representation can be studied separately from the data pattern.

A more detailed consideration of psychological examples may help make the distinction between data pattern and representational space clearer. Sternberg (1986) proposed a three-dimensional theory of love, which he depicted using a triangle where the length of each side represented the score on the corresponding three dimensions of passion, intimacy, and commitment. I am not sure why Sternberg chose the metaphor of a triangle (maybe he thought it would be cute to talk about “love triangles”), but it turns out that this representational space has serious problems. Suppose passion in a relationship is increased. This would require an increase in the length of the side of the triangle representing passion. However, in order to maintain a triangle, one or both of the remaining sides also needs to change. Thus the choice of a triangle as a representational space imposes a limitation on the theory because a change in one dimension must be accompanied by a change on one or both of the other dimensions. This “predicted change” is not part of the theory because Sternberg claims that the three dimensions are independent but it is an inescapable artifact of the particular representational space he chose.

Sternberg could eliminate this problem by changing the representational space to polar coordinates. Three angles (say, 90, 210, and 330) could be used to represent the three dimensions, and the distance from the origin could be used to represent the value on each dimension. This structure resembles the peace sign from the sixties—an upside-down Y embedded in a circle. The center of the circle represents 0 on all three dimensions. Three “axes” radiate out from the center to the perimeter of a circle according to one of the three angles. Within this structure, a given relationship is represented by three ordered pairs (i.e., an angle/length pair for each dimension), and the three points define a triangle. Note that it is possible to vary independently the length of any side in this structure, unlike the metaphor used by Sternberg.

Another example involves the subjective similarity of colors. In a now classic analysis, Shepard (1962) examined Ekman's data on 14 color patches. The nonmetric multidimensional analysis revealed two dimensions, and the data points were arranged in a circular pattern. Even though the data pattern was circular, experience has shown that it makes sense in this domain to use a two-dimensional Cartesian grid as the representational space. It is interesting to note that Shepard cautions that a circular arrangement of the stimuli is one diagnostic sign that something may be going wrong with nonmetric multidimensional scaling. Sometimes a circular arrangement is diagnostic that too many dimensions are being fitted (Shepard, 1974).

WHY AM I MAKING THESE POINTS?

Tracey and Rounds (this issue) did a fine job at describing the data pattern they observed, but the same data pattern could have been described in terms of a cube, and the same randomization tests could have been performed on the implied order relations. Tracey and Rounds (this issue) extol the virtues of circumplex models as capturing similarity between octants, but note that the same can be said of the Cartesian grid when it is divided into octants. Precise relations among any pair of octants can be specified and it is possible to perform the randomization test of Hubert and Arabie (1987) on the ordering within a square as well as the ordering within a circle. It should be understood, therefore, that the circumplex-based spherical model proposed by Tracey and Rounds (this issue) has no mathematical advantages over a cube model formed by using a Cartesian coordinate system.

Thus, it is important to ask, What's so special about a sphere? Does a spherical data pattern facilitate the development of theory or practical interventions in a way that a cube would not? It is important to consider how the representational space adopted influences theory and application. Much like my example of Sternberg's triangle theory of love where a different representational space helped the expression of the underlying theory, I wonder whether an analogous benefit would come by thinking about vocational interests in terms of other representational spaces such as spherical coordinates. For attempts to use a different representation space in the context of circumplex models see Anderson (1960) and Browne (1992).

SIMPLE DIAGNOSTICS FOR CIRCUMPLEX DATA PATTERNS

Matrices that have special patterns, such as the circumplex, can be exploited in interesting and illuminating ways (e.g., Davis, 1979; Graybill, 1983). In the case of the perfect circumplex (what some call the symmetric circulant matrix, see Browne, 1992), the eigenvalues themselves have particular patterns. For instance, when a correlation matrix displays a perfect circumplex pattern, the sum of a row in the correlation matrix is identical to one of the eigenvalues (Graybill, 1983). Further, the sum of each row (and column for that matter) in a perfect circumplex will have the identical value. Another

property of a perfect circumplex is that not all eigenvalues are unique. For instance, a 6×6 correlation matrix that conforms to the perfect circumplex will have four (not six) unique eigenvalues; there will be two pairs of identical eigenvalues. A perfect 6×6 circumplex that has the circ(1, a, b, c, b, a) pattern (see Cudeck, 1986, for a definition of circ), where $a > b > c$, will have the following four unique eigenvalues: $1 - a - b + c$, $1 + a - b - c$, $1 - 2a + 2b - c$, and $1 + 2a + 2b + c$. Other properties of the correlation matrix (e.g., values are bounded between -1 and 1) place additional restriction on the possible eigenvalues. It is possible to extend these arguments to circumplex matrices of any size.

Of course, these properties apply to the assumed, underlying population correlation matrix, so a sample correlation matrix will likely deviate from such properties. However, developing statistical tests for these properties (i.e., tests that treat the perfect circumplex pattern as the null hypothesis) is relatively routine. In this paper I will illustrate these properties with an example, focusing primarily on the descriptive value of these properties. I will skip some of the detail in order to keep this paper nontechnical. Note that the diagnostic properties refer to the description of the data pattern (i.e., a hypothesized pattern in the population correlation matrix). The implications of these properties for the representational space (i.e., the underlying vector space) is beyond the scope of this short note.

I examined a 6×6 correlation matrix recently reported by Tracey and Rounds (Rounds, Tracey, & Hubert, 1992, Table 1, p. 246). The six variables correspond to Holland's six vocational interest types. For the examples below I make use of the correlations for the male subjects reported by Rounds et al., (1992). That is, I took the correlations reported in the lower triangle in Table 1, which are based on $N = 358$ male subjects, and created the complete, square, symmetric 6×6 correlation matrix.

First, I examine the property that the sum of each row is equal to a constant. The sum of the six rows in the correlation matrix were 2.35, 2.53, 2.65, 2.98, 2.85, and 2.82. Without knowing the sampling distribution of a sum of six dependent correlations (which have a common subscript) it is difficult to test this property. It should be fairly routine to derive, however, a test for this property.

Second, I examine the constraint imposed by the perfect circumplex on the uniqueness of the eigenvalues. The eigenvalues for the observed correlation matrix were 2.72, 1.16, 0.91, 0.55, 0.38, and 0.28. We can now check a second property of the perfect circumplex: the sum of each row should equal one of the eigenvalues. A descriptive check of this property can be conducted by taking an average of the six row sums ($[2.35 + 2.53 + 2.65 + 2.98 + 2.85 + 2.82]/6 = 2.7$). This is close to one of the observed eigenvalues (2.72). Again, it is fairly routine to derive a significance test for this property.

Finally, recall that in a perfect 6×6 correlation matrix there will not be six unique eigenvalues. In the Rounds et al., correlation matrix, are there two

TABLE 1

Eigenvalues, Their Standard Errors, and Confidence Intervals for the Rounds, Tracey, and Hubert (1992) 6×6 Correlation Matrix for the Male Subjects ($N = 358$)

| | Factors | | | | | |
|-------------------------|---------|------|------|-----|-----|-----|
| | 1 | 2 | 3 | 4 | 5 | 6 |
| Observed eigenvalue | 2.72 | 1.16 | .91 | .55 | .38 | .28 |
| Standard errors | .20 | .09 | .07 | .04 | .03 | .04 |
| Approx. lower 95% bound | 2.32 | .99 | .78 | .47 | .32 | .24 |
| Approx. upper 95% bound | 3.12 | 1.33 | 1.04 | .63 | .44 | .32 |

pairs of identical (population) eigenvalues for this observed correlation matrix? We need to know the distributional theory of eigenvalues, but we can make use of a result found in Anderson (1984). The standard error of an eigenvalue is approximately the eigenvalue multiplied by the square root of $2/N$, where N is the number of subjects. I should note that the standard error given here should be taken as a first approximation because this result depends on all eigenvalues being unique and is based on the covariance matrix. A 95% confidence interval can be constructed around each eigenvalue by adding to the observed eigenvalue plus/minus the standard error times 1.96. This technique can also be extended to simultaneous confidence intervals to deal with the problem of multiple intervals, see Anderson (1984).

Applying this result to the Rounds et al., (1992) correlation matrix for male subjects leads to the confidence intervals reported in Table 1. Using the heuristic that nonoverlapping confidence intervals indicates a significant difference, there are two pairs of eigenvalues in the Rounds et al., correlation matrix having overlapping confidence intervals. Note that the criterion of nonoverlapping 95% confidence intervals leads to a very conservative test because a $Z = 1.96$ is demanded around each eigenvalue rather than the difference of the eigenvalues. I chose the more conservative 95% confidence interval in part to deal with the problem of multiple confidence intervals. Further research will have to be done on the appropriate test. Note that under the conservative heuristic test of nonoverlapping 95% confidence intervals, the correlation matrix for the male subjects in the Rounds et al., (1992) matrix does satisfy the requirements for the perfect circumplex pattern.

Even though I sympathize with Tracey and Rounds' general critique that circumplex models tend to be applied with little statistical testing, I think that better descriptive diagnostics are needed. A more detailed analysis of the constraints imposed by circumplexes on the correlation matrix should be undertaken. Tracey and Rounds (this issue) report only the first four eigenvalues for their three circumplexes so it is not possible to examine whether their data satisfy these criteria for a perfect circumplex. I discussed three constraints

(the rows must sum to a constant, that constant must equal one of the eigenvalues, and subsets of some eigenvalues must be identical), but additional analysis might uncover additional testable and diagnostic properties.

Further, I wonder whether their claim that the combination of three circumplexes leads to a sphere is correct. A one-dimensional solution is associated with a correlation matrix having a particular pattern (simplex), and a circular arrangement in a two-dimensional solution is associated with a correlation matrix having a particular pattern (correlations decreasing away from the diagonal and then increasing). It seems to me that a three-dimensional structure should have its own "signature pattern" in the correlation matrix. To my knowledge the diagnostic properties of a sphere have not yet been proposed in the literature, but uncovering diagnostic properties would be helpful in evaluating applications of this representational space.

Much progress has been made in the past 30 years to improve our understanding of circumplex models. Joreskog (1978) extended Guttman's (1954) original analysis, and researchers have continued extending the techniques and solving nagging problems such as permitting the analysis to handle negative correlations and developing focused tests of the predicted patterns (e.g., Browne, 1992; Cudeck, 1986; Hubert & Arabie, 1987). I think research in that area should continue. Nevertheless, applied researchers should think carefully before using new techniques. From an applied perspective, the measure of a new tool for data analysis should be whether the new technique helps to advance intuition and theory development in ways that were not possible given the more widely used techniques.

I am pessimistic about the utility of using circumplex models merely to describe data patterns. Although we can describe data patterns that are circular, or even spherical, we must to keep in mind that describing a data pattern is all we are doing. Anything deeper would probably involve a change in coordinates to a representational space that provides a natural language for discussing the phenomenon under study. In the present instance, I believe that theory in interest measurement could benefit by a careful look at how representational spaces are used, whether a different representational space might lead to better theory development, and an examination of the intuitions and predictions would fall out of studying new representational spaces.

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