Problem 7.16

Let us consider plane waves propagating in a homogenous, nonpermeable but anisotropic dielectric. The dielectric is characterized by the tensor ϵ_{ij}. We will assume that the coordinate axes have been chosen so that $\mathbf{D}_i = \epsilon_i \mathbf{E}_i$ where ϵ_i are the eigenvalues of the tensor.

a) We are to show that plane waves with frequency ω and wave vector $\mathbf{k}$ satisfy

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) + \mu_0 \omega^2 \mathbf{D} = 0.$$

This is relatively obvious enough. We know from our general studies of electromagnetic waves that in this principle axis system, $\mathbf{E} \propto \epsilon_i e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$. Using simple vector identities we have

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) = (\mathbf{k}\cdot\mathbf{E})\mathbf{k} - (\mathbf{k}\cdot\mathbf{k})\mathbf{E}.$$

But $\mathbf{k}\cdot\mathbf{E}$—the wave vector is always orthogonal to the field. Hence the first term above vanishes.

Also notice that $\mathbf{k}\cdot\mathbf{k} = \mu_0\omega^2\epsilon_i$ for the i^{th} component of $\mathbf{E}$. Therefore,

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) = -\mu_0\omega^2\epsilon_i\mathbf{E}_i = -\mu_0\omega^2\mathbf{D}.$$

Hence,

$$\boxed{\therefore \mathbf{k} \times (\mathbf{k} \times \mathbf{E}) + \mu_0 \omega^2 \mathbf{D} = 0.}$$

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c) We are to show that $\mathbf{D}_a \cdot \mathbf{D}_b = 0$ where $\mathbf{D}_a, \mathbf{D}_b$ are the displacements associated with the two modes of propagation.

This is quite obvious: the two modes of propagation are naturally orthogonal—we choose orthogonal bases—and so it is natural that $\mathbf{D}_a \cdot \mathbf{D}_b = 0$.

Problem 7.2

Let us consider a plane wave that is normally incident on two parallel layers with refraction indices of each layer being n_1, n_2. We are to determine the thickness d of the second layer such that no wave is reflected from the system at the frequency ω in terms of the refractive indices.

Let us work in the system where the z-axis is normal to the planes—and hence parallel to the direction of motion of the wave—with the origin at the surface of the first surface. From rather general considerations the fields $\mathbf{E}_1, \mathbf{E}_2$, and $\mathbf{E}_3$ for the first medium, second medium and the air on the right are given by

$$\mathbf{E}_1 = ae^{ik_1z} + be^{-ik_1z},$$
$$\mathbf{E}_2 = \alpha e^{ik_2z} + \beta e^{-ik_2z},$$
$$\mathbf{E}_3 = \eta e^{ikz},$$

where $k_1 = n_1c/\omega, k_2 = n_2c/\omega$ and $\kappa = c/\omega$ and a, b, α, β, and η are constants.

From continuity requirements of the electric and magnetic field at the interfaces, we have the constraints

$$a + b = \alpha + \beta \qquad \alpha e^{ik_2d} + \beta e^{ik_2d} = \eta e^{ikd};$$
$$a - b = \frac{n_2}{n_1}(\alpha - \beta) \qquad \alpha e^{ik_2d} - \beta e^{ik_2d} = \frac{1}{n_2}\eta e^{ikd}.$$

Notice that the second two imply that

$$\alpha = \frac{1}{2}\left(1 + \frac{1}{n_2}\right)\eta e^{id(k-k_2)};$$
$$\beta = \frac{1}{2}\left(1 - \frac{1}{n_2}\right)\eta e^{id(k+k_2)}.$$

Now, the case in which there is no reflection is that where $b = 0$. We can solve for b using the above system trivially and see that

$$2b = \left(1 - \frac{n_2}{n_1}\right)\alpha + \left(1 + \frac{n_2}{n_1}\right)\beta.$$