
Physics 523, General Relativity

Homework 1

Due Wednesday, 27th September 2006

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Problem 1

a) We are to use the spacetime diagram of an observer $\mathscr{O}$ to describe an 'experiment' specified by the problem 1.5 in Schutz' text.

We have shown the spacetime diagram in Figure 1 below.

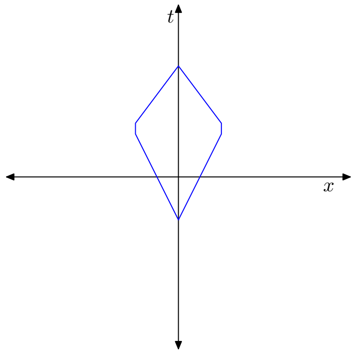

Figure 1. A spacetime diagram representing the experiment which was required to be described in Problem 1.a.

b) The experimenter observes that the two particles arrive back at the same point in spacetime after leaving from equidistant sources. The experimenter argues that this implies that they were released 'simultaneously;' comment.

In his frame, his reasoning is just, and implies that *his* t-coordinates of the two events have the same value. However, there is no absolute simultaneity in spacetime, so a different observer would be free to say that in *her frame*, the two events were not simultaneous.

c) A second observer $\overline{\mathscr{O}}$ moves with speed $v = 3c/4$ in the *negative* x-direction relative to $\mathscr{O}$. We are asked to draw the corresponding spacetime diagram of the experiment in this frame and comment on simultaneity.

Calculating the transformation by hand (so the image is accurate), the experiment observed in frame $\overline{\mathscr{O}}$ is shown in Figure 2. Notice that observer $\overline{\mathscr{O}}$ does not see the two emission events as occurring simultaneously.

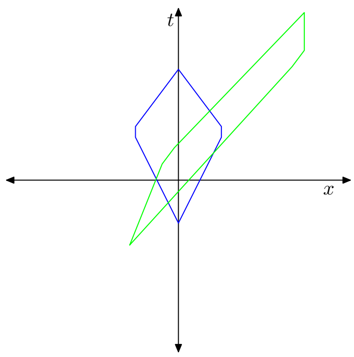

Figure 2. A spacetime diagram representing the experiment in two different frames. The worldlines in blue represent those recorded by observer $\mathscr{O}$ and those in green represent the event as recorded by an observer in frame $\overline{\mathscr{O}}$. Notice that there is obvious 'length contraction' in the negative x-direction and time dilation as well.

d) We are to show that the invariant interval between the two emission events is invariant.

We can proceed directly. It is necessary to know that in frame $\mathscr{O}$ the events have coordinates $p_1 = (5/2, -2)$ and $p_2 = (5/2, 2)$ while in frame $\overline{\mathscr{O}}$ they have coordinates $\overline{p}_1 = \gamma(1, -1/8)$ and $\overline{p}_2 = \gamma(4, 31/8)$ where $\gamma^2 = \frac{16}{7}$.

$$\Delta s^2 = (p_1 - p_2)^2 = 16;$$
$$\Delta \overline{s}^2 = (\overline{p}_1 - \overline{p}_2)^2 = \gamma^2 (-9 + 16) = 16.$$

We see that the invariant interval is indeed invariant in this pointless example.

Problem 2.

a) We are to show that rapidity is additive upon successive boosts in the same direction.

We may as well introduce the notation used in the problem: let $v = \tanh\alpha$ and $w = \tanh\beta$; this allows us to write $\gamma = \frac{1}{\sqrt{1-\tanh^2\alpha}} = \cosh\alpha$ and $v\gamma = \sinh\alpha$, and similar expressions apply for β. We see that using this language, the boost transformations are realized by the matrices[1]

$$\begin{pmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{pmatrix} \mapsto \begin{pmatrix} \cosh\alpha & -\sinh a \\ -\sinh\alpha & \cosh\alpha \end{pmatrix}, \tag{a.1}$$

and similarly for the boost with velocity w. Two successive boosts are then composed by[2]:

$$\begin{aligned}\begin{pmatrix} \cosh\beta & -\sinh\beta \\ -\sinh\beta & \cosh\beta \end{pmatrix}\begin{pmatrix} \cosh\alpha & -\sinh\alpha \\ -\sinh\alpha & \cosh\alpha \end{pmatrix} &= \begin{pmatrix} \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta & -\sinh\alpha\cosh\beta - \cosh\alpha\sinh\beta \\ -\sinh\alpha\cosh\beta - \cosh\alpha\sinh\beta & \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta \end{pmatrix} \\ &= \begin{pmatrix} \cosh(\alpha+\beta) & -\sinh(\alpha+\beta) \\ -\sinh(\alpha+\beta) & \cosh(\alpha+\beta) \end{pmatrix}\end{aligned}$$

This matrix is itself a boost matrix, now parameterized by a rapidity parameter $(\alpha + \beta)$. Therefore, successive boosts are additive for rapidity.

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b) Consider a star which observes a second star receding at speed $9c/10$; this star measures a third moving in the same direction, receding with the same relative speed; this build up continues consecutively N times. What is the velocity of the N^{th} star relative to the first? Give the explicit result for all N.

From the additivity of the rapidity, we see immediately that $\eta' = N\eta$ where $\eta = \operatorname{arctanh}(9/10)$ and η' is the rapidity of the resulting velocity. That is $\eta' = \operatorname{arctanh}(\beta)$ where β is the recession velocity of the N^{th} star relative to the first.

Recall a nice identity easily obtainable from the canonical definitions of $\tanh(x)$:

$$\operatorname{arctanh}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right). \tag{a.2}$$

Therefore, we have that

$$\begin{aligned}\log\left(\frac{1+\beta}{1-\beta}\right) &= N\log\left(\frac{1+\frac{9}{10}}{1-\frac{9}{10}}\right), \\ &= \log\left(19^N\right), \\ \therefore \frac{1+\beta}{1-\beta} &= 19^N,\end{aligned}$$

$$\boxed{\therefore \beta = \frac{19^N - 1}{1 + 19^N}.} \tag{a.3}$$

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[1]Because we are considering successive boosts in one direction, the problem really lives in $1+1$-dimensions and we can make life easier with only 2×2 matrices.

[2]We make use of some obvious identities in hyperbolic geometry.

Problem 3.

a) Consider a boost in the x-direction with speed $v_A = \tanh\alpha$ followed by a boost in the y-direction with speed $v_B = \tanh\beta$. We are to show that the resulting Lorentz transformation can be written as a pure rotation followed by a pure boost and determine the rotation and boost.

This is a $2+1$-dimensional problem—the entire problem involves only the $SO(2,1)$ subgroup of the Lorentz group. Now, although there must certainly be easy ways of solving this problem without setting up a system of equations and using trigonometric identities, we will stick with the obvious answer/easy math route—indeed, the algebra is not that daunting and the equations are easily solved.

The brute-force technique involves writing out the general matrices for both operations and (consistently) matching terms. The two successive boosts result in

$$\begin{pmatrix} \cosh\beta & 0 & -\sinh\beta \\ 0 & 1 & 0 \\ -\sinh\beta & 0 & \cosh\beta \end{pmatrix}\begin{pmatrix} \cosh\alpha & -\sinh\beta & 0 \\ -\sinh\alpha & \cosh\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh\alpha\cosh\beta & -\sinh\alpha\cosh\beta & -\sinh\beta \\ -\sinh\alpha & \cosh\alpha & 0 \\ -\cosh\alpha\sinh\beta & \sinh\alpha\sinh\beta & \cosh\beta \end{pmatrix} \tag{a.4}$$

And a rotation about $\hat{z}$ through the angle θ followed by a boost in the $(\cos\lambda, \sin\lambda)$-direction with rapidity η is given by[3]

$$\begin{pmatrix} \cosh\eta & -\sinh\eta\cos\lambda & -\sinh\eta\sin\lambda \\ -\sinh\eta\cos\lambda & 1+\cos^2\lambda(\cosh\eta-1) & \cos\lambda\sin\lambda(\cosh\eta-1) \\ -\sinh\eta\sin\lambda & \cos\lambda\sin\lambda(\cosh\eta-1) & 1+\sin^2\lambda(\cosh\eta-1) \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\eta & -\sinh\eta\,(\cos\lambda\cos\theta+\sin\lambda\sin\theta) & -\sinh\eta\,(\sin\lambda\cos\theta-\cos\lambda\sin\theta) \\ -\sinh\eta\cos\lambda & \cos\theta+(\cosh\eta-1)(\cos^2\lambda\cos\theta+\cos\lambda\sin\lambda\sin\theta) & -\sin\theta+(\cosh\eta-1)(\cos\lambda\sin\lambda\cos\theta-\cos^2\lambda\sin\theta) \\ -\sinh\eta\sin\lambda & \sin\theta+(\cosh\eta-1)(\sin^2\lambda\sin\theta+\cos\lambda\sin\lambda\cos\theta) & \cos\theta+(\cosh\eta-1)(\sin^2\lambda\cos\theta-\cos\lambda\sin\lambda\sin\theta) \end{pmatrix}$$

The system is over-constrained, and it is not hard to find the solutions. For example, the (00)-entry in both transformation matrices must match,

$$\boxed{\therefore \cosh\eta = \cosh\alpha\cosh\alpha.} \tag{a.5}$$

Looking at the (10) and (20) entries in each box, we see that

$$\sinh\eta\cos\lambda = \sinh\alpha;$$
$$\sin\eta\sin\lambda = \cosh\alpha\sinh\beta;$$

which together imply

$$\boxed{\therefore \tan\lambda = \frac{\sinh\beta}{\tanh\alpha}.} \tag{a.6}$$

Lastly, we must find θ; this can be achieved via the equation matching for the (12) entry:

$$\sin\theta = (\cosh\eta-1)\left(\cos\lambda\sin\lambda\cos\theta-\cos^2\lambda\sin\theta\right);$$
$$\implies \tan\theta\left(1+\cos^2\lambda\,(\cosh\eta-1)\right) = (\cosh\eta-1)\cos\lambda\sin\lambda,$$

$$\boxed{\therefore \tan\theta = \frac{(\cosh\eta-1)\cos\lambda\sin\lambda}{1+\cos^2\lambda\cosh\eta-\cos^2\lambda}.} \tag{a.7}$$

b) A spaceship A moves with velocity v_A along $\hat{x}$ relative to $\mathscr{O}$ and another, B, moves with speed v_B along $\hat{y}$ relative to A. Determine the direction and velocity of the frame $\mathscr{O}$ relative to B.

To map this exactly to the previous problem, we do things backwards and transform $B \to A$ followed by $A \to \mathscr{O}$ relative to B. That is, let $\tanh\alpha = -v_B$ and $\tanh\beta = v_A$. Now, the magnitude of the velocity of frame $\mathscr{O}$ relative to B has rapidity given by equation (a.5), and is moving in the direction an angle $\pi-(\theta+\lambda)$ relative to A where λ and θ are given by equations (a.6) and (a.7), respectively.

[3]This required a bit of algebra, but it isn't worth doing in public.

Physics 523, General Relativity

Homework 2

Due Monday, 9th October 2006

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Problem 1

Let frame $\overline{\mathscr{O}}$ move with speed v in the x-direction relative to frame $\mathscr{O}$. A photon with frequency ν measured in $\mathscr{O}$ moves at an angle θ relative to the x-axis.

a) We are to determine the frequency of the photon in $\overline{\mathscr{O}}$'s frame.

From the set up we know that the momentum of the photon in $\mathscr{O}$ is[1] $(E, E\cos\theta, E\sin\theta)$—that this momentum is null is manifest. The energy of the photon is of course $E = h\nu$ where h is Planck's constant and ν is the frequency in $\mathscr{O}$'s frame.

Using the canonical Lorentz boost equation, the energy measured in frame $\overline{\mathscr{O}}$ is given by

$$\begin{aligned}\overline{E} &= E\gamma - E\cos\theta v\gamma,\\ &= h\nu\gamma - h\nu\cos\theta v\gamma.\end{aligned}$$

But $\overline{E} = h\overline{\nu}$, so we see

$$\boxed{\therefore \frac{\overline{\nu}}{\nu} = \gamma\left(1 - v\cos\theta\right).} \tag{a.1}$$

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b) We are to find the angle θ at which there is no Doppler shift observed.

All we need to do is find when $\overline{\nu}/\nu = 1 = \gamma\left(1 - v\cos\theta\right)$. Every five-year-old should be able to invert this to find that the angle at which no Doppler shift is observed is given by

$$\boxed{\therefore \cos\theta = \frac{1}{v}\left(1 - \sqrt{1 - v^2}\right).} \tag{b.1}$$

Notice that this implies that an observer moving close to the speed of light relative to the cosmic microwave background[2] will see a narrow 'tunnel' ahead of highly blue-shifted photons and large red-shifting outside this tunnel. As the relative velocity increases, the 'tunnel' of blue-shifted photons gets narrower and narrower.

c) We are asked to compute the result in part **a** above using the technique used above.

This was completed already. We made use of Schutz's equation (2.35) when we wrote the four-momentum of the photon in a manifestly light-like form, and we made use of Schutz's equation (2.38) when used the fact that $E = h\nu$.

[1] We have aligned the axes so that the photon is travelling in the xy-plane. This is clearly a choice we are free to make.

[2] The rest frame of the CMB is defined to be that for which the CMB is mostly isotropic—specifically, the relative velocity at which no dipole mode is observed in the CMB power spectrum.

Problem 2

Consider a very high energy cosmic ray proton, with energy $10^9 m_p = 10^{18}$ eV as measured in the Sun's rest frame, scattering off of a cosmic microwave background photon with energy 2×10^{-4}eV. We are to use the Compton scattering formula to determine the maximum energy of the scattered photon.

We can guide out analysis by some simple heuristic heuristic. First of all, we are going to be interested in high momentum transfer interactions. In the proton rest frame we know from e.g. the Compton scattering formula that the hardest type of scattering occurs when the photon is fully 'reflected' with a scattering angle of $\theta = \pi$; this is also what we would expect from classical physics[3].

Now, imagine the proton travelling toward an observer at rest in the solar frame; any photons that scatter off the proton, ignoring their origin for the moment, will be blue-shifted (enormously) like a star would be, but only in the very forward direction of the proton. This means that the most energetic photons seen by an observer in the solar rest frame will be coming from those 'hard scatters' for which the final state photon travels parallel to the proton. Combining these two observations, we expect the most energetic scattering process will be that for which the photon and proton collide 'head-on' in the proton rest frame such that the momentum direction of the incoming photon is opposite to the incoming momentum of the proton in the solar frame.

We are now ready to verify this intuition and compute the maximum energy of the scattered photon. Before we start, it will be helpful to clear up some notation. We will work by translating between the two relevant frames in the problem, the proton rest frame and the solar rest frame. We may without loss of generality suppose that the proton is travelling in the positive x-direction with velocity v—with $\gamma = \left(1 - v^2\right)^{-1/2}$—in the solar frame. Also in the solar frame, we suppose there is some photon with energy $E_\gamma^i = 2 \times 10^{-4}$ eV. This is the photon which we suppose to scatter off the proton.

The incoming photon's energy in the proton's rest frame we will denote $\overline{E}_\gamma^i$; *in the proton frame*, we say that the angle between the photon's momentum and the positive x-axis is $\overline{\theta}$. After the photon scatters, it will be travelling at an angle $\overline{\theta} - \overline{\varphi}$ relative to the x-axis, where $\overline{\varphi}$ is the angle between the incoming and outgoing photon in the proton's rest frame. This outgoing photon will have energy denoted $\overline{E}_\gamma^f$. We can then boost this momentum back to the solar rest frame where its energy will be denoted E_γ^f.

From our work in problem 1 above, we know how to transform the energy of a photon between two frames with relative motion not parallel to the photon's direction. Let us begin our analysis by considering a photon in the proton's rest frame and determine what energy that photon had in the solar rest frame. Boosting along the $(-x)$-direction from the proton frame, we see that

$$E_\gamma^i = \overline{E}_\gamma^i \gamma \left(1 + v \cos\overline{\theta}\right) \quad \Longrightarrow \quad \overline{E}_\gamma^i = \frac{E_\gamma^i}{\gamma\left(1 + v\cos\overline{\theta}\right)}. \tag{a.1}$$

We can relate the energy and scattering angle of the final-state photon in the proton rest frame using the Compton formula. Indeed, we see that

$$\overline{E}_\gamma^f = \frac{\overline{E}_\gamma^i m_p}{m_p + \overline{E}_\gamma^i\left(1 - \cos\overline{\varphi}\right)}, \tag{a.2}$$

where $\overline{\varphi}$ is the scattering angle in this frame.

Finally, we need to reverse-boost the outgoing photon from the proton frame to the solar frame. Here, it is necessary to note that the relative angle between the outgoing

[3] In the proton rest frame, however, this process does not look like what we're after: this process minimizes the out-state photon's energy in that frame. Nevertheless, it is the hardest type of scattering available—any other collision transfers less momentum between the proton and the photon.

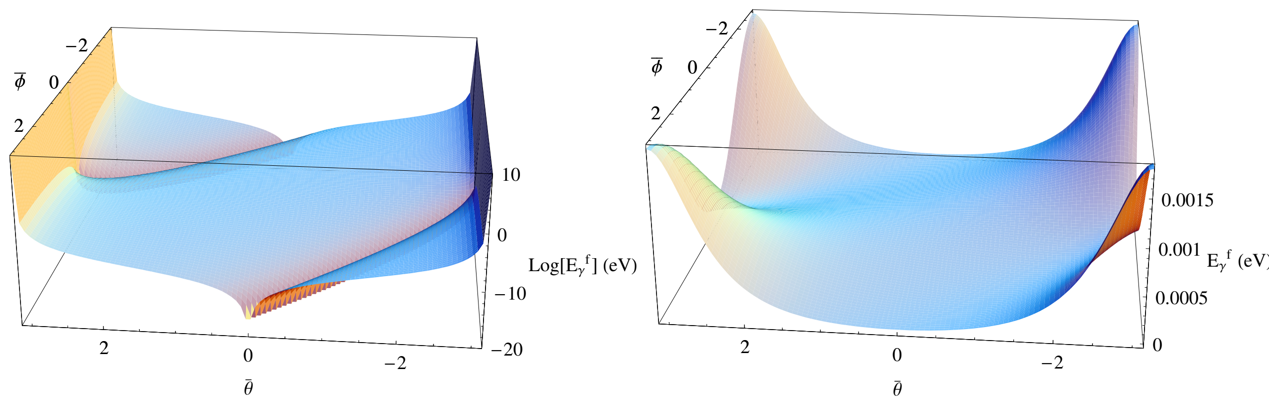

FIGURE 1. These graphs indicate the scattering energy of an incoming 2×10^{-4} eV photon, as measured in the solar frame, as a function of the incoming and outgoing angles $\overline{\theta}$ and $\overline{\varphi}$ as measured in the proton rest frame. The plot on the left is for the situation presently under investigation, where the proton energy is $10^9 m_p$; because of the extremely wide-range of out-going photon energies, this is plotted on a log-scale. On the right is a simpler example where the cosmic ray proton is travelling only semi-relativistically with velocity $v = 4c/5$. In both cases it is clear that the maximal energy observed for scattering takes place when $\overline{\varphi} = \overline{\theta} = \pi$.

photon and the x-axis is now $\overline{\theta} - \overline{\varphi}$. Recalling that we are boosting in the $(-x)$-direction again, we see

$$E_\gamma^f = \overline{E}_\gamma^f \gamma \left\{1 + v\cos\left(\overline{\theta} - \overline{\varphi}\right)\right\}. \tag{a.3}$$

Putting all these together, we see that

$$\begin{aligned} E_\gamma^f &= \overline{E}_\gamma^f \gamma \left\{1 + v\cos\left(\overline{\theta} - \overline{\varphi}\right)\right\}, \\ &= \frac{\overline{E}_\gamma^i m_p}{m_p + \overline{E}_\gamma^i\left(1 - \cos\overline{\varphi}\right)} \gamma \left\{1 + v\cos\left(\overline{\theta} - \overline{\varphi}\right)\right\}, \\ &= \frac{E_\gamma^i m_p}{m_p + E_\gamma^i \frac{\left(1-\cos\overline{\varphi}\right)}{\gamma\left(1+v\cos\overline{\theta}\right)}} \frac{\gamma\left\{1 + v\cos\left(\overline{\theta} - \overline{\varphi}\right)\right\}}{\gamma\left(1 + v\cos\overline{\theta}\right)}, \end{aligned}$$

$$\boxed{\therefore E_\gamma^f = \frac{E_\gamma^i m_p \left\{1 + v\cos\left(\overline{\theta} - \overline{\varphi}\right)\right\}}{m_p\left(1 + v\cos\overline{\theta}\right) + E_\gamma^i \frac{1-\cos\overline{\varphi}}{\gamma}}.} \tag{a.4}$$

The function above is plotted in Figure 1 along with the analogous result for a less-energetic cosmic ray proton.

At any rate, it is clear from the plot or a simple analysis of the second derivatives of E_γ^f that the global maximum is precisely at $\overline{\theta} = \overline{\varphi} = \pi$. This is exactly what we had anticipated—when the collision is head-on and the photon is scattered at an angle π. We can use this to strongly simplify the above equation (a.4),

$$\max\left\{E_\gamma^f\right\} = \frac{E_\gamma^i m_p\left(1 + v\right)}{m_p\left(1 - v\right) + 2E_\gamma^i\sqrt{1 - v^2}}. \tag{a.5}$$

To actually compute the maximum energy allowed, we will need to put in numbers. We know that the energy of the proton is $10^9 m_p = \gamma m_p$ so $\gamma = 10^9$. This is easily translated into a velocity of approximately $1 - 5 \times 10^{-19}$. Knowing that the mass of a proton is roughly 10^9 eV, the first term in the denominator of equation (a.5) is $\mathcal{O} \sim 10^{-10}$ whereas the second term is $\mathcal{O} \sim \times 10^{-13}$, so to about a 1 percent accuracy (which is better than our proton mass figure anyway), we can approximate equation (a.5) as

$$\max\left\{E_\gamma^f\right\} \simeq E_\gamma^i \frac{1+v}{1-v} \approx \frac{2E_\gamma^i}{1-v} = 8 \times 10^{14} \text{ eV}. \tag{a.6}$$

Therefore, the maximum energy of a scattered CMB photon from a 10^{18} eV cosmic ray proton is about 400 TeV—much higher than collider-scale physics. However, the rate of these types of hard-scatters is *enormously* low. Indeed, recalling the picture of a narrowing tunnel of blue-shift at high boost, we can use our work from problem 1 to see that only photons within a $0.0025°$ cone about the direction of motion of the proton are blue-shifted at all—and these are the only ones that can gain any meaningful energy from the collision. This amounts to a phase-space suppression of around 10^{-10} even before we start looking at the small rate and low densities involved.

Problem 3

Consider the coordinates $u = t - x$ and $v = t + x$ in Minkowski spacetime.

a) We are to define a u, v, y, z-coordinate system with the origin located at $\{u = 0, v = 0, y = 0, z = 0\}$ with the basis vector $\vec{e}_u$ connecting between the origin and the point $\{u = 1, v = 0, y = 0, z = 0\}$ and similarly for $\vec{e}_v$. We are to relate these basis vectors to those in the normal Minkowski frame, and draw them on a spacetime plot in t, x-coordinates.

We can easily invert the defining equations $u = t - x$ and $v = t + x$ to find

$$t = \frac{u+v}{2} \qquad \text{and} \qquad x = \frac{v-u}{2}. \tag{a.1}$$

Therefore, the the origin in u, v-coordinates is also the origin in tx-space. Also, the point where $u = 1, v = 0$ which defines $\vec{e}_u$ has coordinates $t = \frac{1}{2}, x = -\frac{1}{2}$ in tx-space; the point $u = 0, v = 1$ corresponds to $t = \frac{1}{2}, x = \frac{1}{2}$ so that

$$\boxed{\vec{e}_u = \frac{\vec{e}_t - \vec{e}_x}{2} \qquad \text{and} \qquad \vec{e}_v = \frac{\vec{e}_t + \vec{e}_x}{2}.} \tag{a.2}$$

These basis vectors are labeled on Figure 2.

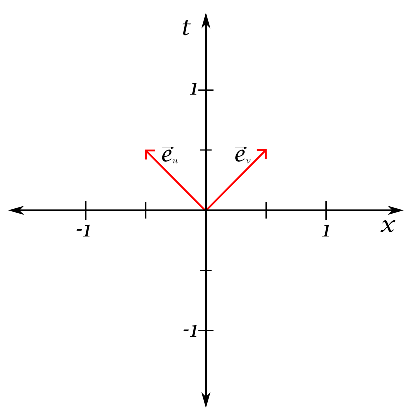

FIGURE 2. Figure required for problem 3 which shows the vectors $\vec{e}_u$ and $\vec{e}_v$ on the tx-plane.

b) We are to show that $\{\vec{e}_u, \vec{e}_v, \vec{e}_y, \vec{e}_z\}$ span all of Minkowski space.

Because the map (a.2) is a bijection, the linear independence of $\vec{e}_t$ and $\vec{e}_x$ implies linear independence of $\vec{e}_u$ and $\vec{e}_v$. And because these are manifestly linearly independent of $\vec{e}_y$ and $\vec{e}_z$, the four vectors combine to form a linearly-independent set—which is to say that they span all of space.

c) We are to find the components of the metric tensor in this basis.

The components of the metric tensor in any basis $\{\vec{e}_i\}$ is given by the matrix $\widetilde{g}_{ij} = g(\vec{e}_i, \vec{e}_j)$ where $g(\cdot,\cdot)$ is the metric on spacetime. Because we have equation (a.2) which relates $\vec{e}_u$ and $\vec{e}_v$ to the tx-bases, we can compute all the relevant inner products using the canonical Minkowski metric. Indeed we find,

$$\widetilde{g}_{ij} = \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{c.1}$$

d) We are to show that $\vec{e}_u$ and $\vec{e}_v$ are null but they are not orthogonal.

In part **c** above we needed to compute the inner products of all the basis vectors, including $\vec{e}_u$ and $\vec{e}_v$. There we found that $g(\vec{e}_u, \vec{e}_v) = 1/2$, so $\vec{e}_u$ and $\vec{e}_v$ are not orthogonal. However, $g(\vec{e}_u, \vec{e}_u) = g(\vec{e}_u, \vec{e}_v) = 0$, so they are both null.

e) We are to compute the one-forms $\mathbf{d}u$, $\mathbf{d}v$, $g(\vec{e}_u, \cdot)$, and $g(\vec{e}_v, \cdot)$.

As scalar functions on spacetime, it is easy to compute the exterior derivatives of u and v. Indeed, using their respective definitions, we find immediately that

$$\mathbf{d}u = dt - dx \qquad \text{and} \qquad \mathbf{d}v = dt + dx. \tag{e.1}$$

The only difference that arises when computing $g(\vec{e}_u, \cdot)$, for example, is that the *components* of $\vec{e}_u$ are given in terms of the basis vectors $\vec{e}_t$ and $\vec{e}_x$ as in equation (a.2). Therefore in the usual Minkowski component notation, we have $\vec{e}_u = (\frac{1}{2}, -\frac{1}{2}, 0, 0)$ and $\vec{e}_v = (\frac{1}{2}, \frac{1}{2}, 0, 0)$. Using our standard Minkowski metric we see that

$$g(\vec{e}_u, \cdot) = -\frac{1}{2}dt - \frac{1}{2}dx \qquad \text{and} \qquad g(\vec{e}_v, \cdot) = -\frac{1}{2}dt + \frac{1}{2}dx. \tag{e.2}$$

Problem 4

We are to give an example of four linearly independent null vectors in Minkowski space and show why it is not possible to make them all mutually orthogonal.

An easy example that comes to mind uses the coordinates $\{x_-, x_+, y_+, z_+\}$ given by

$$x_- = t - x \qquad x_+ = t + = x \qquad y_+ = t + y \qquad z_+ = t + z. \tag{a.1}$$

In case it is desirable to be condescendingly specific, this corresponds to taking basis vectors $\{\vec{e}_{x-}, \vec{e}_{x+}, \vec{e}_{y+}, \vec{e}_{z+}\}$ where

$$\vec{e}_{x-} = \frac{\vec{e}_t - \vec{e}_x}{2} \qquad \vec{e}_{x+} = \frac{\vec{e}_t + \vec{e}_x}{2} \qquad \vec{e}_{y+} = \frac{\vec{e}_t + \vec{e}_y}{2} \qquad \vec{e}_{z+} = \frac{\vec{e}_t + \vec{e}_z}{2}. \tag{a.2}$$

It is quite obvious that each of these vectors is null, and because they are related to the original basis by an invertible map they still span the space. Again, to be specific[4], notice that $\vec{e}_t = \vec{e}_{x-} + \vec{e}_{x+}$ and so we may invert the other expressions by $\vec{e}_i = 2\vec{e}_{i+} - \vec{e}_t$ where $i = x, y, z$.

Let us now show that four linearly independent, null vectors cannot be simultaneously mutually orthogonal. We proceed via *reductio ad absurdum*: suppose that the set $\{\vec{v}_i\}_{i=1,\ldots 4}$ were such linearly independent, mutually orthogonal and null. Because they are linearly independent, they can be used to define a basis which has an associated metric, say $\widetilde{g}$. Now, as a matrix the entries of $\widetilde{g}$ are given by $\widetilde{g}_{ij} = \widetilde{g}(\vec{v}_i, \vec{v}_j)$; because all the vectors are assumed to be orthogonal and null, all the entries of $\widetilde{g}$ are

[4] Do you, ye grader, actually care for me to be this annoyingly specific?

zero. This means that it has zero positive eigenvalues and zero negative eigenvalues—which implies signature($\widetilde{g}$) = 0. But the signature of Minkowski spacetime must be ± 2 [5], and this is basis-independent.

To go one step further, the above argument actually implies that no null vector can be simultaneously orthogonal to and linearly independent of any three vectors.

Problem 5

The frame $\overline{\mathscr{O}}$ moves relative to $\mathscr{O}$ with speed v in the z-direction.

a) We are to use the fact that the Abelian gauge theory field strength $F_{\mu\nu}$ is a tensor to express the electric and magnetic field components measured in $\overline{\mathscr{O}}$ in terms of the components measured in $\mathscr{O}$.

To determine the components of the field strength measured in frame $\overline{\mathscr{O}}$ in terms of the components of frame $\mathscr{O}$, all we need to do is apply a Lorentz transformation for each of the two indices in F_{ab}:

$$F_{\overline{a}\overline{b}} = \Lambda_{\overline{a}}{}^{c}\Lambda_{\overline{b}}{}^{d}F_{cd}. \tag{a.1}$$

Using some of our work in class to identify the components of F_{cd}, we may write the above expression in matrix notation[6] as

$$F_{\overline{a}\overline{b}} = \begin{pmatrix} \gamma & 0 & 0 & v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ v\gamma & 0 & 0 & \gamma \end{pmatrix} \cdot \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \cdot \begin{pmatrix} \gamma & 0 & 0 & v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ v\gamma & 0 & 0 & \gamma \end{pmatrix},$$

$$= \begin{pmatrix} 0 & \gamma(E_x - vB_y) & \gamma(E_y + vB_x) & \gamma^2 E_z(1-v^2) \\ -\gamma(E_x - vB_y) & 0 & -B_z & -\gamma(vE_x - B_y) \\ -\gamma(E_y + vB_x) & B_z & 0 & -\gamma(vE_y + B_x) \\ \gamma^2 E_z(v^2 - 1) & \gamma(vE_x - B_y) & \gamma(vE_y + B_x) & 0 \end{pmatrix}.$$

Using the fact that $\gamma^2(1 - v^2) = 1$, we see that these imply

$$\overline{E_x} = \gamma\left(E_x - vB_y\right) \qquad \overline{E_y} = \gamma\left(E_y + vB_x\right) \qquad \overline{E_z} = E_z, \tag{a.2}$$

$$\overline{B_x} = \gamma\left(B_x + vE_y\right) \qquad \overline{B_y} = \gamma\left(B_y - vE_x\right) \qquad \overline{B_z} = B_z. \tag{a.3}$$

b) Say a particle of mass m and charge q is subjected to some electromagnetic fields. The particle is initially at rest in $\overline{\mathscr{O}}$'s frame. We are to calculate the components of its four-acceleration as measured in $\overline{\mathscr{O}}$ at that moment, transform these components into those measured in $\overline{\mathscr{O}}$ and compare them with the equation for the particle's acceleration directly in $\mathscr{O}$'s frame.

We will use the fact that the four-acceleration is given by

$$\frac{dU^a}{d\tau} = \frac{q}{m}F^a{}_b U^b, \tag{a.1}$$

where U^a is the four-velocity and τ is some affine parameter along the particle's world-line. Now, the above equation works in any reference frame—we can substitute indices with bars over them if we'd like. Because the particle is initially at rest in frame $\overline{\mathscr{O}}$, it's four velocity is given by $U^{\overline{a}} = (-1, 0, 0, 0)$. Therefore we can easily

[5] The '±' depends on convention. Actually, if you use complexified space *or* complexified time (which is more common, but still unusual these days), then you could get away with signature ± 4.

[6] Here, as everywhere in every situation similar to this, $\gamma = \left(1 - v^2\right)^{-1/2}$ where v is the velocity in question; in this case v.

compute the four-acceleration in frame $\overline{\mathscr{O}}$ as follows:

$$\begin{aligned}\frac{dU^{\overline{a}}}{d\tau} &= \frac{q}{m}F^{a}{}_{b}U^{b} = \frac{q}{m}g^{\overline{a}\wp}F_{\wp\overline{b}}U^{b},\\ &= \frac{q}{m}\begin{pmatrix} 0 & -\overline{E_x} & -\overline{E_y} & -\overline{E_z}\\ -\overline{E_x} & 0 & -\overline{B_z} & \overline{B_y}\\ -\overline{E_y} & \overline{B_z} & 0 & -\overline{B_x}\\ -\overline{E_z} & -\overline{B_y} & \overline{B_x} & 0\end{pmatrix}\cdot\begin{pmatrix}-1\\0\\0\\0\end{pmatrix},\\ &= \frac{q}{m}\left(0,\overline{E_x},\overline{E_y},\overline{E_z}\right).\end{aligned}$$

This results shows us that sometimes relativity is terribly unnecessary—the result is completely obvious from a classical electrodynamics point of view.

To determine the components of the four-acceleration as viewed in frame $\mathscr{O}$, all we need to do is Lorentz transform the components of the four-acceleration back into $\mathscr{O}$ (because it is a vector). We find then that the four-acceleration in $\mathscr{O}$ is given by

$$\begin{aligned}\frac{dU^{a}}{d\tau} = \Lambda^{a}{}_{\overline{a}}\frac{dU^{\overline{a}}}{d\tau} &= \frac{q}{m}\begin{pmatrix}\gamma & 0 & 0 & v\gamma\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ v\gamma & 0 & 0 & \gamma\end{pmatrix}\begin{pmatrix}0\\ \overline{E_x}\\ \overline{E_y}\\ \overline{E_z}\end{pmatrix},\\ &= \frac{q}{m}\begin{pmatrix}v\gamma\overline{E_z}\\ \overline{E_z}\\ \overline{E_y}\\ \gamma\overline{E_z}\end{pmatrix},\end{aligned}$$

which upon substitution of the $\overline{\mathscr{O}}$ field components in terms of the $\mathscr{O}$ components, implies

$$\frac{dU^{a}}{d\tau} = \frac{q\gamma}{m}\begin{pmatrix}vE_z\\ E_x - vB_y\\ E_y + vB_x\\ E_z\end{pmatrix}. \tag{a.2}$$

Now, to compute this directly in frame $\mathscr{O}$, we need only transform the four-velocity vector $U^{\overline{a}}$ into U^{a},

$$U^{a} = \Lambda^{a}{}_{\overline{a}}U^{\overline{a}} = \begin{pmatrix}-\gamma\\0\\0\\-v\gamma\end{pmatrix}, \tag{a.3}$$

and use this in the expression for the four-acceleration for an Abelian field theory as quoted above. So we have

$$\frac{dU^{a}}{d\tau} = \frac{q}{m}F^{a}{}_{b}U^{b} = \frac{q}{m}\begin{pmatrix} 0 & -E_x & -E_y & -E_z\\ -E_x & 0 & -B_z & B_y\\ -E_y & B_z & 0 & -B_x\\ -E_z & -B_y & B_x & 0\end{pmatrix}\cdot\begin{pmatrix}-\gamma\\0\\0\\-v\gamma\end{pmatrix}, \tag{a.4}$$

$$\therefore \frac{dU^{a}}{d\tau} = \frac{q\gamma}{m}\begin{pmatrix}vE_z\\ E_x - vB_y\\ E_y + vB_x\\ E_z\end{pmatrix}. \tag{a.5}$$

Physics 523, General Relativity

Homework 3

Due Monday, 16th October 2006

Jacob Lewis Bourjaily

Problem 1

a) Let us consider the region of the $t-x$-plane which is bounded by the lines $t=0, t=1, x=0$, and $x=1$; we are to find the unit outward normal one-forms and their associated vectors for each of the boundary lines.

It is not hard to see that the unit outward normal one-forms and their associated vectors are given by

$$t=0: \quad -dt \mapsto \vec{e}_t \qquad\qquad t=1: \quad dt \mapsto -\vec{e}_t; \tag{1.a.1}$$

$$x=0: \quad -dx \mapsto -\vec{e}_x \qquad\qquad x=1: \quad dx \mapsto \vec{e}_x. \tag{1.a.2}$$

b) Let us now consider the triangular region bounded by events with coordinates $(1,0),(1,1)$, and $(2,1)$; we are to find the outward normal for the null boundary and its associated vector.

The equation for the null boundary of the region is $t=x+1$, which is specified by the vanishing of the function $t-x-1=0$. The normal to the surface is simply the gradient of this zero-form, and so the normal is

$$dt-dx, \tag{1.b.3}$$

and the associated vector is

$$-\vec{e}_t-\vec{e}_x. \tag{1.b.4}$$

Problem 2

We are to describe the (proper orthochronous) Lorentz-invariant quantities that can be built out of the electromagnetic field strength F_{ab} and express these invariant in terms of the electric and magnetic fields.

Basically, any full contraction of indices will result in a Lorentz-invariant quantity. Furthermore, because we are considering things which are invariant under only proper orthochronous transformation, we are free to consider CP-odd combinations, which mix up components with their Hodge-duals. A list of such invariants are:

$$F^a{}_a=0 \qquad F^{ab}F_{ab}=-\left(\star F^{ab}\right)\left(\star F_{ab}\right)=2\left(\vec{B}^2-\vec{E}^2\right) \qquad \left(\star F^{ab}\right)F_{ab}=-4\vec{E}\cdot\vec{B}. \tag{2.a.1}$$

This does not exhaust the list of invariants, however: we are also free to take a number of derivatives. These start getting rather horrendous, but we can start with an easy example:

$$\left(\partial_a F^{ab}\right)^2=16\pi^2 J^a J_a=16\pi^2\left(\vec{J}^2-\rho^2\right). \tag{2.a.2}$$

Along this vein, we find

$$\left(\partial_a F^{ab}\right)\left(\partial_c F^{cd}\right)F_{bd}=16\pi^2\vec{J}\cdot\left(\vec{B}\times\vec{J}\right); \tag{2.a.3}$$

$$\left[\left(\partial_a F^{ab}\right)F^{ab}\right]^2=16\pi^2\left\{\rho^2\vec{E}^2-\left(\vec{E}\cdot\vec{J}\right)^2+\left(\vec{B}\times\vec{J}\right)^2-2\rho\vec{E}\cdot\left(\vec{B}\times\vec{J}\right)\right\}; \tag{2.a.4}$$

$$\left(\partial_a F^{ab}\right)\left(\partial_c F^{cd}\right)\left(\star F_{bd}\right)=16\pi^2\vec{J}\cdot\left(\vec{E}\times\vec{J}\right). \tag{2.a.5}$$

We could go higher in derivatives, but we know that $\partial_a\star F^{ab}=0$ and we are free to make the Lorentz gauge choice $\Box F^{ab}=0$. I suspect that further combinations will not yield independent quantities.

Problem 3

Consider a pair of twins are born somewhere in spacetime. One of the twins decides to explore the universe; she leaves her twin brother behind and begins to travel in the x-direction with constant acceleration $a = 10$ m/s^2 as measured in her rocket frame. After ten years according to her watch, she reverses the thrusters and begins to accelerate with a constant $-a$ for a while.

a) At what time on her watch should she again reverse her thrusters so she ends up home at rest?

There is an obvious symmetry in this problem: if it took her 10 years by her watch to go from rest to her present state, then 10 years of reverse acceleration will bring her to rest, at her farthest point from home. Because of the constant negative acceleration, after reaching her destination at 20 years, she will begin to accelerate towards home again. In 10 more years, when her watch reads 30 years, she will be in the same state as when her watch read 10 years, only going in the opposite direction.

Therefore, at 30 years, she should reverse her thrusters again so she arrives home in her home's rest frame.

b) According to her twin brother left behind, what was the most distant point on her trip?

To do this, we need only to solve the equations for the travelling twin's position and time as seen in the stationary twin's frame. This was largely done in class but, in brief, we know that her four-acceleration is normal to her velocity: $a^\xi u_\xi = 0$ everywhere along her trip, and $a^\xi a_\xi = a^2$ is constant. This leads us to conclude that

$$a^t = \frac{du^t}{d\tau} = au^x \qquad \text{and} \qquad a^x = \frac{du^x}{d\tau} = au^t, \tag{3.b.1}$$

where τ is the proper time as observed by the travelling twin. This system is quickly solved for an appropriate choice of origin[1]:

$$t = \frac{1}{a}\sinh(a\tau) \qquad \text{and} \qquad x = \frac{1}{a}\cosh(a\tau). \tag{3.b.2}$$

This is valid for the first quarter of the twin's trip—all four 'legs' can be given explicitly by gluing together segments built out of the above.

For the purposes of calculating, it is necessary to make $a\tau$ dimensionless. This is done simply by

$$a = \frac{10 \text{ m}}{\text{sec}^2} = 1.053 \text{ year}^{-1}. \tag{3.b.3}$$

An approximate result[2] could have been obtained by thinking of $c = 3 \times 10^8$ m/s and 3×10^7 sec $= 1$ year.

So the distance at 10 years is simply

$$x(10 \text{ yr}) = \frac{1}{1.053}\cosh(10.53) = 17710 \text{ light years.}$$

The maximum distance travelled by the twin as observed by her (long-deceased) brother is therefore twice this distance, or[3]

$$\max(x) = 35,420 \text{ light years.} \tag{3.b.4}$$

c) When the sister returns, who is older, and by how much?

Well, in the brother's rest frame, his sister's trip took four legs, each requiring

$$t(10 \text{ yr}) = \frac{1}{1.053}\sinh(10.53) = 17710 \text{ years,}$$

which means that

$$t_{\text{total}} = 70,838 \text{ years.} \tag{3.c.5}$$

In contrast, his sister's time was simply her proper time, or 40 years. Therefore the brother who stayed behind is now 70,798 years older than his twin sister.

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[1] We consider the twin to begin at $(t = 0, x = 1)$.

[2] Because cosh goes like an exponential for large argument, our result is exponentially sensitive to the figures; because we know c and the number of seconds per year to rather high-precision, there is no reason not to use the correct value of a—indeed, the approximate value of $a \sim 1$ year^{-1} gives an answer almost 40% below our answer.

[3] If we had used instead $a = 1$/year as encouraged by the problem set, our answer would have been 22,027 light years.

Problem 4[4]

Consider a star located at the origin in its rest frame $\mathscr{O}$ emitting a continuous flux of radiation, specified by luminosity L.

a) We are to determine the non-vanishing components of the stress-energy tensor as seen by an observer located a distance x from the star along the x-axis of the star's frame.

There are many ways to go about determining the components of the stress-energy tensor. We will be un-inspired and compute it directly from the equation for the Maxwell stress-energy tensor (found by looking at metric variations of the Maxwell action):

$$T^{ab} = F^{a}_{\ c}F^{bc} - \frac{1}{4}\eta^{ab}F_{ab}F^{ab}. \tag{4.a.1}$$

We have in previous exercises computed all of the necessary terms, so we may simply quote that

$$T^{00} = \vec{E}^2 + \frac{1}{2}\left(\vec{B}^2 - \vec{E}^2\right) = \frac{1}{2}\left(\vec{B}^2 + \vec{E}^2\right) = |\vec{S}|, \tag{4.a.2}$$

$$T^{0i} = \left(\vec{E}\times\vec{B}\right)^i = |\vec{S}|, \tag{4.a.3}$$

where $\vec{S}$ is the Poynting vector, whose magnitude is just the energy density flux. Now, when we expand T^{xx}, we find a bit more work in for us, at first glance, we see

$$T^{xx} = \frac{3}{2}E_x^2 - \frac{1}{2}B_x^2 + \frac{1}{2}\left(E_y^2 + E_z^2 + B_y^2 + B_z^2\right);$$

but we should note that because the radiation is only reaching the observer along the x-direction, $\vec{S}$ lies along the x-direction and so $B_x = E_x = 0$; therefore, we do indeed see that

$$T^{xx} = \frac{1}{2}\left(\vec{B}^2 + \vec{E}^2\right) = |\vec{S}|. \tag{4.a.4}$$

And making use of the fact that $\vec{S}$ only has components in the x-direction, we see that $T^{0y} = T^{0z} = 0$—with symmetrization implied.

Now, the energy density flux over a sphere centred about the origin of radius x naturally is $\frac{L}{4\pi x^2}$. Therefore, we see that

$$\therefore T^{00} = T^{x0} = T^{0x} = T^{xx} = \frac{L}{4\pi x^2}. \tag{4.a.5}$$

b) Let $\vec{X}$ be the null vector connecting the origin in $\mathscr{O}$ to event at which the radiation is measured. Let $\vec{U}$ be the velocity four-vector of the sun. We are to show that $\vec{X} \to (x, x, 0, 0)$ and that T^{ab} has the form

$$\mathbf{T} = \frac{L}{4\pi}\frac{\vec{X}\otimes\vec{X}}{\left(\vec{U}\cdot\vec{X}\right)^4}.$$

Well, it is intuitively obvious that if an observer sees radiation at $(x, x, 0, 0)$, that, because it is null and forward-propagating, it must have been emitted from a source along the line $\tau(1, 1, 0, 0)$ where τ is an affine parameter for the world line of the photon. If it is the case that the photon was emitted by the sun that is sitting at $x = 0$, then it must have been emitted at $(0, 0, 0, 0)$, which means that $\vec{X} \to (x, x, 0, 0)$.

Now, using the fact that $\vec{U} = (1, 0, 0, 0)$ for the star, we have that $\vec{U}\cdot\vec{X} = x$, and this is frame-independent. Now, we see that $\vec{X}\otimes\vec{X}$ only has components in (t, x)-directions

[4]This is the most poorly worded problem I have encountered thus far in this course. If there is any misunderstanding, I am strongly inclined to blame Schutz.

and furthermore all the coefficients are the same, namely x^2. Therefore

$$\frac{L}{4\pi}\frac{\vec{X}\otimes\vec{X}}{\left(\vec{U}\cdot\vec{X}\right)^4} = \frac{L}{4\pi x^2}\left(\vec{e}_t\otimes\vec{e}_t+\vec{e}_t\otimes\vec{e}_t+\vec{e}_x\otimes\vec{e}_t+\vec{e}_x\otimes\vec{e}_x\right). \tag{4.b.6}$$

Because this matches our explicit calculation in a certain frame and the expression is manifestly frame-independent we see that this is a valid expression for $\mathbf{T}$ in any reference frame[5].

c) Consider an observer $\overline{\mathscr{O}}$ travelling with speed v away from the star's frame $\mathscr{O}$ in the x-direction. In that frame, the observation of radiation is at $\vec{X}\to(R,R,0,0)$. We are to find R as a function of x and express $T^{\bar{0}\bar{x}}$ in terms of R.

There is no need to convert $\vec{U}$ of the sun into $\overline{\mathscr{O}}$'s coordinates because it only appears in $\mathbf{T}$ as a complete contraction—which is to say that $\vec{U}\cdot\vec{X}$ is frame independent. Now, all we need to do then is compute the coordinates of $\vec{X}$ in $\overline{\mathscr{O}}$'s coordinate system. This is done by a simple Lorentz transformation:

$$\vec{X}\underset{\overline{\mathscr{O}}}{\to}(x\gamma(1-v),x\gamma(1-v),0,0)\equiv(R,R,0,0), \tag{4.c.7}$$

which is to say, $R=x\gamma(1-v)$.

Bearing in mind that the numerator in the expression of $\mathbf{T}$ was invariant, we see that

$$T^{\bar{0}\bar{x}}=\frac{L}{4\pi}\frac{R^2}{x^4}. \tag{4.c.8}$$

Now, inverting our expression for R, we see that

$$x^2=\frac{R^2}{\gamma^2(1-v)^2}=R^2\left(\frac{1+v}{1-v}\right),$$

and so

$$\therefore T^{\bar{0}\bar{x}}=\frac{L}{4\pi R^2}\left(\frac{1-v}{1+v}\right)^2. \tag{4.c.9}$$

[5]Well, specifically, the difference between the T^{ab} calculated above and the coordinate-free tensor vanishes identically at x; this tensor identity is obviously frame independent and so the tensors are identical.

Physics 523, General Relativity

Homework 4

Due Wednesday, 25th October 2006

Jacob Lewis Bourjaily

Problem 1

Recall that the worldline of a continuously accelerated observer in flat space relative to some inertial frame can be described by

$$t(\lambda,\alpha)=\alpha\sinh(\lambda)\qquad\text{and}\qquad x(\lambda,\alpha)=\alpha\cosh(\lambda),\tag{1.a.1}$$

where λ is an affine parameter of the curve with $\alpha\lambda$ its proper length—i.e. the 'time' as measured by an observer in the accelerated frame. Before, we considered α to be constant and only varied λ. We are now going to consider the entire (non-surjective) curvilinear map from two-dimensional Minkowski to-space to itself defined by equation (1.a.1).

a) Consider the differential map from t, x-coordinate charts to λ, α-coordinate charts implied by equation (1.a.1)—lines of constant α are in the λ-direction, and lines of constant λ are in the α-direction. We are to show that wherever lines of constant α meet lines of constant λ, the two curves are orthogonal.

To show that the two curves cross 'orthogonally,' we must demonstrate that their tangent vectors are orthogonal at points of intersection. This is not particularly hard. Because orthogonality is a frame independent notion, we may as well compute this in t, x-space. The lines of constant λ parameterized by α are given by $\ell(\alpha)=(\alpha\sinh\lambda,\alpha,\cosh\lambda)$, which has the associated tangent vector

$$\vec{\ell}\equiv\frac{\partial\ell(\alpha)}{\partial\alpha}=(\sinh\lambda,\cosh\lambda).\tag{1.a.2}$$

Similarly, lines of constant α parameterized by λ are $\vartheta(\lambda)=(\alpha\sinh\lambda,\alpha\cosh\lambda)$, which obviously has the associated tangent

$$\vec{\vartheta}\equiv\frac{\partial\vartheta(\lambda)}{\partial\lambda}=(\alpha\cosh\lambda,\alpha\sinh\lambda).\tag{1.a.3}$$

We see at once that

$$g(\vec{\ell},\vec{\vartheta})=-\alpha\cosh\lambda\sinh\lambda+\alpha\sinh\lambda\cosh\lambda=0.\tag{1.a.4}$$

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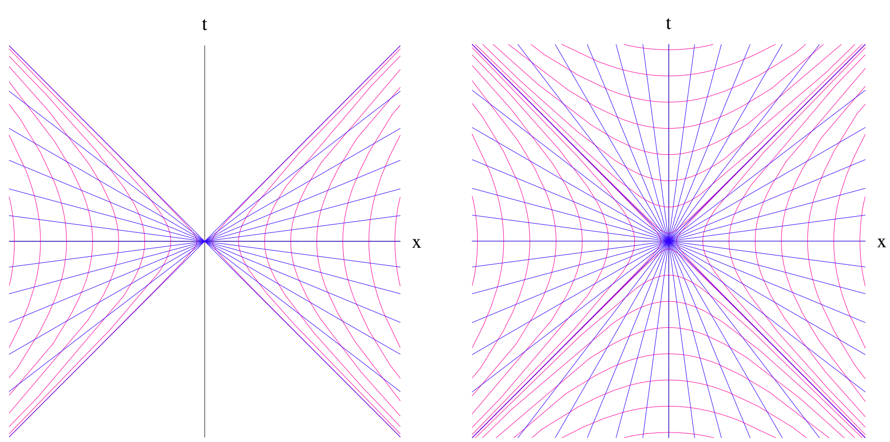

Figure 1. The orthogonal curvilinear coordinate charts which could be used by a uniformly accelerated observer in Minkowski spacetime. The red curves indicate surfaces of constant α and the blue curves indicate surfaces of constant λ. The diagram on the left shows the coordinate patch explicitly constructed in Problem 1, and the diagram on the right extends this construction to the whole of Minkowski space—minus lightcone of an observer at the origin.

b) We are to show that the map specified by equation (1.a.1) gives rise to an orthogonal coordinate system that covers half of Minkowski space in two disjoint patches. We should also represent this coordinate system diagrammatically.

From our work in part (a) above, we know that the tangent vectors to the lines of constant λ and α are given by

$$\vec{e}_\lambda = \alpha\cosh\lambda\;\vec{e}_t + \alpha\sinh\lambda\;\vec{e}_x \qquad \text{and} \qquad \vec{e}_\alpha = \sinh\lambda\;\vec{e}_t + \cosh\lambda\;\vec{e}_x. \tag{1.b.1}$$

Therefore, the differential map (where greek letters are used to indicate λ, α-coordinates) is given by

$$\Lambda^\mu{}_n = \begin{pmatrix} \alpha\cosh\lambda & \alpha\sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix}. \tag{1.b.2}$$

We see immediately that the Jacobian, $\det(\Lambda) = \alpha \neq 0$ which implies that the λ, α coordinate system is good generically (where it is defined). That it is 'orthogonal' is manifest because $\vec{e}_\lambda \cdot \vec{e}_\alpha = 0$ by part (a) above.

Note that the charts of (1.a.1) are not well-defined on or within the past or future lightcones of an observer at the origin: the curves of $\alpha =$ constant, the hyperbolas, are all time-like and outside the past and future lightcones of an observer at the origin; and the lines of $\lambda =$ constant are all spacelike and coincident at the origin. It does not take much to see that these coordinates have no overlap within the past and future lightcones of the Minkowski origin.

The coordinate system spanned by λ, α is shown in Figure 1.

c) We are to find the metric tensor and its associated Christoffel symbols of the coordinate charts described above.

Using equation (1.b.1), we can directly compute the components of the metric tensor in λ, α coordinates—$g_{\mu\nu} = g(\vec{e}_\mu, \vec{e}_\nu)$ where λ is in the '0'-position—

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.c.1}$$

The Christoffel symbols can be computed by hand rather quickly in this case; but we will still show some rough steps. Recall that the components of the Christoffel symbol $\Gamma^\mu_{\nu\rho}$ are given by

$$\Gamma^\mu_{\nu\rho}\vec{e}_\mu = \left(\frac{\partial \vec{e}_\nu}{\partial x^\rho}\right)^\mu \vec{e}_\mu.$$

Again making use of equation (1.b.1), we see that

$$\frac{\partial \vec{e}_\alpha}{\partial \alpha} = 0 \qquad \Longrightarrow \qquad \Gamma^\alpha_{\alpha\alpha} = \Gamma^\lambda_{\alpha\alpha} = 0. \tag{1.c.2}$$

Slightly less trivial, we see

$$\frac{\partial \vec{e}_\alpha}{\partial \lambda} = \cosh\lambda\;\vec{e}_t = \sinh\lambda\;\vec{e}_x = \frac{1}{a}\vec{e}_\lambda \qquad \Longrightarrow \qquad \Gamma^\lambda_{\alpha\lambda} = \Gamma^\lambda_{\lambda\alpha} = \frac{1}{\alpha}; \tag{1.c.3}$$

and,

$$\frac{\partial \vec{e}_\lambda}{\partial \lambda} = \alpha\sinh\lambda\;\vec{e}_t + \alpha\cosh\lambda\vec{e}_x = \alpha\vec{e}_\alpha \qquad \Longrightarrow \qquad \Gamma^\alpha_{\lambda\lambda} = \alpha, \quad \text{and} \quad \Gamma^\lambda_{\lambda\lambda} = 0. \tag{1.c.4}$$

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Problem 2

We are to find the Lie derivative of a tensor whose components are $T^{ab}{}_{c}$.

Although we are tempted to simply state the result derived in class and found in numerous textbooks, we will at least feign a derivation. Let us begin by recalling that the components of the tensor $\mathbf{T}$ are given by $T^{ab}{}_{c} = \mathbf{T}(\mathbf{E}^a, \mathbf{E}^b, \mathbf{E}_c)$ where the $\mathbf{E}$'s are basis vector- and one-form fields. Now, by the Leibniz rule for the Lie derivative we know that

$$\pounds_X\big(\mathbf{T}(\mathbf{E}^a,\mathbf{E}^b,\mathbf{E}_c)\big) = \pounds_X(T)\left(\mathbf{E}^a,\mathbf{E}^b,\mathbf{E}_c\right) + \mathbf{T}\left(\pounds_X(\mathbf{E}^a),\mathbf{E}^b,\mathbf{E}_c\right) + \mathbf{T}\left(\mathbf{E}^a,\pounds_X(\mathbf{E}^b),\mathbf{E}_c\right) + \mathbf{T}\left(\mathbf{E}^a,\mathbf{E}^b,\pounds_X(\mathbf{E}_c)\right). \tag{2.a.1}$$

Now, the first term on the right hand side of equation (2.a.1) gives the components of $\pounds_X(\mathbf{T})$, which is exactly what we are looking for. Rearranging equation (2.a.1) and converting our notation to components, we see

$$\left(\pounds_X(\mathbf{T})\right)^{ab}{}_{c} = \pounds_X\big(T^{ab}{}_{c}\big) - T^{\alpha b}{}_{c}\left(\pounds_X(\mathbf{E}^a)\right)_\alpha - T^{a\beta}{}_{c}\left(\pounds_X(\mathbf{E}^b)\right)_\beta - T^{ab}{}_{\gamma}\left(\pounds_X(\mathbf{E}_c)\right)^\gamma. \tag{2.a.2}$$

Now, we can either use some identities or just simply recall that

$$\left(\pounds_X(\mathbf{E}^a)\right)_\alpha = \frac{\partial X^a}{\partial x^\alpha} \qquad \text{and} \qquad \left(\pounds_X(\mathbf{E}_c)\right)^\gamma = \frac{\partial X^\gamma}{\partial x^c}. \tag{2.a.3}$$

We now have all the ingredients; putting everything together, we have

$$\left(\pounds_X(\mathbf{T})\right)^{ab}{}_{c} = X^\delta \frac{\partial}{\partial x^\delta}\left(T^{ab}{}_{c}\right) - T^{\alpha b}{}_{c}\frac{\partial X^a}{\partial x^\alpha} - T^{a\beta}{}_{c}\frac{\partial X^b}{\partial x^\beta} + T^{ab}{}_{\gamma}\frac{\partial X^\gamma}{\partial x^c}. \tag{2.a.4}$$

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Problem 3

Theorem: Acting on any tensor $\mathbf{T}$, the Lie derivative operator obeys

$$\pounds_U\pounds_V(\mathbf{T}) - \pounds_V\pounds_U(\mathbf{T}) = \pounds_{[U,V]}(\mathbf{T}). \tag{3.a.1}$$

proof: We will proceed by induction. Let us suppose that the theorem holds for all tensors of rank less than or equal to $\binom{r}{s}$ for some $r, s \geq 1$. We claim that this is sufficient to prove the hypothesis for any tensor of rank $\binom{r}{s+1}$ or $\binom{r+1}{s}$. (The induction argument is identical for the two cases—our argument will depend on which index is advancing—so it is not necessary to expound both cases.)

Now, all rank $\binom{r+1}{s}$ tensors can be written as *a sum of* tensor products between $\binom{r}{s}$ rank tensors $\mathbf{T}$ indexed by i and rank $\binom{1}{0}$ tensors $\mathbf{E}$, again indexed by i[1]. That is, we can express an arbitrary $\binom{r+1}{s}$ tensor as a sum of $\sum_i \mathbf{T}_i \otimes \mathbf{E}_i$—where i is just an index label! But this complication is entirely unnecessary: by the linearity of the Lie derivative, it suffices to show the identity for any one tensor product in the sum.

Making repeated use of the linearity of the Lie derivative and the Leibniz rule, we see

$$\begin{aligned}
\left(\pounds_U\pounds_V - \pounds_V\pounds_U\right)(\mathbf{T}\otimes\mathbf{E}) =& \pounds_U\Big((\pounds_V\mathbf{T})\otimes\mathbf{E} + \mathbf{T}\otimes\pounds_V\mathbf{E}\Big) - \pounds_V\Big((\pounds_U\mathbf{T})\otimes\mathbf{E} + \mathbf{T}\otimes\pounds_U\mathbf{E}\Big),\\
=& \Big(\pounds_U\pounds_V\mathbf{T}\Big)\otimes\mathbf{E} + \pounds_V\mathbf{T}\otimes\pounds_U\mathbf{E} + \pounds_U\mathbf{T}\otimes\pounds_V\mathbf{E} + \mathbf{T}\otimes\Big(\pounds_U\pounds_V\mathbf{E}\Big)\\
&- \Big(\pounds_V\pounds_U\mathbf{T}\Big)\otimes\mathbf{E} - \pounds_U\mathbf{T}\otimes\pounds_V\mathbf{E} - \pounds_V\mathbf{T}\otimes\pounds_U\mathbf{E} - \mathbf{T}\otimes\Big(\pounds_V\pounds_U\mathbf{E}\Big),\\
=& \Big(\pounds_{[U,V]}\mathbf{T}\Big)\otimes\mathbf{E} + \mathbf{T}\otimes\Big(\pounds_{[U,V]}\mathbf{E}\Big),\\
=& \pounds_{[U,V]}\Big(\mathbf{T}\otimes\mathbf{E}\Big);
\end{aligned}$$

where in the second to last line we used the induction hypothesis—applicable because both $\mathbf{T}$ and $\mathbf{E}$ are of rank $\binom{r}{s}$ or less.

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[1]The savvy reader knows that an arbitrary $\binom{r}{s}$ tensor can not be written as a tensor product of r contravariant and s covariant pieces; *however* every $\binom{r}{s}$ tensor can be written as a *sum of* such tensor products: indeed, this is exactly what is done when writing 'components' of the tensor.

It now suffices to show that the identity holds for all $\binom{0}{1}$ forms and all $\binom{1}{0}$ tensors[2]. We will actually begin one-step lower and note that equation (3.a.1) follows trivially from the Leibniz rule for 0-forms. Indeed, we see that for any 0-form f,

$$\begin{aligned}
\pounds_U(\pounds_V f) &= \Big(\pounds_U V\Big) f + V\Big(\pounds_U f\Big), \\
&= \pounds_{[U,V]} f + \pounds_V(\pounds_U f)\,, \\
\therefore \Big(\pounds_U\pounds_V - \pounds_V\pounds_U\Big) f &= \pounds_{[U,V]} f. \qquad (3.a.2)
\end{aligned}$$

Now, to finish our proof, we claim that the identity holds for any $\binom{1}{0}$ and $\binom{0}{1}$ tensors, say $\mathbf{X}$ and $\mathbf{Y}$, respectively. Recall that a one form $\mathbf{Y}$ is a function mapping vector fields into scalars—i.e. $\mathbf{Y}(\mathbf{X})$ is a 0-form. For our own convenience, we will write $\mathbf{Y}(\mathbf{X}) \equiv \langle \mathbf{X}, \mathbf{Y}\rangle$. From our work immediately above, we know the identity holds for $\langle \mathbf{X}, \mathbf{Y}\rangle$:

$$\Big(\pounds_U\pounds_V - \pounds_V\pounds_U\Big)\langle \mathbf{X}, \mathbf{Y}\rangle = \pounds_{[U,V]}\Big(\langle \mathbf{X}, \mathbf{Y}\rangle\Big). \qquad (3.a.3)$$

Because the Leibniz rule obeys contraction, we can expand out the equation above similar to as before. Indeed, almost copying the equations above verbatim we find

$$\begin{aligned}
(\pounds_U\pounds_V - \pounds_V\pounds_U)\left(\langle \mathbf{X}, \mathbf{Y}\rangle\right) =& \pounds_U\Big(\langle(\pounds_V\mathbf{X}), \mathbf{Y}\rangle + \langle \mathbf{X}, \pounds_V\mathbf{Y}\rangle\Big) - \pounds_V\Big(\langle(\pounds_U\mathbf{X}), \mathbf{Y}\rangle + \langle \mathbf{X}, \pounds_U\mathbf{Y}\rangle\Big), \\
=& \quad \langle \pounds_U\pounds_V\mathbf{X}, \mathbf{Y}\rangle + \langle \pounds_V\mathbf{X}, \pounds_U\mathbf{Y}\rangle + \langle \pounds_U\mathbf{X}, \pounds_V\mathbf{Y}\rangle + \langle \mathbf{X}, \pounds_U\pounds_V\mathbf{Y}\rangle \\
& - \langle \pounds_V\pounds_U\mathbf{X}, \mathbf{Y}\rangle - \langle \pounds_U\mathbf{X}, \pounds_V\mathbf{Y}\rangle - \langle \pounds_V\mathbf{X}, \pounds_U\mathbf{Y}\rangle - \langle \mathbf{X}, \pounds_V\pounds_U\mathbf{Y}\rangle\,, \\
=& \Big\langle\Big(\pounds_U\pounds_V - \pounds_V\pounds_U\Big)\mathbf{X}, \mathbf{Y}\Big\rangle + \Big\langle \mathbf{X}, \Big(\pounds_U\pounds_V - \pounds_V\pounds_U\Big)\mathbf{Y}\Big\rangle\,, \\
=& \big\langle \pounds_{[U,V]}\mathbf{X}, \mathbf{Y}\big\rangle + \big\langle \mathbf{X}, \pounds_{[U,V]}\mathbf{Y}\big\rangle\,.
\end{aligned}$$

In the last line, we referred to equation (3.a.3) and expanded it using the Leibniz rule. Almost a footnote-comment: the last two lines do not precisely prove our required theorem as they stand: to identify the two pieces of each sum we need one small trick—replace one of $\mathbf{X}$ or $\mathbf{Y}$ with the basis one-forms or vector fields, and the result for the other becomes manifest.

Therefore, because equation (3.a.1) holds for all one-forms and vector fields, our induction work proves that it must be true for all tensor fields of arbitrary rank. ὅπερ ἔδει δεῖξαι

[2]You should probably suspect this is overkill: the induction step seemed to make no obvious use of the fact that $r, s \geq 1$. And, as shown below, the identity is almost trivially true for the case of scalars. Nevertheless, it is better to be over-precise than incorrect. (In the famous words of Blaise Pascal to a mathematician friend: "I have made this letter longer because I have not had the time to make it shorter.")

Problem 4

The torsion and curvature tensors are defined respectively,

$$T(X,Y) = \nabla_X Y - \nabla_Y X - \pounds_X Y \qquad \text{and} \qquad R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}. \tag{4.a.1}$$

We are to prove

a) $T(fX, gY) = fgT(X,Y)$,

b) $R(fX, gY)hZ = fghR(X,Y)Z$,

for arbitrary functions f, g and h, and vector fields X, Y and Z.

Theorem a: $T(fX, gY) = fgT(X,Y)$.

proof: In both of the required proofs, we will make repeated uses of the 'defining' properties of the covariant derivative ∇ and of the Lie derivative. In particular, we will need the following properties of the connection:

(1) $\nabla_X Y$ is a tensor in the argument X. This means that as an operator, $\nabla_{fX+gY} = f\nabla_X + g\nabla_Y$.

(2) $\nabla_X Y$ obeys the Leibniz rule in Y. Specifically, this means $\nabla_X(fY) = X(f)Y + f\nabla_X Y$. This implies that $\nabla_X Y$ is linear in Y—which follows when f is a constant.

We are almost ready to 'prove the identity by brute force in a couple of lines.' Let's just prepare one more trick up our sleeve: we will need

$$\begin{aligned}
[fX, gY] = \pounds_{fX}(gY) &= g\pounds_{fX}Y + \Big(\pounds_{fX}g\Big)Y, \\
&= -g\pounds_Y(fX) + fX(g)Y, \\
&= -gfY(X) - gXY(f) + fX(g)Y.
\end{aligned}$$

Let us begin:

$$\begin{aligned}
T(fX, gY) &= \nabla_{fX}(gY) - \nabla_{gY}(fX) - \pounds_{fX}(gY)\,, \\
&= f\nabla_X(gY) - g\nabla_Y(fX) - \pounds_{fX}(gY)\,, \\
&= fX(g)Y + fg\nabla_X Y - g(Y(f))X - fg\nabla_Y X + gfY(X) + gXY(f) - fX(g)Y, \\
&= fg\nabla_X Y - fg\nabla_Y X - gf\pounds_X Y, \\
&\therefore T(fX, gY) = fgT(X,Y).
\end{aligned} \tag{4.a.2}$$

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Theorem b: $R(fX, gY)hZ = fghR(X,Y)Z$.

proof: We have already collected all of the properties and identities necessary to straightforwardly prove the theorem. Therefore, we may proceed directly.

$$\begin{aligned}
R(fX, gY)hZ =& \Big\{\nabla_{fX}\nabla_{gY} - \nabla_{gY}\nabla_{fX} - \nabla_{[fX,gY]}\Big\}hZ, \\
=& \Big\{f\nabla_X\left(g\nabla_Y\right) - g\nabla_Y\left(f\nabla_X\right) - fg\nabla_{[X,Y]} - f\nabla_{YX(g)} + g\nabla_{XY(f)}\Big\}hZ, \\
=& \Big\{fX(g)\nabla_Y + fg\nabla_X\nabla_Y - gY(f)\nabla_X - fg\nabla_Y\nabla_X - fg\nabla_{[X,Y]} - fX(g)\nabla_Y + gY(f)\nabla_X\Big\}hZ, \\
=& \Big\{fg\nabla_X\nabla_Y - fg\nabla_Y\nabla_X - fg\nabla_{[X,Y]}\Big\}hZ, \\
=& fgR(X,Y)hZ, \\
=& fg\Big\{\nabla_X\Big(Y(h)Z + h\nabla_Y Z\Big) - \nabla_Y\Big(X(h)Z + h\nabla_X Z\Big) - [X,Y]\,(h)Z - h\nabla_{[X,Y]}Z\Big\}, \\
=& fg\Big\{\nabla_X\Big(Y(h)Z\Big) + X(h)\nabla_Y Z + h\nabla_X\nabla_Y Z - \nabla_Y\Big(X(h)Z\Big) - Y(h)\nabla_X Z - h\nabla_Y\nabla_X Z \\
& \qquad - X\left(\nabla_Y h\right)Z + Y\left(\nabla_X h\right)Z - h\nabla_{[X,Y]}Z\Big\}, \\
=& fg\Big\{hR(X,Y)Z + \left(\nabla_X\left(Y(h)\right)\right)Z - \nabla_Y\left(X(h)\right)Z - X\left(Y(h)\right)Z + Y\left(X(h)\right)Z\Big\}, \\
& \therefore R(fX, gY)hZ = fghR(X,Y)Z.
\end{aligned} \tag{4.b.1}$$

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Physics 523, General Relativity
Homework 5
Due Friday, 17th November 2006

Jacob Lewis Bourjaily

Problem 1

Let us consider a manifold with a torsion free connection $R(X,Y)$ which is not necessarily metric compatible. We are to prove that

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0, \tag{1.1}$$

and the Bianchi identity

$$\nabla_X\big(R(X,Y)\big)V + \nabla_Y\big(R(Z,X)\big)V + \nabla_Z\big(R(X,Y)\big)V = 0. \tag{1.2}$$

The first identity is relatively simple to prove—it follows naturally from the Jacobi identity for the Lie derivative. Let us first prove the Jacobi identity:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0. \tag{1.3}$$

Using the antisymmetry of the Lie bracket and our result from last homework problem 3, we have

$$[X,[Y,Z]] = \pounds_X[Y,Z] = -\pounds_{[Y,Z]}X = \pounds_Z\pounds_Y X - \pounds_Y\pounds_Z X = -[Z,[X,Y]] - [Y,[Z,X]].$$

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The condition of a connection being torsion free is that

$$\pounds_X Y = \nabla_X Y - \nabla_Y X. \tag{1.4}$$

Expanding the Lie brackets encountered in the statement of the Jacobi identity,

$$\begin{aligned}
0 =& \pounds_X\pounds_Y Z + \pounds_Y\pounds_Z X + \pounds_Z\pounds_X Y,\\
=& \pounds_X(\nabla_Y Z - \nabla_Z Y) + \pounds_Y(\nabla_Y X - \nabla_X Y) + \pounds_Z(\nabla_X Y - \nabla_Y X),\\
=& \nabla_X\nabla_Y Z - \nabla_X\nabla_Z Y - \nabla_{[Y,Z]}X + \nabla_Y\nabla_Z X - \nabla_Y\nabla_X Z - \nabla_{[Z,X]}Y + \nabla_Z\nabla_X Y - \nabla_Z\nabla_Y X - \nabla_{[X,Y]}Z,\\
=& \big(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}\big)Z + \big(\nabla_Y\nabla_Z - \nabla_Z\nabla_Y - \nabla_{[Y,Z]}\big)X + \big(\nabla_Z\nabla_X - \nabla_X\nabla_Z - \nabla_{[Z,X]}\big)Y;
\end{aligned}$$

$$\therefore R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0. \tag{1.5}$$

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To prove the Bianchi identity, we will 'dirty' our expressions with explicit indices in hope of a quick solution. It is rather obvious to see that (1.2) is equivalent to the component expression

$$R^a{}_{bcd;e} + R^a{}_{bde;c} + R^a{}_{bec;d} = 0. \tag{1.6}$$

Worse than introducing components, let us use our (gauge) freedom to consider the Bianchi identity evaluated at a point p in spacetime in Riemann normal coordinates[1]. If we show that the Bianchi identity (1.6) holds in any particular coordinates at a point p, it necessarily must hold in any other coordinate system—and if p is arbitrary, then it follows that the Bianchi identity holds throughout spacetime.

Recall from lecture or elsewhere that Riemann normal coordinates at p are such that $\Gamma^a_{bc}(p) = 0$. This implies that the covariant derivative of the Riemann tensor is simply a normal derivative at p. Using the definition of $R^a{}_{bcd}$ in terms of the Christoffel symbols, we see at once that

$$R^a{}_{bcd;e}(p) + R^a{}_{bde;c}(p) + R^a{}_{bec;d}(p) = \Gamma^a_{bd,ce}(p) - \Gamma^a_{bc,de}(p) + \Gamma^a_{be,dc}(p) - \Gamma^a_{bd,ec}(p) + \Gamma^a_{bc,ed}(p) - \Gamma^a_{be,cd}(p);$$

$$\therefore R^a{}_{bcd;e}(p) + R^a{}_{bde;c}(p) + R^a{}_{bec;d}(p) = 0. \tag{1.7}$$

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[1] Riemann normal coordinates are constructed geometrically as follows: in a sufficiently small neighbourhood about p, every point can be reached by traversing a certain geodesic through p a certain distance. If we choose to define all families of geodesics through p using the same affine parameter λ then if we fix λ, there is a (smooth) bijection between tangent vectors in T_pM to points in the neighbourhood about p: the direction of $v \in T_pM$ tells the direction to the nearby points and its magnitude (for fixed λ) tells the distance to travel along the geodesic. Needless to say this construction does not require a metric.

Problem 2

We are to compute the Riemann tensor, the Ricci tensor, the Weyl tensor and the scalar curvature of a conformally-flat metric,

$$g_{ab}(x) = e^{2\Omega(x)}\eta_{ab}. \tag{2.1}$$

Using the definition of the Christoffel symbol with our metric above, we find

$$\begin{aligned}\Gamma^a_{bc} &= \frac{1}{2}g^{am}\left\{g_{am,b} + g_{bm,a} - g_{ab,m}\right\},\\ &= \frac{1}{2}e^{-2\Omega}\eta^{am}\left\{\eta_{bm}e^{2\Omega}\partial_c\Omega + \eta_{cm}e^{2\Omega}\partial_b\Omega - e^{2\Omega}\eta_{bc}\partial_m\Omega\right\},\\ \therefore\ \Gamma^a_{bc} &= \delta^a_b\partial_c\Omega + \delta^a_c\partial_b\Omega - \eta_{bc}\eta^{am}\partial_m\Omega.\end{aligned} \tag{2.2}$$

Using this together with the (definition of the) Riemann tensor's components

$$R^a{}_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^m_{bd}\Gamma^a_{cm} - \Gamma^m_{bc}\Gamma^a_{dm}, \tag{2.3}$$

we may compute directly[2],

$$\begin{aligned}R^a{}_{bcd} =& \delta^a_d\partial_c\partial_b\Omega - \eta_{bd}\eta^{am}\partial_c\partial_m\Omega - \delta^a_c\partial_b\partial_d\Omega + \eta_{bc}\eta^{am}\partial_d\partial_m\Omega - \delta^a_d\left(\partial_b\Omega\right)\left(\partial_c\Omega\right) + \eta_{bd}\eta^{am}(\partial_c\Omega)(\partial_m\Omega)\\ &- \delta^a_d(\partial_c\Omega)(\partial_b\Omega) - \delta^a_c(\partial_b\Omega)(\partial_b\Omega) + \eta_{bc}\delta^a_d\eta^{mn}(\partial_m\Omega)(\partial_n\Omega) + \delta^a_c(\partial_b\Omega)(\partial_d\Omega) - \eta_{bc}\eta^{am}(\partial_d\Omega)(\partial_m\Omega)\\ &+ \delta^a_c(\partial_d\Omega)(\partial_b\Omega) + \delta^a_d(\partial_b\Omega)(\partial_c\Omega) - \eta_{bd}d^a_c\eta^{mn}(\partial_m\Omega)(\partial_n\Omega) - \eta_{bd}\eta^{am}(\partial_c\Omega)(\partial_m\Omega) + \eta_{bd}\eta^{am}(\partial_c\Omega)(\partial_m\Omega)\\ =&\Big\{\delta^m_b\left(\delta^a_c\delta^n_d - \delta^a_d\delta^n_c\right) + \eta_{bd}\left(\eta^{an}\delta^m_c - \eta^{mn}\delta^a_c\right) + \eta_{bc}\left(\eta^{mn}\delta^a_d - \eta^{an}\delta^m_d\right)\Big\}\left(\partial_m\Omega\right)\left(\partial_n\Omega\right)\\ &+ \left(\delta^a_d\partial_c - \delta^a_c\partial_d\right)\partial_b\Omega + \eta^{am}\left(\eta_{bc}\partial_d\partial_m\Omega - \eta_{bd}\partial_c\partial_m\Omega\right).\end{aligned}$$

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It will be helpful to recast this into the form where all the indices are lowered. We can do this by acting with the metric tensor. Doing so we find,

$$\begin{aligned}e^{-2\Omega}R_{abcd} =&\Big\{\delta^m_b\left(\eta_{ac}\delta^n_d - \eta_{ad}\delta^n_c\right) + \eta_{bd}\left(\delta^n_a\delta^m_c - \eta_{ac}\eta^{mn}\right) + \eta_{bc}\left(\eta_{ad}\eta^{mn} - \delta^m_a\delta^n_d\right)\Big\}(\delta_m\Omega)(\delta_n\Omega)\\ &+ \eta_{ad}\partial_c\partial_b\Omega - \eta_{ac}\partial_d\partial_b\Omega + \eta_{bc}\partial_d\partial_a\Omega - \eta_{bd}\partial_c\partial_a\Omega,\\ =&\Big\{\eta_{ad}\delta^m_b\delta^n_c - \eta_{ac}\delta^m_b\delta^n_d + \eta_{bc}\delta^m_a\delta^n_d - \eta_{bd}\delta^m_a\delta^n_c\Big\}\Big(\partial_m\partial_n\Omega - (\partial_m\Omega)(\partial_n\Omega)\Big) + \Big(\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}\Big)\eta^{mn}(\partial_m\Omega)(\partial_n\Omega).\end{aligned} \tag{2.4}$$

Although we will not have any use for such frivolities, we can further compress this expression to

$$e^{-2\Omega}R_{abcd} = 4\delta^r_{[a}\,\delta^n_{b]}\delta^s_{[d}\,\delta^m_{c]}\eta_{rs}\Big(\partial_m\partial_n\Omega - (\partial_m\Omega)(\partial_n\Omega)\Big) + \Big(\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}\Big)\eta^{mn}(\partial_m\Omega)(\partial_n\Omega). \tag{2.5}$$

Now, we can then find the Ricci tensor by acting on equation (2.4) with g^{ac}. Letting D be the dimensionality of our manifold, we find

$$\begin{aligned}R_{bd} =&\Big\{\delta^m_d\delta^n_b - D\delta^m_d\delta^n_b + \delta^m_d\delta^n_b - \eta_{bd}\eta^{mn}\Big\}\Big(\partial_m\partial_n\Omega - (\partial_m\Omega)(\partial_n\Omega)\Big) + \eta^{mn}\Big(\eta_{bd} - D\eta_{bd}\Big)(\partial_m\Omega)(\partial_n\Omega),\\ =&(2-D)\Big(\partial_b\partial_d\Omega - (\partial_b\Omega)(\partial_d\Omega)\Big) + (2-D)\eta_{bd}\eta^{mn}(\partial_m\Omega)(\partial_n\Omega) - \eta_{bd}\eta^{mn}\partial_m\partial_n\Omega.\end{aligned} \tag{2.6}$$

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Lastly, contracting this, we find the scalar curvature,

$$\begin{aligned}e^{2\Omega}R =&(2-D)\eta^{mn}\Big(\partial_m\partial_n\Omega - (\partial_m\Omega)(\partial_n\Omega)\Big) + D(2-D)\eta^{mn}(\partial_m\Omega)(\partial_n\Omega) - D\eta^{mn}\partial_m\partial_n\Omega,\\ =&2(1-D)\eta^{mn}\partial_m\partial_n\Omega - (2-D)(1-D)\eta^{mn}(\partial_m\Omega)(\partial_n\Omega).\end{aligned} \tag{2.7}$$

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[2]To be absolutely precise, there are two terms which manifestly cancel that appear when expanding this expression, which we have left out for typographical and aesthetic considerations.

All that remains for us to compute is the Weyl tensor. Any exposure to conformal geometry immediately tells us that the Weyl tensor vanishes. That is, that

$$R_{abcd} = \frac{1}{(D-2)}\Big(g_{ac}R_{db} + g_{db}R_{ac} - g_{ad}R_{bc} - g_{bc}R_{ad}\Big) - \frac{1}{(D-1)(D-2)}R\Big(g_{ac}g_{db} - g_{ad}g_{bc}\Big). \tag{2.8}$$

We will try as hard as possible to avoid actually computing the right hand side by expanding our expressions above. To show that the Weyl tensor vanishes, we must build R_{abcd} out of R_{bc}, R and the metric g_{ab}. This statement alone essentially gives us the expression at first glance.

The first important thing to notice is that R_{abcd} has no term proportional to $\eta^{mn}\partial_m\partial_n\Omega$ while both R_{ab} and R do. This means that if R_{abcd} can only be composed of linear combinations of R_{ab} and R which do not contain $\eta^{mn}\partial_m\partial_n\Omega$. Looking at expressions (2.4) and (2.6), we see that they can only appear in the combination

$$R_{bd} + \frac{e^{2\Omega}\eta_{bd}}{2(1-D)}R = R_{bd} + \frac{g_{bd}}{2(1-D)}R. \tag{2.9}$$

Any multiple of this combination will automatically have no $\eta^{mn}\partial_m\partial_n\Omega$ contribution.

Staring a bit more at equations (2.4) and (2.6), we notice that the first set of terms in (2.4) are all of the form $g_{ac}R_{bd}$. Indeed, we see that

$$\frac{1}{2-D}\Big\{\eta_{ad}R_{bc}-\eta_{ac}R_{bd}+\eta_{bc}R_{ad}-\eta_{bd}R_{ac}\Big\} = \Big\{\eta_{ad}\delta^m_b\delta^n_c-\eta_{ac}\delta^m_b\delta^n_d+\eta_{bc}\delta^m_a\delta^n_d-\eta_{bd}\delta^m_a\delta^n_c\Big\}\Big(\partial_m\partial_n\Omega-(\partial_m\Omega)(\partial_n\Omega)\Big)+\dots . \tag{2.10}$$

Notice that multiplying both sides of the above equation by $e^{2\Omega}$ will convert all of the η_{ab}'s into g_{ab}'s[3] . This is all we need to construct the Riemann tensor from the Ricci tensor and scalar curvature: knowing the combination of Ricci tensors which gives part of the Riemann tensor, we can use (2.9) to determine the rest. Indeed, we see that

$$\begin{aligned}
C_{abcd} + R_{abcd} =& \frac{1}{2-D}\Big\{g_{ab}R_{bc} - g_{ad}R_{bd} + g_{bc}R_{ad} - g_{bd}R_{ac}\Big\} + \frac{R}{2(1-D)(2-D)}\Big(g_{ad}g_{bc} - g_{ac}g_{bd} + g_{bc}g_{ad} - g_{bd}g_{ac}\Big),\\
=& \frac{1}{D-2}\Big\{g_{ac}R_{bd} - g_{ad}R_{bc} - g_{bc}R_{ad} + g_{bd}R_{ac}\Big\} - \frac{R}{(D-1)(D-2)}\Big(g_{ac}g_{bd} - g_{ad}g_{bc}\Big),\\
=& \Big\{g_{ad}\delta^m_b\delta^n_c - g_{ac}\delta^m_b\delta^n_d + g_{bc}\delta^m_a\delta^n_d - g_{bd}\delta^m_a\delta^n_c\Big\}\Big(\partial_m\partial_n\Omega - (\partial_m\Omega)(\partial_n\Omega)\Big) + \Big(g_{ad}g_{bc} - g_{ac}g_{bd}\Big)g^{mn}(\partial_m\Omega)(\partial_n\Omega),\\
=& R_{abcd};
\end{aligned}$$

$$\therefore C_{abcd} = 0. \tag{2.11}$$

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[3]The conversion from $\eta_{ab} \to g_{ab}$ is completely natural. The only possibly non-trivial step comes from the last term in the expression (2.4) for the Riemann tensor: bringing $e^{2\Omega}$ to the right hand side of (2.4), we have a term which has two lowered η_{ab}'s and one upper η_{ab}; now, $e^{2\Omega}\eta^{mn} = e^{4\Omega}g^{mn}$ and how these two factors of $e^{2\Omega}$ can be absorbed into the lowered η's as desired.

Problem 3

We are to show that if $\varphi(x)$ satisfies the flat-space, massless Klein-Gordon equation, then if $g_{ab} = e^{2\Omega(x)}\eta_{ab}$, the transformed field $e^{\beta\Omega(x)}\varphi(x) \equiv \varphi'(x)$ satisfies the equation

$$g^{ab}\varphi'_{;ab} - \alpha R\varphi' = 0, \tag{3.1}$$

for appropriate values of α and β—dependant on the spacetime dimension but independent of $\Omega(x)$.

Let us agree to call $\Box \equiv \eta^{ab}\partial_a\partial_b$. Then the flat-space Klein-Gordon equation is simply $\Box\varphi(x) = 0$. Recall the expression for the scalar curvature R in D spacetime dimensions for a metric which is conformally-related to the Minkowski metric (2.7):

$$R = 2(1-D)e^{-2\Omega}\Box\Omega - (2-D)(1-D)e^{-2\Omega}\eta^{mn}(\partial_m\Omega)(\partial_n\Omega). \tag{3.2}$$

We would like to explicitly state $g^{ab}\nabla_b\nabla_a$ in terms of $\Box$ and Ω. This can be done quite explicitly, recalling the Christoffel symbols for a conformally-flat spacetime(2.2),

$$\begin{aligned} g^{ab}\nabla_b\nabla_a &= g^{ab}\partial_a\partial_b - g^{ab}\Gamma^c_{ab}\partial_c, \\ &= e^{-2\Omega}\Big\{\Box - \eta^{ab}\Big(\delta^c_a(\partial_b\Omega)\partial_c + \delta^c_b(\partial_a\Omega)\partial_c - \eta_{ab}\eta^{cm}(\partial_m\Omega)\partial_c\Big)\Big\}, \\ &= e^{-2\Omega}\Big\{\Box - \eta^{cb}(\partial_b n\Omega)\partial_c - \eta^{ac}(\partial_a\Omega)\partial_c + D\eta^{cm}(\partial_m\Omega)\partial_c\Big\}, \\ &= e^{-2\Omega}\Big\{\Box - (D-2)\eta^{ab}(\partial_a\Omega)\partial_b\Big\}. \end{aligned}$$

Acting with $g^{ab}\nabla_b\nabla_a$ on φ' we find,

$$\begin{aligned} g^{ab}\nabla_b\nabla_a\varphi' &= e^{-2\Omega}\Big\{\Box\left(e^{\beta\Omega}\varphi\right) + (D-2)\eta^{ab}(\partial_a\Omega)\left(\partial_b\left(e^{\beta\Omega}\varphi\right)\right)\Big\}, \\ &= e^{-2\Omega}\Big\{\beta\varphi'\Box(\Omega) + \beta(\beta+D-2)\varphi'\eta^{ab}(\partial_a\Omega)(\partial_b\Omega) + 2\beta e^{\beta\Omega}\eta^{ab}(\partial_a\varphi)(\partial_b\Omega) + (D-2)e^{\beta\Omega}\eta^{ab}(\partial_a\varphi)(\partial_b\Omega)\Big\}. \end{aligned}$$

Although only one equation, if (3.1) is to hold for arbitrary $\Omega(x)$, there are actually three constraints implied by (3.1)—one for each functionally distinct contribution. Actually, we'll find that there are only two independent conditions—just enough to uniquely determine α and β.

First, notice that R does not contain any derivatives of $\varphi(x)$. Therefore equation (3.1) implies that

$$2\beta e^{\beta\Omega}\eta^{ab}(\partial_a\varphi)(\partial_b\Omega) + (D-2)e^{\beta\Omega}\eta^{ab}(\partial_a\varphi)(\partial_b\Omega) = 0, \tag{3.3}$$

arising from the $g^{ab}\nabla_b\nabla_a\varphi'$ term in (3.1). This obviously implies that

$$\therefore \beta = -\frac{D-2}{2}. \tag{3.4}$$

The next condition(s) come form matching the remaining two functionally distinct terms in (3.1), namely[4]

$$g^{ab}\nabla_b\nabla_a\varphi' - \alpha R\varphi' \propto \beta\Box\Omega + \beta(\beta+D-2)\eta^{ab}(\partial_a\Omega)(\partial_b\Omega) - 2\alpha(1-D)\Box\Omega + \alpha(D-2)(D-1)\eta^{ab}(\partial_a\Omega)(\partial_b\Omega). \tag{3.5}$$

Matching the corresponding terms, we see that

$$\alpha = \frac{\beta}{2(1-D)} \qquad \text{and} \qquad \alpha = \frac{-\beta(\beta+D-2)}{(D-2)(D-1)}. \tag{3.6}$$

We see that $\beta = \frac{1}{2}(D-2)$ is consistent with both of these—more concretely, any two of these three constraints is sufficient to imply the third. Therefore, we have shown that $\varphi' = e^{\beta\Omega}\varphi$ will satisfy the modified Klein-Gordon equation (3.1) for any $\Omega(x)$ if

$$\therefore \beta = \frac{2-D}{2} \qquad \text{and} \qquad \alpha = \frac{1}{4}\frac{D-2}{D-1}. \tag{3.7}$$

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[4]We are not including those pieces eliminated by the choice (3.4).

Physics 523, General Relativity
Homework 6
Due Wednesday, 29th November 2006

Jacob Lewis Bourjaily

Problem 1[1]

We are asked to determine the ratio of frequencies observed at two fixed[2] points in a spacetime with a static metric g_{ab}; we should use this to determine the redshift of light emitted from the surface of the Sun which is observed on the surface of the Earth.

Imagine a clock at a fixed point x_1 which ticks with a regular interval Δs. Because the point is stationary, we may use the definition of the spacetime metric g_{ab} to see that this interval is related to the coordinate time interval Δt by[3]

$$\Delta s^2 = \Delta t_1^2 g_{00}(x_1). \tag{1.1}$$

We have included a subscript on the coordinate time interval to make its position-dependence manifest. The invariant interval Δs, however, must certainly be position-independent for any reliable clock. Therefore, we naturally have that

$$\Delta s^2 = \Delta t_1^2 g_{00}(x_1) = \Delta t_2^2 g_{00}(x_2), \tag{1.2}$$

for any other point x_2. This implies that

$$\therefore \frac{\Delta t_1^2}{\Delta t_2^2} = \frac{g_{00}(x_2)}{g_{00}(x_1)}. \tag{1.3}$$

It is important to note that this discussion is not limited to clocks ticking regularly: any process with a well-defined, constant time interval observed at two distinct points will obey equation (1.3). Indeed, consider an atomic transition which emits photons with frequency $\nu_1 \equiv \frac{1}{\Delta t_1}$ at point x_1. Equation (1.3) implies that the frequency at x_1 will be related to the frequency ν_2 at x_2 by

$$\therefore \frac{\nu_2}{\nu_1} = \sqrt{\frac{g_{00}(x_2)}{g_{00}(x_1)}}. \tag{1.4}$$

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To determine the redshift of light emitted from the Sun and observed on the Earth we recall that in the Newtonian (weak-field) approximation,

$$g_{00}(x) = -1 - 2\varphi_N(x), \tag{1.5}$$

where $\varphi_N(x)$ is the Newtonian potential at x. The only subtlety is that we should make sure to be careful about units when computing $\varphi_N(x)$. Notice that because the '1' in $-1 - 2\varphi_N(x)$ is dimensionless, so should $\varphi_N(x)$ be. This will be the case if we judiciously set $c = 1$. In these units, we find

$$\varphi_N(R_\odot) = -2.12 \times 10^{-6} \qquad \text{and} \qquad \varphi_N(R_\oplus) = -1.06 \times 10^{-8}, \tag{1.6}$$

which gives a redshift of 2.11 parts per million.

[1] Note added in revision: this solution is bad. The argument presented for equation (1.4) is not valid (even though the right answer emerges). One should be very careful about the thought experiment under consideration (because the inverse result is easy to obtain under a different situation).

[2] The equation which the problem set asks us to demonstrate is only valid for *stationary* sources and observers—otherwise there would be a doppler-shift term obfuscating the equation.

[3] In his textbook, Weinberg has an interesting discussion on why it is fundamentally not possible to disentangle Δs from Δt at a particular point. However, it *is* possible to compare the metric at two distinct points—by observing a gravitational redshift—as described presently.

Problem 2

We are to find the 'natural' generally covariant generalization of the flat-space Klein-Gordon Lagrangian (which was shown to be Weyl invariant in the last problem set). We should use this to determine the matter stress-energy tensor and show that it is traceless.

The striking similarity between

$$g^{ab}\nabla_a\nabla_b\varphi - \frac{1}{6}R\varphi = 0, \tag{2.1}$$

and the massive Klein-Gordon equation makes us guess that the action from which this is derived is

$$S = \frac{1}{2}\int d^4x\sqrt{-g}g^{ab}\left(\nabla_a\varphi\nabla_b\varphi + \frac{1}{6}R_{ab}\varphi^2\right). \tag{2.2}$$

Our intuition is confirmed by calculating the equation of motion:

$$\begin{aligned}0 &= \nabla_a\left(\frac{\partial\mathscr{L}}{\partial\nabla_a\varphi}\right) - \frac{\partial\mathscr{L}}{\partial\varphi} = \nabla_a\left(g^{ab}\nabla_b\varphi\right) - \frac{1}{6}R\varphi,\\ &= g^{ab}\nabla_a\nabla_b\varphi - \frac{1}{6}R\varphi.\end{aligned} \tag{2.3}$$

Therefore the action (2.2) does indeed give rise to the desired equation of motion for φ as desired.

We now must compute the stress-energy tensor for this matter Lagrangian. Recall that the stress-energy tensor T^{ab} of a system with action S is defined according to

$$\delta S = \frac{1}{2}\int d^4x\sqrt{-g}\; T^{ab}\delta g_{ab}, \tag{2.4}$$

from the variation $g_{ab}\mapsto g_{ab}+\delta g_{ab}$. To compute the metric variation for the action given in (2.2) we first recall some useful identities:

$$\delta g^{ab} = -g^{ac}g^{bd}\delta g_{cd}; \qquad \delta(\sqrt{-g}) = \frac{1}{2}\sqrt{-g}g^{ab}\delta g_{ab}; \tag{2.5}$$

$$\text{and}\qquad g^{ab}\delta R_{ab} = \nabla^a w_a, \qquad \text{where} \qquad w_a \equiv \nabla^b\left(\delta g_{ab}\right) - g^{cd}\nabla_a\left(\delta g_{cd}\right). \tag{2.6}$$

This last identity, (2.6), follows from work done in lecture. Although brevity tempts us to simply quote Wald's textbook, it is sufficiently important to warrant a full derivation. Therefore, to please the reader, a proof of this identity has been included as an Appendix to this problem set.

We are now prepared to compute the metric variation of the action (2.2). As we proceed, any total divergence will be assumed to integrate to zero.

$$\begin{aligned}\delta S =& \frac{1}{2}\int d^4x\sqrt{-g}\left\{\frac{1}{2}g^{ab}\delta g_{ab}g^{cd}\left(\nabla_c\varphi\nabla_d\varphi + \frac{1}{6}R_{cd}\varphi^2\right) + \delta g^{ab}\left(\nabla_a\varphi\nabla_b\varphi + \frac{1}{6}R_{ab}\varphi^2\right) + \frac{1}{6}g^{ab}\delta R_{ab}\varphi^2\right\},\\ =& \frac{1}{2}\int d^4x\sqrt{-g}\left[\delta g_{ab}\left\{\frac{1}{2}g^{ab}g^{cd}\left(\nabla_c\varphi\nabla_d\varphi + \frac{1}{6}R_{cd}\varphi^2\right) - \left(\nabla^a\varphi\nabla^b\varphi + \frac{1}{6}R^{ab}\varphi^2\right)\right\} + \frac{1}{6}g^{cd}\delta R_{cd}\varphi^2\right],\\ =& \frac{1}{2}\int d^4x\sqrt{-g}\left[\delta g_{ab}\left\{\frac{1}{2}g^{ab}g^{cd}\left(\nabla_c\varphi\nabla_d\varphi + \frac{1}{6}R_{cd}\varphi^2\right) - \left(\nabla^a\varphi\nabla^b\varphi + \frac{1}{6}R^{ab}\varphi^2\right)\right\} + \frac{1}{6}\left(\nabla^c w_c\right)\varphi^2\right],\\ =& \frac{1}{2}\int d^4x\sqrt{-g}\left[\delta g_{ab}\left\{\frac{1}{2}g^{ab}g^{cd}\left(\nabla_c\varphi\nabla_d\varphi + \frac{1}{6}R_{cd}\varphi^2\right) - \left(\nabla^a\varphi\nabla^b\varphi + \frac{1}{6}R^{ab}\varphi^2\right)\right\} - \frac{1}{6}\left(\nabla^c\varphi^2\right)w_c\right].\end{aligned} \tag{2.7}$$

The last term in the expression above is qualitatively different from the first two. Let us try to recast it into a form which makes the δg_{ab}-dependence manifest. Using the definition of w_c and making repeated use of integration by parts, we see

$$\begin{aligned}\int d^4x\sqrt{-g}\;\nabla^a(\varphi^2)w_a &= \int d^4x\sqrt{-g}\;\nabla^a(\varphi^2)\Big(\nabla^b\left(\delta g_{ab}\right) - g^{cd}\nabla_a\left(\delta g_{cd}\right)\Big),\\ &= \int d^4x\sqrt{-g}\;\left(\nabla^b\Big(\delta g_{ab}\nabla^a(\varphi^2)\Big) - \delta g_{ab}\nabla^b\Big(\nabla^a(\varphi^2)\Big) - g^{cd}\nabla_a\Big(\delta g_{cd}\nabla^a(\varphi^2)\Big) + g^{ab}\delta g_{ab}\nabla_c\Big(\nabla^c(\varphi^2)\Big)\right),\\ &= \int d^4x\sqrt{-g}\;\delta g_{ab}\Big(g^{ab}g^{cd}\nabla_c\nabla_d\varphi^2 - \nabla^a\nabla^b\varphi^2\Big).\end{aligned} \tag{2.8}$$

We are now ready to put everything together and find T^{ab}. To make our result a bit more transparent, let us agree to call $g^{cd}\nabla_c\nabla_d \equiv \Box$. Also, the identity $g^{cd}\nabla_c\varphi\nabla_d\varphi = \frac{1}{2}\Box\varphi^2 - \varphi\Box\varphi$ will allow us to tidy up our expressions substantially. Combining all of this, we can continue our work on the total variation (2.7) using the result from (2.8) to find

$$\begin{aligned}\delta S &= \frac{1}{2}\int d^4x\sqrt{-g}\delta g_{ab}\left\{\frac{1}{2}g^{ab}g^{cd}\left(\nabla_c\varphi\nabla_d\varphi + \frac{1}{6}R_{cd}\varphi^2\right) - \left(\nabla^a\varphi\nabla^b\varphi + \frac{1}{6}R^{ab}\varphi^2\right) - \frac{1}{6}\left(g^{ab}g^{cd}\nabla_c\nabla_d\varphi^2 - \nabla^a\nabla^b\varphi^2\right)\right\},\\
&= \frac{1}{2}\int d^4x\sqrt{-g}\delta g_{ab}\left\{\frac{1}{2}g^{ab}g^{cd}\left(\nabla_c\varphi\nabla_d\varphi - \frac{1}{3}\nabla_c\nabla_d\varphi^2 + \frac{1}{6}R_{cd}\varphi^2\right) - \nabla^a\varphi\nabla^b\varphi - \frac{1}{6}R^{ab}\varphi^2 + \frac{1}{6}\nabla^a\nabla^b\varphi^2\right\},\\
&= \frac{1}{2}\int d^4x\sqrt{-g}\delta g_{ab}\left\{\frac{1}{2}g^{ab}\left(\frac{1}{6}\Box\varphi^2 + \frac{1}{6}R\varphi^2 - \varphi\Box\varphi\right) - \frac{1}{3}\nabla^a\nabla^b\varphi^2 + \varphi\nabla^a\nabla^b\varphi - \frac{1}{6}R^{ab}\varphi^2\right\}. \qquad (2.9)\end{aligned}$$

This allows us to read-off

$$\therefore T^{ab} = \frac{1}{2}g^{ab}\left(\frac{1}{6}\Box\varphi^2 + \frac{1}{6}R\varphi^2 - \varphi\Box\varphi\right) - \frac{1}{3}\nabla^a\nabla^b\varphi^2 + \varphi\nabla^a\nabla^b\varphi - \frac{1}{6}R^{ab}\varphi^2. \qquad (2.10)$$

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As anyone who's seen conformal field theory knows, the trace of the stress-energy tensor must vanish. Let's see how this 'magically' works out in the situation considered presently.

$$\begin{aligned}g_{ab}T^{ab} &= \frac{1}{3}\Box\varphi^2 + \frac{1}{3}R\varphi^2 - 2\varphi\Box\varphi - \frac{1}{3}\Box\varphi^2 + \varphi\Box\varphi - \frac{1}{6}R\varphi^2,\\
&= \frac{1}{6}R\varphi^2 - \varphi\Box\varphi,\\
&= -\varphi\left(\Box\varphi - \frac{1}{6}R\varphi\right),\\
&= 0.\end{aligned}$$

Notice that the last line required using the equations of motion—which wasn't entirely anticipated—at least by us.

Problem 3: Killing Vectors

a. If $\zeta_a(x)$ is a Killing field and $p^a(\lambda)$ is the tangent vector to a geodesic curve $\gamma(\lambda)$, then $p^a\zeta_a(x)$ is constant along γ.

proof: The derivative of $p^a\zeta_a$ along γ is

$$p^b\nabla_b\left(p^a\zeta_a\right) = p^a p^b\nabla_b\zeta_a + \zeta_a p^b\nabla_b p^a. \tag{3.1}$$

The first term vanishes because $p^a p^b$ is symmetric while $\nabla_b\zeta_a$ is antisymmetric (because it is Killing). The second term vanishes because p^a is the tangent of a geodesic, which practically by definition implies that it obeys the geodesic equation, $p^b\nabla_b p^a = 0$.

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b. We are to list the ten independent Killing fields of Minkowski spacetime.

The ten independent Killing fields correspond to the ten generators of the Poincaré algebra: four translations, three rotations, and three boosts. Given in terms of the basis vectors $\vec{e}_a$, we the Killing *vector* fields are therefore

Translations :	$\vec{e}_t,$	$\vec{e}_x,$	$\vec{e}_y,$	$\vec{e}_z;$
Rotations :	$y\vec{e}_x - x\vec{e}_y,$	$z\vec{e}_y - y\vec{e}_z,$	$x\vec{e}_z - z\vec{e}_x;$	
Boosts :	$x\vec{e}_t + t\vec{e}_x,$	$y\vec{e}_t + t\vec{e}_y,$	$z\vec{e}_t + t\vec{e}_z;$	

Each of these ten vector fields manifestly satisfies Killing's equation. That they are linearly independent is also manifest[4].

c. If ζ_a and η_a are Killing fields and α, β constants, then $\alpha\zeta_a + \beta\eta_a$ is Killing.

proof: As should be obvious to all but the most casual observer,

$$\nabla_b\left(\alpha\zeta_a + \beta\eta_z\right) = \alpha\nabla_b\zeta_a + \beta\nabla_b\eta_a = -\alpha\nabla_a\zeta_b - \beta\nabla_a\eta_b = -\nabla_a\left(\alpha\zeta_b + \beta\eta_b\right), \tag{3.2}$$

because, being constants, α, β commute with the gradient and ζ_a, η_a are Killing. Therefore equation (3.2) implies that $(\alpha\zeta_a + \beta\eta_a)$ is Killing.

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d. We are to show that Lorentz transformations of the Killing vector fields listed in part (**b**) above give rise to linear recombinations of the same fields with constant coefficients.

Because every Lorentz transformation can be built from infinitesimal ones, it is sufficient to demonstrate the claim for infinitesimal Lorentz transformations. And this makes our work exceptionally easy. Infinitesimal Lorentz transformations are simply the identity plus a constant multiple of the generators of the Lorentz algebra; but (the last six of) the Killing fields listed in part (**b**) are nothing but these Lorentz generators.

Therefore, any infinitesimal Lorentz transformation of the Killing fields listed in part (**b**) is a linear combination of those same Killing fields with constant coefficients. And by extension, the same is true for any finite Lorentz transformation.

[4] Although we should add that we were not requested to demonstrate this—so our lack of exposition here should be forgiven.

Appendix

In problem 2 we made use of an identity that didn't obviously follow from the work in lecture. We remedy that deficiency presently[5].

Lemma: Under the variation $g_{ab} \mapsto g_{ab} + \delta g_{ab}$,

$$g^{ab}\delta R_{ab} = \nabla^a w_a, \qquad \text{where} \qquad w_a \equiv \nabla^b\left(\delta g_{ab}\right) - g^{cd}\nabla_a\left(\delta g_{cd}\right). \tag{A.1}$$

proof: We may begin with the related expression derived in lecture,

$$g^{ab}\delta R_{ab} = \nabla_a\Big(g^{bc}\delta\Gamma^a_{bc} - g^{ac}\delta\Gamma^b_{cb}\Big). \tag{A.2}$$

Rearranging this we find

$$g^{ab}\delta R_{ab} = \nabla^a\Big\{g^{bc}g_{ad}\delta\Gamma^d_{bc} - \delta\Gamma^d_{ad}\Big\}; \tag{A.3}$$

therefore, it suffices to show that the term in brackets is equal to w_a. Expanding this expression and using symmetry to collect and cancel terms, we find

$$\begin{aligned}
g^{bc}g_{ad}\delta\Gamma^d_{bc} - \delta\Gamma^d_{ad} =& \frac{1}{2}g^{bc}g_{ad}\delta g^{de}\Big(g_{be,c} + g_{ce,b} - g_{bc,e}\Big) + \frac{1}{2}g^{bc}\delta^e_a\Big(\delta g_{be,c} + \delta g_{ce,b} - \delta g_{bc,e}\Big)\\
&- \frac{1}{2}\delta g^{be}\Big(g_{ae,b} + g_{be,a} - g_{ab,e}\Big) - \frac{1}{2}g^{be}\Big(\delta g_{ae,b} + \delta g_{be,a} - \delta g_{ab,e}\Big),\\
=& -\frac{1}{2}g^{bc}g^{ef}\delta^d_a\delta g_{de}\Big(g_{bf,c} + g_{cf,b} - g_{bc,f}\Big) + \frac{1}{2}g^{bc}\Big(\delta g_{ba,c} + \delta g_{ca,b} - \delta g_{bc,a}\Big) + \frac{1}{2}\delta g_{de}g^{dc}g^{ef}g_{cf,a} - \frac{1}{2}g^{be}\delta g_{be,a},\\
=& -g^{bc}g^{ef}\delta^d_a\delta g_{de}g_{bf,c} - \frac{1}{2}g^{bc}g^{ef}\delta^d_a\delta g_{de}g_{bc,f} + g^{bc}\delta g_{ba,c} - \frac{1}{2}g^{bc}\delta g_{bc,a} + \frac{1}{2}\delta g_{de}g^{dc}g^{ef}g_{cf,a} - \frac{1}{2}g^{bc}\delta g_{bc,a},\\
=& g^{bc}\delta g_{ba,c} - g^{bc}\delta g_{bc,a} - g^{bc}g^{ef}g_{bf,c}\delta g_{ae} - \frac{1}{2}g^{bc}g^{ef}g_{bc,f}\delta g_{ae} + \frac{1}{2}g^{dc}g^{ef}g_{cf,a}\delta g_{de}.
\end{aligned}$$

Expanding the first two terms in the expression above in terms of covariant derivatives and Christoffel symbols, we observe

$$g^{bc}\delta g_{ab,c} = \nabla^b\left(\delta g_{ab}\right) + g^{bc}\delta g_{eb}\Gamma^e_{ac} + g^{bc}\delta g_{ae}\Gamma^e_{bc}, \tag{A.4}$$

and

$$g^{bc}\delta g_{bc,a} = g^{bc}\nabla_a\left(\delta g_{bc}\right) + g^{bc}\delta g_{be}\Gamma^e_{ac} + g^{bc}\delta g_{ec}\Gamma^e_{ba}. \tag{A.5}$$

Noting that the terms with the covariant derivatives are what we are looking for—together, they give w_a. Putting everything together,

$$g^{bc}g_{ad}\delta\Gamma^d_{bc} - \delta\Gamma^d_{ad} = w_a + \left\{g^{bc}\delta g_{ae}\Gamma^e_{bc} - g^{bc}\delta g_{ec}\Gamma^e_{ba} + g^{bc}g^{ef}\left(\frac{1}{2}g_{fc,a}\delta g_{be} + \frac{1}{2}g_{bc,f}\delta g_{ae} - g_{bf,c}\delta g_{ae}\right)\right\}. \tag{A.6}$$

All that remains is for us to show that the terms in curly brackets above vanish. To do this, we will expand our expressions one last time—this time using the definition of the Christoffel symbols for a metric connection. Doing so, we find

$$\begin{aligned}
g^{bc}g_{ad}\delta\Gamma^d_{bc} - \delta\Gamma^d_{ad} - w_a = g^{bc}g^{ef}\Big\{& \frac{1}{2}g_{fc,a}\delta g_{be} + \frac{1}{2}g_{bc,f}\delta g_{ae} - g_{bf,c}\delta g_{ae}\\
&- \frac{1}{2}g_{bf,a}\delta g_{ec} - \frac{1}{2}g_{af,b}\delta g_{ec} + \frac{1}{2}g_{ba,f}\delta g_{ec}\\
&+ \frac{1}{2}g_{bf,c}\delta g_{ae} + \frac{1}{2}g_{cf,b}\delta g_{ae} - \frac{1}{2}g_{bc,f}\delta g_{ae}\Big\},\\
=& \, 0.
\end{aligned}$$

Here we have indicated the terms that cancel together in matching colours. With this, we have shown that

$$\therefore g^{ab}\delta R_{ab} = \nabla^a\Big\{g^{bc}g_{ad}\delta\Gamma^d_{bc} - \delta\Gamma^d_{ad}\Big\} = \nabla^a\Big\{\nabla^b\left(\delta g_{ab}\right) - g^{cd}\nabla_a\left(\delta g_{cd}\right)\Big\} = \nabla^a w_a. \tag{A.7}$$

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[5] We hope that there is an easier way to prove the following Lemma. But alas! too little time to be brief. Breviloquence is a time-consuming luxury.

Physics 523, General Relativity

Homework 7

Due Wednesday, 6$^{\text{th}}$ December 2006

Jacob Lewis Bourjaily

Problem 1

Consider a gyroscope moving in circular orbit of radius R about a static, spherically-symmetric planet of mass m.

a. We are to derive the equations of motion for the gyroscopic spin vector as a function of azimuthal angle and show that the spin precesses about the direction normal to the orbital plane.

This calculation will be far from elegant, and will probably not give rise to much insight. Nevertheless, we start by recalling the Lagrangian describing a particle's worldline(in the $\theta = \frac{\pi}{2}$ plane) in a static, isotropic spacetime,

$$\mathscr{L} = -g_{ab}u^a u^b = f(r)(u^t)^2 - \frac{1}{f(r)}(u^r)^2 - r^2(u^\varphi)^2, \tag{1.1}$$

where $u^a \equiv \frac{dx^a}{d\tau}$ for some affine parameter τ. Because our analysis will be limited to circular geodesics, we will not have much use for the u^r coordinate; however, its equation of motion will be necessary to relate the various integrals of motion. First observe that u^φ is non-dynamical in the Lagrangian and so it gives us our first integral of motion,

$$J \equiv r^2 u^\varphi. \tag{1.2}$$

For circular geodesics, u^a will of course only have 0 and φ components; u^t is also non-dynamical, and so we are free to set u^t by the normalization of the affine parameter τ:

$$u^2 = -g_{ab}u^a u^b = f(R)(u^t)^2 - \frac{J^2}{R^2} \equiv 1, \qquad \Longrightarrow \qquad u^t = \sqrt{\frac{1}{f(R)}\left(1 + \frac{J^2}{R^2}\right)}. \tag{1.3}$$

Now, it is easy to see that the equation of motion for the r-component is

$$-2\frac{\ddot{r}}{f(r)} + 2\frac{\dot{r}^2}{f^2(r)}f'(r) = -\frac{\dot{r}^2}{f^2(r)} - 2\frac{J^2}{r^3} + f'(r)(u^t)^2. \tag{1.4}$$

Because we are looking for solutions where both $\dot{r}$ and $\ddot{r}$ vanish—and $r = R$—we see at once that this implies the relation

$$\begin{aligned} J^2 = \frac{1}{2}f'(R)(u^t)^2 R^3 &= \frac{m}{R^2}\frac{R^4}{(R-2m)}\left(1 + \frac{J^2}{R^2}\right), \\ &= \frac{mR^2}{R-2m}\frac{1}{1 - \frac{m}{R-2m}}, \\ &= \frac{mR^2}{R-3m}. \end{aligned} \tag{1.5}$$

Above, we made use of the definition of the Schwarzschild metric's $f(r) = 1 - \frac{2m}{r}$. We have now completely specified the circular geodesic of radius R in which we are interested.

The direction of a gyroscope's spin is therefore simply a vector S^a which satisfies the orthogonality condition $u^a S^b g_{ab} = 0$ along the geodesic. Recall that two parallelly-transported vectors have the property that the gradient of their scalar product vanishes. This immediately allows us to write down the equation for the evolution of the components of S^a along τ,

$$\frac{dS_a}{d\tau} = \Gamma^b_{ac} S_b u^c, \tag{1.6}$$

which, upon using the Christoffel symbols for the Schwarzschild metric[1], becomes

$$\frac{dS_t}{d\tau} = \Gamma^r_{tt} S_r u^t = \frac{1}{2} f(R) f'(R) S_r u^t = \frac{1}{2}\sqrt{\frac{1}{f(R)}\left(1+\frac{J^2}{R^2}\right)} S_r; \tag{1.7}$$

$$\frac{dS_r}{d\tau} = \Gamma^t_{rt} S_t u^t + \Gamma^\varphi_{r\varphi} S_\varphi u^\varphi = \frac{f'(R)}{2f(R)}\sqrt{\frac{1}{f(R)}\left(1+\frac{J^2}{R^2}\right)} S_t + \frac{J}{R^3} S_\varphi; \tag{1.8}$$

$$\frac{dS_\theta}{d\tau} = \Gamma^\varphi_{\theta\varphi} S_\varphi u^\theta + \Gamma^\theta_{\theta r} S_\theta u^r + \Gamma^r_{\theta\theta} S_r u^\theta = 0; \tag{1.9}$$

$$\frac{dS_\varphi}{d\tau} = \Gamma^\theta_{\varphi\varphi} S_\theta u^\varphi + \Gamma^r_{\varphi\varphi} S_r u^\varphi = -\frac{J}{R} f(R) S_r. \tag{1.10}$$

This almost completes our analysis. Indeed, notice that the above system of equations implies that the θ-component of the gyroscope's spin is fixed. All the motion of S^a as it is transported along τ is confined to the plane normal to $\hat{\theta}$. Therefore, we may conclude that the gyroscope will precess about the axis normal to its orbital plane.

The finicky reader may object that the system of equations (1.6-9) are over-specified. To be thorough we should eliminate redundancy. The first of the relations among these expressions comes from the orthogonality condition on the spin vector $S_a u^a = 0$. In components this reads

$$S_t u^t + S_\varphi u^\varphi = 0 \qquad \Longrightarrow \qquad S_t \sqrt{\frac{1}{f(R)}\left(1+\frac{J^2}{R^2}\right)} = -\frac{J}{R^2} S_\varphi. \tag{1.11}$$

Also, it is more physically interesting to compute evolution relative to the angle φ as observed by a stationary observer on the planet. Replacing S_t in favour of S_φ and making us of the fact $\frac{d\tau}{d\varphi} = \frac{R^2}{J}$,

$$\begin{aligned}
\frac{dS_t}{d\varphi} &= \frac{R^2}{2J}\sqrt{\frac{1}{f(R)}\left(1+\frac{J^2}{R^2}\right)} S_r;\\
\frac{dS_r}{d\varphi} &= \left(\frac{1}{R} - \frac{f'(R)}{2f(R)}\right) S_\varphi;\\
\frac{dS_\varphi}{d\varphi} &= -R f(R) S_r;\\
\frac{dS_\theta}{d\varphi} &= 0.
\end{aligned}$$

The last redundancy to take care of comes from the geodesic equation for $S^a S^b g_{ab}$—namely, that this scalar is preserved. Let us choose to normalize $S^a S^b g_{ab} = +1$ so that

$$\begin{aligned}
1 &= -\frac{1}{f(R)} S_t^2 + f(R) S_r^2 + \frac{1}{R^2} S_\theta^2 + \frac{1}{R^2} S_\varphi^2,\\
&= \frac{1}{R^2} S_\varphi^2 \left(1 - \frac{J^2}{R^2}\frac{1}{(1+\frac{J^2}{R^2})}\right) + f(R) S_r^2 + \frac{1}{R^2} S_\theta^2,\\
&= \frac{S_\varphi^2}{(R^2+J^2)} + f(R) S_r^2 + \frac{1}{R^2} S_\theta^2.
\end{aligned}$$

Bearing in mind that S_θ is a constant of motion, me may therefore write

$$S_\varphi^2 = f(R)\left(R^2+J^2\right)\left(\frac{1}{f(R)} - \frac{1}{f(R)R^2} S_\theta^2 - S_r^2\right) \qquad \text{or} \qquad S_r^2 = \frac{1}{f(R)\left(R^2+J^2\right)}\left((R^2+J^2) - \frac{(R^2+J^2)}{R^2} S_\theta^2 - S_\varphi^2\right). \tag{1.12}$$

The two substantive equations of motion are clearly $\frac{dS_r}{d\varphi}$ and $\frac{dS_\varphi}{d\varphi}$. Squaring the equations derived above, and using the normalization condition to reexpress unlike components,

[1]And specializing to the obvious coordinate choice $\theta \mapsto \frac{\pi}{2}$ everywhere it is encountered.

we find

$$\left(\frac{dS_r}{d\varphi}\right)^2 = \left(\frac{1}{R} - \frac{f'(R)}{2f(R)}\right)^2 f(R)\left(R^2 + J^2\right)\left(\frac{1}{f(R)} - \frac{1}{f(R)R^2}S_\theta^2 - S_r^2\right), \tag{1.13}$$

$$\left(\frac{dS_\varphi}{d\varphi}\right)^2 = R^2 f(R)^2 \frac{1}{f(R)\left(R^2 + J^2\right)}\left((R^2 + J^2) - \frac{(R^2 + J^2)}{R^2}S_\theta^2 - S_\varphi^2\right). \tag{1.14}$$

Despite how horrendous these equations look at first glance, the structure present is very simple. Notice that any function $g(\varphi) \equiv \alpha\cos(\beta\varphi)$ (or $g(\varphi) = \alpha\sin(\beta\varphi)$) satisfies the differential equation

$$\left(\frac{d}{d\varphi}\alpha\cos(\beta\varphi)\right)^2 = \left(\frac{dg}{d\varphi}\right)^2 = \beta^2\left(\alpha^2 - g^2(\varphi)\right). \tag{1.15}$$

The initial conditions will determine the coefficients β, α, but the general result is now complete[2].

b. If the gyroscope studied in part (a) is observed to have its spin entirely in the orbital plane, then how much precession is observed? What is the precession observed in the case of a satellite in low-earth orbit?

When we finished part (a), we had done everything necessary to determine the precession of a gyroscope in circular orbit given suitable boundary conditions. In the case of a gyroscope with spin lying in its orbital plane, $S_\theta = 0$. This greatly simplifies our algebra. Let us proceed to simplify the expressions (1.13) and (1.14).

Using the expression for the angular momentum J (1.5) derived above, expanding, and collecting terms, we find

$$\begin{aligned}
\left(\frac{dS_r}{d\varphi}\right)^2 &= \frac{\left(2f(R) - Rf'(R)\right)^2}{4R^2 f(R)}\left(R^2 + J^2\right)\left(\frac{1}{f(R)} - S_r^2\right),\\
&= \frac{\left(2f(R) - Rf'(R)\right)^2}{4R(R-2m)}\frac{R^2\left(R - 2m\right)}{(R-3m)}\left(\frac{1}{f(R)} - S_r^2\right),\\
&= \frac{\left(R - 2m - m\right)^2}{R}\frac{1}{(R-3m)}\left(\frac{1}{f(R)} - S_r^2\right),\\
&= \frac{R-3m}{R}\left(\frac{1}{f(R)} - S_r^2\right).
\end{aligned}$$

As described in part (a), a solution to this differential equation is of the form $\alpha\cos(\beta\varphi)$.If we define φ so that S_r is maximum when $\varphi = 0$, we have

$$\therefore S_r(\varphi) = \sqrt{\frac{R}{R-2m}}\cos\left(\sqrt{\frac{R-3m}{R}}\varphi\right). \tag{1.16}$$

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We can follow the same analysis, or simply differentiate this to obtain S_φ. Either way, one finds that

$$\therefore S_\varphi(\varphi) = -R\sqrt{\frac{R-2m}{R-3m}}\sin\left(\sqrt{\frac{R-3m}{R}}\varphi\right). \tag{1.17}$$

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Using the fact that S_a is a unit covector, we know that the angle between $S^a(0)$ and $S^a(2\pi n)$ after n orbits will be given by

$$\cos(\vartheta) = f^{-1}(R)f(R)\cos\left(\sqrt{\frac{R-3m}{R}}2\pi n\right), \qquad \text{or,} \qquad \vartheta = 2\pi n\sqrt{\frac{R-3m}{R}}, \tag{1.18}$$

which is well-approximated by

$$\delta\vartheta \simeq \frac{3m}{R}\pi n. \tag{1.19}$$

For a low-earth orbit satellite in circular motion, we therefore expect the gyroscopic precession to be on the order of 1.66×10^{-9} degrees per orbit.

[2] If you had hoped to see us simplify these expressions enormously, please read our solution to part (b) below.

Problem 2

Consider a 4+1-dimensional AdS spacetime described by the metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_3^2, \qquad \text{where} \qquad f(r) = 1 + r^2 - \frac{\mu}{r^2}. \tag{2.1}$$

We are to determine the radial coordinate of the black hole horizon, calculate the proper time of a massive object to free-fall from the surface of the black hole to the singularity at $r = 0$, and determine the radius and period of the null-circular orbit.

The horizon radius is that for which $f(r)$ vanishes. A child's experience with the quadratic formula is sufficient to see that there is exactly one real root of $f(r)$ and this corresponds to a horizon radius of

$$\therefore r_h = \sqrt{\frac{1}{2}\left(\sqrt{4\mu^2+1}-1\right)}. \tag{2.2}$$

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To calculate the proper time for free-fall from the horizon, we need to quickly derive the equations for motion in only the r-direction. Because we'll need the angular dependence later, we'll start a bit more generally. First, look at the Lagrangian for the particle's motion (its worldline),

$$\mathscr{L} = -g_{ab}u^a u^b = f(r)\dot{t}^2 - \frac{1}{f(r)}\dot{r}^2 - r^2\dot{\varphi}^2. \tag{2.3}$$

Now, as always, a '˙' indicates differentiation with respect to an affine parameter, say τ, along the worldline of the particle. We will eventually impose the normalization condition (think Lagrange multipliers)

$$\kappa \equiv -g_{ab}u^a u^b, \tag{2.4}$$

where $\kappa = 1$ for time-like worldlines and $\kappa = 0$ for null. The first thing that should be apparent form the Lagrangian is that there are two non-interacting degrees of freedom, $\dot{t}$ and $\dot{\varphi}$, giving rise to two integrals of motion[3]

$$E \equiv f(r)\dot{t}, \qquad \text{and} \qquad J \equiv r^2\dot{\varphi}. \tag{2.5}$$

Now, in the case of purely radial motion of a massive object, $J = 0$ and $\kappa = 1$; so we are left with only

$$1 = \frac{1}{f(r)}E^2 - \frac{1}{f(r)}\dot{r}^2, \qquad \Longrightarrow \dot{r}^2 = E^2 - f(r). \tag{2.6}$$

Notice that this means that E must be chosen so that $\dot{r}^2 = 0 = E^2 - f(R)$ for some R. In the problem under consideration, we want to find the motion of an object dropped *from rest* at $R = r_h$—and r_h is defined to be such that $f(r_h) = 0$. Therefore, $E^2 = 0$ for our present problem, and

$$\frac{dr}{d\tau} = \sqrt{-f(r)}; \tag{2.7}$$

which is easy enough to formally invert:

$$\tau = \int_0^{r_h} \frac{dr}{\sqrt{-f(r)}}. \tag{2.8}$$

Our computer algebra software had no difficulty evaluating this, showing that

$$\therefore \tau = \frac{\pi}{4} - \frac{1}{2}\text{arccot}\left(2\sqrt{\mu}\right). \tag{2.9}$$

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[3]We could have framed this discussion in terms of Killing fields, but we'll stick to Euler while we can.

Lastly, we are asked to find the radius at which light can orbit circularly, and determine the coordinate time of this orbit's period. To do this, we need only to re-instate J into our expression for $\dot{r}^2$ and set $\kappa \to 0$ for null geodesics. Written suggestively, this gives

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}f(r)\frac{J^2}{r^2} = \frac{1}{2}E^2. \tag{2.10}$$

Reminiscent of effective potentials, we are inspired to consider an analogue problem in 1+1-dimensions governed by the effective potential

$$V_{eff} = \frac{J^2}{2r^2}\left(1 + r^2 - \frac{\mu}{r^2}\right). \tag{2.11}$$

This effective potential has only one turning point, at

$$-\frac{J^2}{r^3} + 2\frac{\mu J^2}{r^5} = 0 \qquad \Longrightarrow \qquad r = \sqrt{2\mu}. \tag{2.12}$$

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This is the radius at which there are circular, null geodesics—as evidenced by the fact that $\dot{r} = 0$ at this radius. Inserting $r = \sqrt{2\mu}$ into (2.10),

$$E^2 = f(\sqrt{2\mu})\frac{J^2}{2\mu} = J^2\left(1 + \frac{1}{4\mu}\right) \qquad \Longrightarrow \qquad \frac{J^2}{E^2} = \frac{4\mu}{4\mu+1}. \tag{2.13}$$

This is needed for us to compute the coordinate-time orbit period. Recall from our definitions of E and J that

$$\frac{d\varphi}{dt} = \frac{d\varphi}{d\tau}\frac{d\tau}{dt} = \frac{J}{r^2}\frac{f(r)}{E}, \tag{2.14}$$

—which when combined with the above implies

$$\frac{d\varphi}{dt} = \frac{1}{2}\sqrt{\frac{4\mu+1}{\mu}}. \tag{2.15}$$

This is trivially integrated. We find that the coordinate time of one orbit is

$$\therefore t_p = 4\pi\sqrt{\frac{\mu}{4\mu+1}}. \tag{2.16}$$

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Problem 3

Consider a clock in circular orbit at radius $R = 10m$ about a spherically symmetric star.

a. We are to determine the proper time of the $R = 10m$ orbit.

We can draw heavily on our work above. Using the notation and conventions of problem one, we see that

$$\tau_p = \int d\tau = \int \frac{d\tau}{d\varphi}d\varphi = \frac{R^2}{J}\int d\varphi = 2\pi\frac{R^2}{J}.$$

Using our equation (1.5) for J at a given R, we find

$$\therefore \tau_p = 2\pi\frac{R\sqrt{R-3m}}{\sqrt{m}}. \tag{3.a.1}$$

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For the particular question at hand, $r = 10m$, we find the period to be

$$\tau_p = 20\sqrt{7}\pi m. \tag{3.a.2}$$

b. If once each orbit the clock transmits a signal to a distant observer, what time interval does this observer observe?

The time coordinate t is precisely the time observed by a distant observer in Schwarzschild geometry. Therefore, we simply modify the calculation above as follows.

$$t_p = \int dt = \int \frac{dt}{d\tau}\frac{d\tau}{d\varphi}d\varphi = 2\pi\frac{R^2}{J}\sqrt{\frac{1}{f(R)}\left(1+\frac{J^2}{R^2}\right)} = 2\pi\frac{R^{3/2}}{\sqrt{m}}. \tag{3.b.1}$$

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We point out that this agrees identically with Kepler's third law.

For $R = 10m$ we find

$$t_p = 20\sqrt{10}\pi m. \tag{3.b.2}$$

c. If another observer is stationed in stationary orbit at $R = 10m$, what time do their clocks report as the orbit period?

The proper time of a shuttle on a fixed distance from the origin is given by

$$\Delta\tau^2 = \left(1 - \frac{2m}{R}\right)\Delta t^2 + 0; \tag{3.c.1}$$

$$\therefore \tau_p = \frac{2\pi R^{3/2}}{\sqrt{m}}\sqrt{1-\frac{2m}{R}} = 40\sqrt{2}\pi. \tag{3.c.2}$$

d. We are to redux the calculation of part (b), this time for the case of an orbit at $R = 6m$ where $m = 14M_\odot$ and explain why this bound is interesting.

It is not very challenging to simply put real numbers into our calculation above; we find

$$t_p = 2\pi\frac{R^{3/2}}{\sqrt{m}} = 2\times 10^{-8}\text{ s}. \tag{3.d.1}$$

The reason why this is the minimum for fluctuations to be observed from x-ray sources is that $R = 6m$ is the minimum radius at which there is a stable circular orbit.

e. If forty years go by according to the watch of a distant observer, how long has passed on a spaceship orbiting at $R = 6m$ for $m = 14M_\odot$.

The one thing that both observers will agree on is that during the interval in question the orbiting observer made 6.6×10^{16} orbits; this was of course calculated using the result of part (b) above. Using part (a), we learn that the person living inside the orbiting spaceship observed 28 years pass to complete these 6.6×10^{16} orbits.

Physics 523, General Relativity

Homework 8

Due Wednesday, 13th December 2006

Jacob Lewis Bourjaily

Problem 1

Consider a flat FRW universe, governed by the metric

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right), \tag{1.1}$$

filled with only relativistic material and a cosmological constant Λ; say the space has a big bang at coordinate time $t = 0$. We are to calculate the cosmic scale factor $a^2(t)$ up to an overall normalization and describe the asymptotic motion of photon travelling along the positive $\hat{x}$-axis.

Let us begin by quickly reviewing the Einstein equations for this universe[1]. We note that the components of the Ricci tensor and scalar curvature for this metric are

$$R_{tt} = 3\frac{\ddot{a}}{a}, \qquad R_{ij} = -\delta_{ij}\left(a\dot{a} + 2\dot{a}^2\right), \quad \text{and} \quad R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right). \tag{1.2}$$

And recall that a (single-component) perfect fluid with equation of state $p = w\rho$ has a stress energy tensor given by

$$T^a{}_b = \rho \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & w & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & w \end{pmatrix}. \tag{1.3}$$

The Einstein field equations are then

$$R_{ab} - \frac{1}{2}g_{ab}R = -8\pi G\left(T_{ab} - \frac{\Lambda}{8\pi G}g_{ab}\right), \tag{1.4}$$

where Λ is the cosmological constant[2]. Writing out the 'tt' Einstein equation, we find

$$3\frac{\ddot{a}}{a} - 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = -3\frac{\dot{a}^2}{a^2} = -8\pi G\left(\rho + \frac{\Lambda}{8\pi G}\right), \tag{1.5}$$

which implies

$$\therefore \dot{a}^2 = \frac{8\pi G}{3}a^2\left(\rho + \frac{\Lambda}{8\pi G}\right). \tag{1.6}$$

It turns out that we won't actually need any of the other Einstein equations.

The last equation we need relates the energy density to the cosmic scale factor. This comes about from the conservation of energy[3],

$$\frac{d}{da}\left(\rho a^3\right) = -3pa^2. \tag{1.7}$$

This equation is implied by the divergencelessness of T_{ab}, which is itself just a restatement of the Bianchi identity for G_{ab}.

At any rate, we can use the conservation of energy for a fluid with equation of state $p = w\rho$ to determine how ρ varies as a function of $a(t)$. We see

$$\frac{d}{da}\left(\rho a^3\right) = 3a^2\rho + a^3\frac{d\rho}{da} = -3w\rho a^2,$$
$$\implies \frac{d\rho}{da} = -3(1+w)a^{-1}\rho.$$

Solving this equation by simple integration, we have

$$\log(\rho) = -3(1+w)\log(a) + \text{const.} \qquad \implies \qquad \rho \propto a^{-3(1+w)}. \tag{1.8}$$

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[1] Because we are more familiar with the notation used by Weinberg—despite its oddities—our derivations will follow his. *However*, we will use $a(t)$ to denote the cosmic scale factor so as to avoid confusion with R.

[2] It is quite common to see Λ defined with the $8\pi G$ absorbed into its definition. We prefer to keep it structurally more similar to the metric than the stress-energy (which follows from the paradigm that Λ is a metric parameter as opposed to a vacuum energy).

[3] That this is a statement of the conservation of energy can be understood as follows: the amount of energy in a comoving box of size a^3 is just ρa^3; because we consider only a perfect fluid, as the box expands, the only leakage arises from the 'pressure' at the sides of the box—which has surface area $6a^2$. However, only half of this is lost because only half of the pressure along the faces of the box is due to 'outgoing' flow; so there is a net loss of $3pa^2$ worth of energy.

Using the fact that a relativistic fluid has an equation of state $w = \frac{1}{3}$, we observe that

$$\rho = \frac{3\beta^2}{8\pi G}a^{-4}, \tag{1.9}$$

where β^2 is a constant of integration.

Putting this into the Einstein equation (1.6),

$$\dot{a}^2 = \frac{8\pi G}{3}a^2\left(\frac{3\beta^2}{8\pi G}a^{-4} + \frac{\Lambda}{8\pi G}\right) = \frac{1}{a^2}\left(\beta^2 + \frac{\Lambda}{3}a^4\right). \tag{1.10}$$

This ordinary differential equation can be integrated directly[4].

$$t = \int_0^t dt = \int_0^a \frac{a'\ da'}{\sqrt{\beta^2 + \frac{\Lambda}{3}a'^4}} = \sqrt{\frac{3}{4\Lambda}}\text{arcsinh}\left(\frac{a^2}{\beta}\sqrt{\frac{\Lambda}{3}}\right);$$

$$\therefore a^2(t) = \beta\sqrt{\frac{3}{\Lambda}}\sinh\left(2t\sqrt{\frac{\Lambda}{3}}\right). \tag{1.11}$$

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Let us now check that our solution agrees with the required boundary conditions. First, $a(t=0) = 0$, as required; this shows that we were not unjustified in our organization of constants of integration when solving the differential equations above. Also, at very early times or when Λ is very small,

$$a^2(t) = \beta\sqrt{\frac{3}{\Lambda}}\sinh\left(2t\sqrt{\frac{\Lambda}{3}}\right) \approx 2\beta t, \qquad \text{for } t \to 0, \tag{1.12}$$

which is precisely what we would have obtained if setting $\Lambda = 0$ in (1.6). Similarly, in late times

$$a^2(t) \approx \beta\sqrt{\frac{3}{4\Lambda}}e^{2t\sqrt{\Lambda/3}}, \qquad \text{for } t \to \infty, \tag{1.13}$$

which is what we would have obtained if we had neglected the radiation density ρ altogether in equation (1.6).

Now, the motion of a photon in this space is entirely controlled by the condition that its worldline is null. For motion along the $\hat{x}$-axis, this is simply the statement that

$$0 = -dt^2 + a^2(t)dx^2, \qquad \Longrightarrow \qquad dx = \frac{dt}{a(t)}. \tag{1.14}$$

Again, this can be integrated—at least formally—so that if motion starts at the origin at time $t = 0$ then

$$x(t) = \int_0^t \frac{dt'}{a(t')}. \tag{1.15}$$

Although this integral can be done analytically in terms of hypergeometric functions—(what can't?)—it is far from illuminating. Therefore, rather than computing the light trajectory $x(t)$ analytically for a generic two-component universe, let us analyze its motion in the asymptotic regions of interest.

We showed above that for very early times,

$$a(t) \approx \sqrt{2\beta t}, \qquad \Longrightarrow \qquad x(t) = \sqrt{\frac{2}{\beta}}\sqrt{t}. \tag{1.16}$$

Comparing this with the notation of the problem set, we have

$$\sqrt{\frac{2}{\beta}} = \left(\frac{3}{8\pi GB}\right)^{1/4} \qquad \Longrightarrow \qquad B = \frac{3\beta^2}{8\pi G}; \tag{1.17}$$

so the problem set's B is such that $\rho = Ba^{-4}$—which we could have guessed.

[4]Several horrendous integrals appear in this problem set; most were solved with the aide of a computer algebra package.

Alternatively, for late times we should use the approximation (1.13) which gives

$$\Delta x = \left(\frac{4\Lambda}{3\beta^2}\right)^{1/4} \int_{t_i}^{t_i+\Delta t} dt'\; e^{-t'\sqrt{\Lambda/3}} = \left(\frac{12}{\beta^2\Lambda}\right)^{1/4} e^{-t_i\sqrt{\Lambda/3}} \left(1 - e^{-\Delta t\sqrt{\Lambda/3}}\right). \tag{1.18}$$

This implies that at late times the photon will essentially freeze it's position—advancing exponentially slower and slower as coordinate time goes to infinity. Indeed, if t_i is a time late enough[5] for the universe to be virtually dominated by Λ, then within the infinitude of time to the end of the universe, the photon will travel only the finite distance

$$x(\infty) - x(t_i) = \left(\frac{12}{\beta^2\Lambda}\right) e^{-t_i\sqrt{\Lambda/3}}. \tag{1.19}$$

Problem 2

We are to study a closed FRW universe which is 'radiation dominated for only a negligibly short fraction of its life' and determine how many times a photon released at the big bang can encircle the universe before the big crunch. Although it is quite likely that the author of the problem had a mostly-matter-dominant universe in mind, there are certainly other ways of interpreting the problem[6]. We will consider here only the most obvious interpretation of the problem—the one of a universe with matter and relativistic energy components.

Unfortunately, our analysis in problem 1 above was not sufficiently general to consider a closed universe with the metric

$$ds^2 = -dt^2 + a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2(\theta)d\varphi^2\right\}, \tag{2.1}$$

where $k = 1$ for a closed universe. Therefore, we will need to quickly generalize that discussion to include $k \neq 0$.

Notice that the coordinate 'r' here is not a radius in the sense of a usual spherical geometry: by setting $k = 1$ we are forced to restrict r to the range $r \in [-1, 1]$—it is an angular coordinate. Indeed, that $k = 1$ describes the geometry of a three-sphere is made manifest by the change of variables $r = \sin(\lambda)$ so that the metric becomes

$$ds^2 = -dt^2 + a^2(t)\left(d\lambda^2 + \sin^2(\lambda)d\theta^2 + \sin^2(\lambda)\sin^2(\theta)d\varphi^2\right), \tag{2.2}$$

which by inspection is the metric of a three-sphere with fixed radius $a(t)$.

The only reason why we so digress is to clarify that fixed-λ and fixed-θ trajectories are only geodesics when $\lambda = \theta = \pi/2$—otherwise the orbit will not describe a great-circle on the sphere. The moral is that if we would like to study simple photon geodesics in a closed spacetime, we must set $\lambda = \theta = \pi/2$—or, equivalently, we must set the coordinate $r \to 1$.

Now, let us return to the metric (2.1) and derive the Einstein field equations. If we write the metric in the form

$$ds^2 = -dt^2 + a^2(t)\widetilde{g}_{jk}dx^j dx^k, \tag{2.3}$$

then we find that

$$R_{tt} = 3\frac{\ddot{a}}{a}, \qquad R_{ij} = -\widetilde{g}_{ij}\left(a\dot{a} + 2\dot{a}^2 + 2k\right), \quad \text{and} \quad R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right). \tag{2.4}$$

Now, the universe under investigation has a stress-energy tensor which is the sum of those for 'radiation' ($w = \frac{1}{3}$) and 'matter' ($w = 0$) components. Therefore, like above, the 'tt'-Einstein equation is simply

$$3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = 8\pi G\left(\rho_m + \rho_r\right), \tag{2.5}$$

which implies

$$\therefore \dot{a}^2 = \frac{8\pi G}{3}a^2\left(\rho_m + \rho_r\right) - k, \tag{2.6}$$

[5]This happens approximately when the small angle expansion of sinh breaks down: when $t \sim \sqrt{\frac{3}{4\Lambda}}$.

[6]Consider, for example a universe with radiation and a $w = -\frac{1}{2}$ quintessence field: such a universe would certainly have the property that radiation is dominant for a negligibly short time in the early/late universe.

where ρ_m and ρ_r are the densities of matter and radiation components of the universe, respectively.

In problem 1 equation (1.8) we derived the relationship $\rho \propto a^{-3(1+w)}$ using only the conservation of energy for a perfect fluid. This therefore most certainly applies for both matter and radiation components of the universe. Expressing the constants of proportionality as

$$\rho_m = \frac{2\beta}{8\pi G}a^{-3} \qquad \text{and} \qquad \rho_r = \frac{3\zeta}{8\pi G}a^{-4}, \tag{2.7}$$

the Einstein equation becomes

$$\dot{a}^2 = \frac{\beta}{a} + \frac{\zeta}{a^2} - k. \tag{2.8}$$

This differential equation is generally solvable in terms of hypergeometric functions, but these are far from enlightening. Rather, we are told to consider the limit that the universe is radiation-dominated for a vanishingly small fraction of its lifetime. This is equivalent to considering 'the age of the universe' to consist almost entirely of that time for which $\beta/a \gg \zeta/a^2$. In this limit, for a closed universe, we have

$$\dot{a}^2 = \frac{\beta}{a} - 1. \tag{2.9}$$

This differential equation can be solved by a clever trick: we know that 1. $a(t)$ has a maximum at $a(t) = \beta$—because then $\dot{a} = 0$—and 2. that $a(t_0) = a(t_f) = 0$. Therefore, we are free to parameterize $a = \frac{\beta}{2}(1 - \cos\eta)$ for some new parameter η. In terms of η, we find that

$$\dot{a}^2 = \frac{1}{1-\cos\eta}(2 - 1 + \cos\eta) = \frac{1+\cos\eta}{1-\cos\eta} = \frac{1-\cos^2\eta}{(1-\cos\eta)^2} = \frac{\sin^2\eta}{(1-\cos\eta)^2}; \tag{2.10}$$

$$\therefore \frac{da}{dt} = \frac{\sin\eta}{1-\cos\eta}. \tag{2.11}$$

Notice that t and η are related by the equation

$$\frac{dt}{d\eta} = \frac{dt}{da}\frac{da}{d\eta} = \frac{1-\cos\eta}{\sin\eta}\frac{\beta}{2}\sin\eta = \frac{\beta}{2}(1-\cos\eta) = a(\eta). \tag{2.12}$$

We are now prepared to determine how many times a photon released at the big bang can encircle the universe before the big crunch. As described above, geodesics which encircle the universe are those for which $r = 1$, $\theta = \pi/2$ in terms of the coordinates of the metric (2.1). Therefore, the condition for a null light ray is simply

$$ds^2 = 0 = -dt^2 + a^2(t)d\varphi^2, \qquad \Longrightarrow \qquad d\varphi = \frac{dt}{a(t)}. \tag{2.13}$$

The total angular distance such a photon can travel during the total time of the universe is then given by

$$\varphi_{tot} = \int d\varphi = \int_{t=0}^{t=t_f} \frac{dt}{a(t)} = \int_0^{2\pi} \frac{dt}{d\eta}\frac{d\eta}{a(\eta)} = \int_0^{2\pi} \frac{a(\eta)}{a(\eta)}d\eta = 2\pi. \tag{2.14}$$

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Problem 3

We are asked to study the asymptotic evolution of a universe filled with matter together with another form of energy[7], termed 'quintessence' with an 'exotic' equation of state $p_Q = w\rho_Q$.

a. We are to determine the equation of state for which quintessence energy density will eventually dominate the universe.

In problem 1, we worked out the dependence of an energy density component in terms of the cosmic scale factor function $a(t)$ and the component's equation of state w (1.8):

$$\rho \propto a(t)^{-3(1+w)}. \tag{3.b.1}$$

Matter, with equation of state $w_m = 0$ is easily seen to evolve according to $\rho_m \propto a(t)^{-3}$. Therefore, any energy component with equation of state $w < 0$ will eventually dominate over matter—as $a(t)$ becomes sufficiently large at late times.

b. We are to solve for $a(t)$ assuming a universe in which quintessence dominates, and find the condition which the equation of state must satisfy so that $a(t)$ remains finite for any finite time.

Assuming that that quintessence is the dominant source of energy density in the universe, we may safely ignore the matter and radiation contributions to Einstein's equation; then, in accordance with (1.6) and (1.8), we find

$$\dot{a}^2(t) = \frac{8\pi G}{3} a^2 \rho_Q \equiv \beta^2 a^{-(1+3w)}(t). \tag{3.c.1}$$

This implies

$$\int a^{(1+3w)/2} da = \beta \int dt. \tag{3.c.2}$$

Now, there are three relevant cases to consider:

- If $w > -1$, this system can be integrated directly: setting $a(0) = 0$[8], we obtain

$$a(t) \propto t^{2/(3(1+w))} \qquad \text{for} \quad w > -1. \tag{3.c.3}$$

- When $w = -1$, we have

$$\int_{a_0}^{a} \frac{da'}{a'} = \beta \int_0^t dt' = \log\left(\frac{a(t)}{a_0}\right), \qquad \Longrightarrow \qquad a(t) = a_0 e^{\beta t} \qquad \text{for} \quad w = -1. \tag{3.c.4}$$

  Notice that this agrees with our results obtained above for a universe with a cosmological constant (for which $w = -1$).

- The (pathological) case of $w < -1$, a bit more care must be taken to evaluate the integral. We find

$$\int_{a_0}^{a} a'^{(1+3w)/2} da' = \beta, t \qquad \Longrightarrow \qquad a(t) \propto \left\{\frac{3(1+w)}{2} t + a_0^{3(1+w)/2}\right\}^{2/(3(1+w))}, \tag{3.c.5}$$

  and bearing in mind that $w < -1$, this has the structure of

$$a(t) \propto \frac{1}{(\eta - \zeta t)^{1/\zeta}} \qquad \text{for} \quad w < -1. \tag{3.c.6}$$

  Clearly, for $w < -1$, $a(t)$ diverges in finite time.

c. We are asked to determine the condition for which the universe has a future horizon.

The null condition on the worldline of a photon travelling in, e.g., the $\hat{x}$-direction is

$$ds^2 = 0 = -dt^2 + a^2(t) dx^2, \qquad \Longrightarrow \qquad dx = \frac{dt}{a(t)}. \tag{3.d.1}$$

[7]The problem explicitly calls this exotic energy density quintessence despite it having nothing at all to do with a model of quintessence. Indeed, there are no models of quintessence for which $w < -1$, but these are considered here anyway.

[8]Actually, this is probably not the boundary conditions we would like to set: because the early universe will be either matter or radiation dominated, it would be more natural to integrate from some some value $a(t_0)$ from whence quintessence dominates. This, however, would not change our primary results.

Therefore, the coordinate distance which a photon can travel is given by

$$x(t=\infty)-x_i=\int_{t_0}^{\infty}\frac{dt}{a(t)}. \tag{3.d.2}$$

It is obvious to anyone with an education including first-semester calculus that this integral is finite only if $a(t)\propto t^{\lambda}$ for $\lambda>1$—and finiteness of the total distance travelled during an infinite time span indicates the existence of a horizon. Using our work above, we see that there is a horizon if

$$\frac{2}{3(1+w)}>1, \qquad \Longrightarrow \qquad w<-\frac{1}{3}. \tag{3.d.3}$$

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