

PHYSICS 505, CLASSICAL ELECTRODYNAMICS

Homework 12

Due Friday, 10th December 2004

JACOB LEWIS BOURJAILY

Problem 7.28

Let us consider a circularly polarized plane wave moving in the z direction that has a finite extent in the x and y directions. Assuming that the amplitude modulation is slowly varying, we are to give approximations for the electric and magnetic fields.

Because the wave is circularly polarized, we can assume that its polarization has only a small longitudinal part (which is independent of z because it is plane wave). Therefore, the electric field should be of the form,

$$\mathbf{E} = [E_0(\mathbf{e}_1 \pm i\mathbf{e}_2) + F(x, y)\mathbf{e}_3] e^{i(kz - \omega t)}.$$

Now by Maxwell's equations we know that \mathbf{e} must be divergenceless in free space—or in a neutral medium—so that $\nabla \cdot \mathbf{E} = 0$. But this is simply the requirement that

$$\nabla \cdot \mathbf{E} = \left[\frac{\partial E_0}{\partial x} \pm i \frac{\partial E_0}{\partial y} + ikF(x, y) \right] e^{i(kz - \omega t)} = 0.$$

It is clear that this implies that

$$F(x, y) = \frac{i}{k} \left(\frac{\partial E_0}{\partial x} \pm i \frac{\partial E_0}{\partial y} \right).$$

Hence, we have that

$$\boxed{\therefore \mathbf{E} = \left[E_0(x, y)(\mathbf{e}_1 \pm i\mathbf{e}_2) + \frac{i}{k} \left(\frac{\partial E_0}{\partial x} \pm i \frac{\partial E_0}{\partial y} \right) \mathbf{e}_3 \right] e^{i(kz - \omega t)}} \quad \check{\delta}\pi\epsilon\rho \check{\delta}\delta\epsilon\iota \delta\epsilon\check{\iota}\xi\alpha\iota$$

We can find the field \mathbf{B} using Maxwell's equations. If \mathbf{E} is slowly varying, then we may neglect second derivatives of E_0 . Solving directly, we have

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E} = -\nabla \times \left[E_0(x, y)(\mathbf{e}_1 \pm i\mathbf{e}_2) + \frac{i}{k} \left(\frac{\partial E_0}{\partial x} \pm i \frac{\partial E_0}{\partial y} \right) \mathbf{e}_3 \right] e^{i(kz - \omega t)}, \\ &\simeq - \left[\pm k E_0(x, y) \mathbf{e}_1 + ik E_0(x, y) \mathbf{e}_2 + \left(\pm i \frac{\partial E_0}{\partial x} - \frac{\partial E_0}{\partial y} \right) \mathbf{e}_3 \right] e^{i(kz - \omega t)}, \end{aligned}$$

Integrating out time, we see that

$$\begin{aligned} \mathbf{B} &= -\frac{ik}{w} \left[\pm E_0(x, y) \mathbf{e}_1 + i E_0(x, y) \mathbf{e}_2 + \frac{1}{k} \left(\pm i \frac{\partial E_0}{\partial x} - \frac{\partial E_0}{\partial y} \right) \mathbf{e}_3 \right] e^{i(kz - \omega t)}, \\ &= \mp \frac{ik}{w} \left[E_0(x, y) \mathbf{e}_1 \pm i E_0(x, y) \mathbf{e}_2 + \frac{1}{k} \left(i \frac{\partial E_0}{\partial x} \mp \frac{\partial E_0}{\partial y} \right) \mathbf{e}_3 \right] e^{i(kz - \omega t)}, \\ &= \mp i \sqrt{\mu\epsilon} \left[E_0(x, y) \mathbf{e}_1 \pm i E_0(x, y) \mathbf{e}_2 + \frac{i}{k} \left(\frac{\partial E_0}{\partial x} \pm i \frac{\partial E_0}{\partial y} \right) \mathbf{e}_3 \right] e^{i(kz - \omega t)}, \end{aligned}$$

$$\boxed{\therefore \mathbf{B} \simeq \mp i \sqrt{\mu\epsilon} \mathbf{E}} \quad \check{\delta}\pi\epsilon\rho \check{\delta}\delta\epsilon\iota \delta\epsilon\check{\iota}\xi\alpha\iota$$

Problem 7.29

For the circularly-polarized plane wave of problem 7.28 above and assuming that E_0 is a real function, we are to calculate the time-averaged component of angular momentum parallel to the direction of propagation. We should compare this with the energy of the wave in vacuum and interpret this in terms of photons.

Recall that in in vacuum, the time averaged energy (or simply the energy density) is given by $u = \frac{1}{2} \left(\epsilon |\mathbf{E}|^2 + \frac{1}{\mu} |\mathbf{B}|^2 \right)$. In our situation, because $\mathbf{B} \simeq i \sqrt{\mu\epsilon} \mathbf{E}$, we have that $u = \epsilon |\mathbf{E}|^2$.

For the angular momentum, we have that $\ell = \epsilon \mathbf{r} \times (\mathbf{E} \times \mathbf{B})$ and because \mathbf{E} and \mathbf{B} are orthogonal vectors (in complex space) we have that $|\mathbf{E} \times \mathbf{B}| = \pm \sqrt{\mu\epsilon} |\mathbf{E}|^2$. The direction of propagation is given by the wave vector vk and so $\ell = \pm \epsilon \sqrt{\mu\epsilon} k |\mathbf{E}|^2$.

The ratio of the energy density to the angular momentum is therefore

$$\frac{\ell}{u} = \pm k \sqrt{\epsilon\mu} = \pm \frac{c}{\omega} = \pm \omega^{-1}. \quad \check{\delta}\pi\epsilon\rho \check{\delta}\delta\epsilon\iota \delta\epsilon\check{\iota}\xi\alpha\iota$$

This reminds us of the quantization of the photon whose angular momentum can only take on discrete values (and only has two helicities).