Problem 6.9

We are to discuss the conservation of energy and linear momentum for a macroscopic system of sources and electromagnetic fields in a uniform, isotropic medium described by a permittivity ϵ and a permeability μ. We are to derive the canonical energy density, Poynting vector, field-momentum density, and Maxwell stress tensor.

A little bit surprised by the problem, we recall from chapters 4 and 5 of Jackson's text that the energy density of the electric and magnetic fields are, respectively,

$$W_E = \frac{1}{2}\int_\Omega \mathbf{E}\cdot\mathbf{D}\; d^3x \quad \text{and} \quad W_B = \frac{1}{2}\int_\Omega \mathbf{H}\cdot\mathbf{B}\; d^3x.$$

In a linear medium with the given permittivity and permeability, we have that $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$. It is quite obvious therefore that the energy density will be given by

$$u = \frac{1}{2}\left(\epsilon\mathbf{E}^2 + \mu\mathbf{H}^2\right).$$

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Although we could simply restate the exact arguments presented in Jackson's section 6.7, we believe this would be an utter waste of time. We hope that Ben will understand our reluctance to copy obvious statements from the text. We find that the Poynting vector is simply

$$\mathbf{S} = \mathbf{E}\times\mathbf{H}.$$

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For the field-momentum density, we similarly refer to Jackson's section 6.7. Following the discussion up to equation 6.117, we see that

$$\mathbf{P}_{\text{field}} = \mu\epsilon\int_\Omega \mathbf{E}\times\mathbf{H}\; d^3x,$$

which of course implies that the field-momentum density is simply

$$\mathbf{g} = \mu\epsilon\mathbf{E}\times\mathbf{H}.$$

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As we have done above, we notice that Jackson derives the Maxwell stress tensor in section 6.7 and arrives at the expression

$$T_{\alpha\beta} = \epsilon\left[E_\alpha E_\beta + \frac{1}{\mu\epsilon}B_\alpha B_\beta - \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{E} + \frac{1}{\mu\epsilon}\mathbf{B}\cdot\mathbf{B}\right)\delta_{\alpha\beta}\right].$$

Inserting our expression $\mathbf{B} = \mu\mathbf{H}$, we see that this quickly reduces to the desired

$$T_{\alpha\beta} = \left[\epsilon E_\alpha E_\beta + \mu H_\alpha H_\beta - \frac{1}{2}\delta_{\alpha\beta}\left(\epsilon E^2 + \mu H^2\right)\right].$$

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Problem 6.10

We are to discuss the conservation of angular momentum and derive the differential and integral forms of the conservation law.

It is not entirely clear to what extent we are supposed to 'show' this fact. It is rather clear that the exact analogue of the momentum conservation law is given by

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \left(\mathbf{r}\times\mathbf{g}_{\text{mech}} + \mathbf{r}\times\mathbf{g}_{\text{field}}\right) d^3x &= \oint_{\partial\Omega} \mathbf{r}\times T_{\alpha\beta}n_\beta\; da,\\
\frac{d}{dt}\int_\Omega \left(\mathscr{L}_{\text{mech}} + \mathscr{L}_{\text{field}}\right) d^3x &= \oint_{\partial\Omega} \mathbf{r}\times\overleftrightarrow{\mathbf{T}}\cdot\mathbf{n}\; da,\\
\frac{d}{dt}\int_\Omega \left(\mathscr{L}_{\text{mech}} + \mathscr{L}_{\text{field}}\right) d^3x &= -\oint_{\partial\Omega} \mathbf{n}\cdot\overleftrightarrow{\mathbf{M}}\; da,\\
\therefore\ \frac{d}{dt}\int_\Omega \left(\mathscr{L}_{\text{mech}} + \mathscr{L}_{\text{field}}\right) d^3x &+ \oint_{\partial\Omega} \mathbf{n}\cdot\overleftrightarrow{\mathbf{M}}\; da = 0.
\end{aligned}$$

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It is fairly obvious that the differential form of the above is simply given by

$$\therefore\ \frac{\partial}{\partial t}\left(\mathscr{L}_{\text{mech}} + \mathscr{L}_{\text{field}}\right) + \nabla\cdot\overleftrightarrow{\mathbf{M}} = 0.$$

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