Physics 505, Classical Electrodynamics

Homework 2

Due Thursday, 23rd September 2004

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2.1 A point charge q is brought a distance d away from an infinite, conducting, grounded plane. We are to solve the following using the method of images.

a) Let us find the surface-charge density induced on the plane and plot it.

From the symmetry of the problem, it is clear that the potential $\varphi(x)$ is equivalent to that produced by the charge q together with an image charge $q' = -q$ located a distance d on the opposite side of the plane. Specifically, the potential is given by

$$\varphi(x) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{(x_1^2+x_2^2+(x_3-d)^2)^{1/2}} - \frac{1}{(x_1^2+x_2^2+(x_3+d)^2)^{1/2}}\right),$$

in cartesian coordinates where the point charge is located at the point $z \equiv (0,0,d)$.

The charge density is then computed,

$$\begin{aligned}\sigma(x) &= -\epsilon_0 \frac{\partial\varphi}{\partial x_3}\bigg|_{x_3=0}, \\ &= -\frac{q}{4\pi}\left(-\frac{x_3-d}{(x_1^2+x_2^2+(x_3-d)^2)^{3/2}} + \frac{x_3+d}{(x_1^2+x_2^2+(x_3+d)^2)^{3/2}}\right)\bigg|_{x_3=0}, \\ &= -\frac{q}{4\pi}\left(\frac{d}{(x_1^2+x_2^2+d^2)^{3/2}} + \frac{d}{(x_1^2+x_2^2+d^2)^{3/2}}\right),\end{aligned}$$

$$\boxed{\therefore \sigma(x) = -\frac{qd}{2\pi(x_1^2+x_2^2+d^2)^{3/2}}}$$

In Figure 1 below, we have plotted the charge density of the plane.

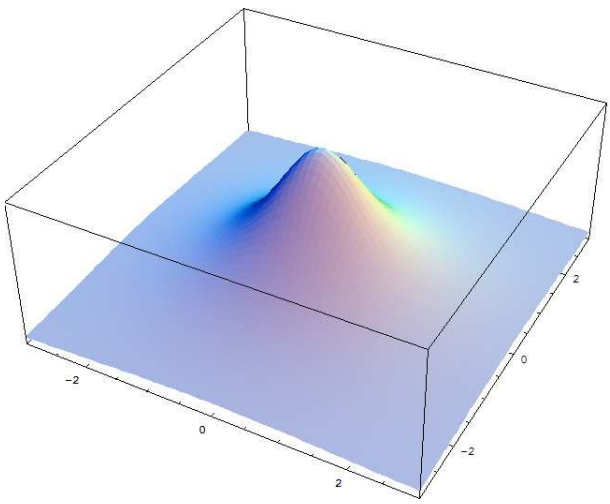

Figure 1. Problem (2.1): Induced charge density as a function of position on the plane. The z-axis labels the density in arbitrary units. Specifically, $\{-qd/(2\pi), d\} \mapsto 1$

b) We are to compute the force between the plane and the charge using Coulomb's law for the force between the charge and its image.

Quite directly we see that

$$\boxed{F = -\frac{q^2}{4\pi\epsilon_0}\frac{1}{(2d)^2}\hat{z}.}$$

c) We are to compute the total force acting on the plane by directly integrating over $\sigma^2/2\epsilon_0$.
Let us proceed to compute the force directly

$$\begin{aligned} F &= \int da\ \frac{\sigma^2}{2\epsilon_0} = \frac{q^2 d^2}{8\pi^2\epsilon_0}\int \frac{r dr d\theta}{(r^2+d^2)^{3/2}}, \\ &= \frac{q^2 d^2}{8\pi^2\epsilon_0}\int \frac{du d\theta}{u^{3/2}}, \\ &= -\frac{q^2 d^2}{16\pi\epsilon_0}\ \frac{1}{u^2}\bigg|_{d^2}^{\infty}, \\ &= -\frac{q^2 d^2}{16\pi\epsilon_0 d^4}, \end{aligned}$$

$$\boxed{\therefore F = -\frac{q^2}{4\pi\epsilon_0}\frac{1}{(2d)^2}\hat{z}.}$$

d) We are to compute the work required to remove the charge from its position to infinity.
By direct computation, we see that the work required is

$$W = -\int_d^\infty F d\ell = -\frac{q^2}{16\pi\epsilon_0}\int_d^\infty \frac{dz}{z^2} = \frac{q^2}{16\pi\epsilon_0}\ \frac{1}{z}\bigg|_d^\infty = \frac{q^2}{16\pi\epsilon_0 d}.$$

e) Let us compute the potential energy of the charge-image-charge system and compare this result with that of part (d) above.
From the definition of potential energy, we see that

$$U = \frac{q^2}{8\pi\epsilon_0 d}.$$

This is precisely twice the energy required to pull the charge q to infinity. This is expected: the energy is twice that of the original system because we also have energy associated with the electric field of the image charge. In the original system, there is only the electric field of a single charge. Furthermore, notice that the position and hence energy of the image charge depends on the charge and position of q. This implies that it is negligent to consider q moving in a static field produced by q'.

2.2 Let us consider the problem of a point charge q located *inside* a hollow, grounded, conducting sphere of inner radius a.

a) We are to find the potential function describing the inside of the sphere.
This problem is superficially similar to that discussed in Jackson's text regarding a charge located outside a conducting, grounded sphere. It should be clear that if the charge is located at the point r, then, by axial symmetry, the image charge must be located along the direction of r at a distance of $r' > a$—if the method of image charges applies. Therefore, the potential at a point x will be given in the form

$$\varphi(x) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|x-r|} + \frac{q'}{|x-r'|}\right),$$

Because the sphere is grounded, we must have $\varphi(x) = 0\ \forall x|\ x = a$. This situation is identical to that required for the situation described in Jackson and therefore we must have that

$$\frac{q}{a} = -\frac{q'}{r'} \quad \text{and} \quad \frac{r}{a} = \frac{a}{r'}.$$

In particular, this implies that

$$\boxed{\therefore \varphi(x) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{(x^2+r^2-2xr\cos\theta)^{1/2}} - \frac{a}{r\left(x^2+\frac{a^4}{r^2}-2x\frac{a^2}{r}\cos\theta\right)^{1/2}}\right).}$$