Physics 523, Quantum Field Theory II

Homework 8

Due Wednesday, 10th March 2004

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Renormalization of Pseudo-Scalar Yukawa Theory

Let us consider the theory generated by the Lagrangian

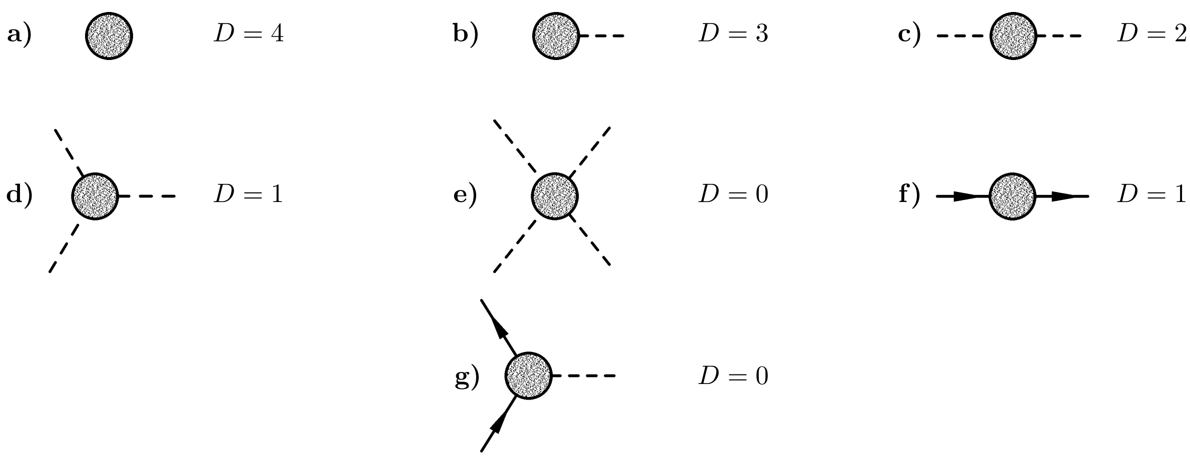
$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi_o)^2 - \frac{1}{2}m_{\phi_o}^2\phi_o^2 + \overline{\psi}_o(i\partial\!\!\!/ - m_{e_o})\psi_o - ig_o\overline{\psi}_o\gamma^5\psi_o\phi_o.$$

Superficially, this theory will diverge very similarly to quantum electrodynamics because the fields and the coupling constant have the same dimensions as in quantum electrodynamics. Therefore, we see that the superficial divergence is given by $D = 4L - 2P_\phi - P_e$ where L represents the number of loops and P_ϕ and P_e represent the number of pseudo-scalar and fermion propagator particles, respectively. Furthermore, we see that this can be reduced to

$$\boxed{D = 4 - N_\phi - \frac{3}{2}N_e,} \tag{a.1}$$

where N_ϕ and N_e represent the number of external pseudo-scalar and fermion lines, respectively.

We see that this implies that the following diagrams are superficially divergent:



Although vacuum energy is an extraordinarily interesting problem of physics, we will largely ignore diagram (a) which is quite divergent. We note that because the Lagrangian is invariant under parity transformations $\phi(t, \mathbf{x}) \to -\phi(t, -\mathbf{x})$ any diagram with an odd number of external ϕ's will give zero. In particular, the divergent diagrams (b) and (d) will be zero.

The first divergent diagram we will consider, (c), is clearly $\sim a_0\Lambda^2 + a_1 p^2 \log\Lambda$ where we note that the term proportional to p in the expansion vanishes by parity symmetry. Similarly, we naïvely suspect that the divergence of diagram (f) would be $\sim a_0\Lambda + p\!\!\!/ \log\Lambda$ but the term linear in Λ is reduced to $m_e \log\Lambda$ by the symmetry of the Lagrangian of chirality inversion of ψ together with $\phi \to -\phi$. The diagrams (e) and (g) are both $\sim \log\Lambda$. All together, there are six divergent constants in this theory.

We note that because the diagram (e) diverges, we must introduce a counterterm δ_λ which implies that our original Lagrangian should have included a term $\frac{\lambda}{4!}\phi^4$.

We define renormalized fields, $\phi_o \equiv Z_\phi^{1/2}\phi$ and $\psi_o \equiv Z_2^{1/2}\psi$, where Z_ϕ and Z_2 are as would be defined canonically. Using these our Lagrangian can be written as

$$\mathscr{L} = \frac{1}{2}Z_\phi(\partial_\mu\phi)^2 - \frac{1}{2}Z_\phi m_{\phi_o}^2\phi^2 - Z_2\overline{\psi}(i\partial\!\!\!/ - m_{e_o})\psi - -ig_oZ_2Z_\phi^{1/2}\overline{\psi}\gamma^5\psi\phi - \frac{\lambda}{4!}Z_\phi^2\phi^4.$$

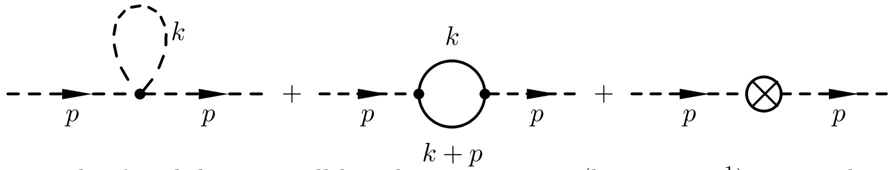
Let us define the counterterms,

$$\delta_{m_\phi} \equiv Z_\phi m_{\phi_o}^2 - m_\phi^2, \quad \delta_{m_e} \equiv Z_2 m_{e_o} - m_e, \quad \delta_\phi \equiv Z_\phi - 1, \quad \delta_\lambda \equiv \lambda_o Z_\phi^2 - \lambda, \quad \delta_1 \equiv \frac{g_o}{g}Z_2Z_\phi^{1/2} - 1, \quad \delta_2 \equiv Z_2 - 1.$$

Therefore, we may write our renormalized Lagrangian

$$\begin{aligned}\mathscr{L} &= \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_\phi^2\phi^2 + \overline{\psi}(i\partial\!\!\!/ - m_e)\psi - ig\overline{\psi}\gamma^5\psi\phi - \frac{\lambda}{4!}\phi^4 \\ &+ \frac{1}{2}\delta_\phi(\partial_\mu\phi)^2 - \frac{1}{2}\delta_{m_\phi}\phi^2 + \overline{\psi}(i\delta_2\partial\!\!\!/ - \delta_{m_e})\psi - ig\delta_1\overline{\psi}\gamma^5\psi\phi - \frac{\delta_\lambda}{4!}\phi^4.\end{aligned} \tag{a.4}$$

Let us compute the pseudo-scalar self-energy diagrams to the one-loop order, keeping only the divergent pieces. This corresponds to:

$-iM^2(p^2) =$ 

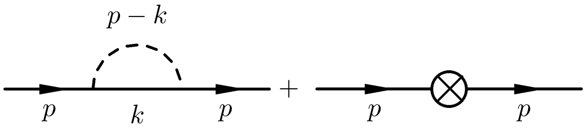
Using the 'canonical procedure' and dropping all but divergent pieces (linear in ϵ^{-1}) we see that

$$\begin{aligned}
-iM^2(p^2) &= -i\frac{\lambda}{2}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m_\phi^2} - g^2\int\frac{d^dk}{(2\pi)^d}\mathrm{Tr}\left[\frac{\gamma^5 i(\not{k}+\not{p}+m_e)i\gamma^5(\not{k}+m_e)}{((k+p)^2-m_e^2)(k^2-m_e^2)}\right] + i(p^2\delta_\phi - \delta_{m_e}),\\
&= -i\frac{\lambda}{2}\frac{1}{(4\pi)^{d/2}}\frac{\Gamma\left(1-\frac{d}{2}\right)}{(m_\phi^2)^{1-d/2}} - 4g^2\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{\ell^2 - x(1-x)p^2 - m_e^2}{(\ell^2-\Delta)^2} + i(p^2\delta_\phi - \delta_{m_2}),\\
&= -i\frac{\lambda}{2}\frac{1}{(4\pi)^{d/2}}\frac{m_\phi^2}{(1-d/2)}\frac{\Gamma\left(2-\frac{d}{2}\right)}{(m^2)^{2-d/2}} - 4g^2\int_0^1 dx\left[-\frac{i}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma\left(1-\frac{d}{2}\right)}{\Delta^{1-d/2}} + \frac{i}{(4\pi)^{d/2}}\frac{\Gamma\left(2-\frac{d}{2}\right)}{\Delta^{2-d/2}}\left(x(1-x)p^2+m_e^2\right)\right]\\
&\quad + i(p^2\delta_\phi - \delta_{m_2}),\\
&\sim i\frac{\lambda m_\phi^2}{32\pi^2}\frac{2}{\epsilon} - 8g^2\frac{i}{(4\pi)^2}\frac{2}{\epsilon}\int_0^1 dx\left(m_e^2 - x(1-x)p^2\right) + 4g^2\frac{i}{(4\pi)^2}\frac{2}{\epsilon}\int_0^1 dx\left(m_e^2 + x(1-x)p^2\right) + i(p^2\delta_\phi - \delta_{m_2}),\\
&= i\frac{\lambda m_\phi^2}{16\pi^2}\frac{1}{\epsilon} + i\frac{g^2}{4\pi^2}\frac{2}{\epsilon}\left(-2m_e^2 + \frac{2}{6}p^2 + \frac{1}{6}p^2 + m_e^2\right) + i(p^2\delta_\phi - \delta_{m_2}),\\
&= i\left(\frac{\lambda m_\phi^2}{16\pi^2} + \frac{g^2p^2}{4\pi^2} - \frac{g^2m_e^2}{2\pi^2}\right)\frac{1}{\epsilon} + i(p^2\delta_\phi - \delta_{m_2}).
\end{aligned}$$

Therefore, applying our renormalization conditions, we see that[1]

$$\boxed{\therefore\ \delta_{m_\phi} = \left(\frac{\lambda m_\phi^2}{16\pi^2} - \frac{g^2m_e^2}{2\pi^2}\right)\frac{1}{\epsilon}, \qquad \delta_\phi = -\left(\frac{g^2}{4\pi^2}\right)\frac{1}{\epsilon}.} \tag{b.1}$$

Similarly, let us compute the fermion self-energy diagrams to one-loop order, keeping only divergent parts. This corresponds to:

$-i\Sigma_2^2(\not{p}) =$ 

Again, using the 'canonical procedure' and dropping all but divergent pieces (linear in ϵ^{-1}) we see that

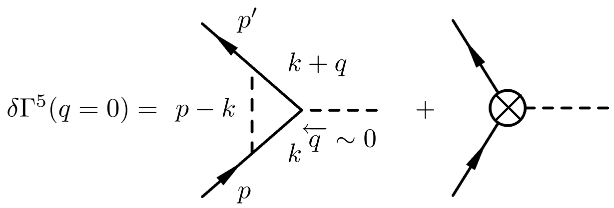
$$\begin{aligned}
-i\Sigma(\not{p}) &= g^2\int\frac{d^dk}{(2\pi)^d}\left[\gamma^5\frac{i}{((p-k)^2-m_\phi^2)}\frac{i(\not{k}+m_e)}{(k^2-m_e^2)}\gamma^5\right] + i(\not{p}\delta_2 - \delta_{m_e}),\\
&= -g^2\int\frac{d^dk}{(2\pi)^d}\frac{\not{k}-m_e}{(k^2-m_e^2)((p-k)^2-m_\phi^2)} + i(\not{p}\delta_2 - \delta_{m_e}),\\
&= -g^2\int_0^1 dz\int\frac{d^d\ell}{(2\pi)^d}\frac{\not{p}z - m_e}{(\ell^2-\Delta)^2} + i(\not{p}\delta_2 - \delta_{m_2}),\\
&\sim -i\frac{g^2}{(4\pi)^2}\frac{2}{\epsilon}\int_0^1 dz\,(\not{p}z - m_e) + i(\not{p}\delta_2 - \delta_{m_e}),\\
&= i\left(\frac{g^2\not{p}}{16\pi^2} - \frac{g^2m_e}{8\pi^2}\right)\frac{1}{\epsilon} + i\not{p}\delta_2 - i\delta_{m_e}.
\end{aligned}$$

Therefore, applying our renormalization conditions, we see that

$$\boxed{\therefore\ \delta_{m_e} = -\left(\frac{g^2m_e}{8\pi^2}\right)\frac{1}{\epsilon}, \qquad \delta_2 = -\left(\frac{g^2}{16\pi^2}\right)\frac{1}{\epsilon}.} \tag{b.2}$$

[1]For renormalization conditions and Feynman rules please see the Appendix.

Let us now compute the δ_1 counterterm by computing $\delta\Gamma^5(q=0)$ given by:



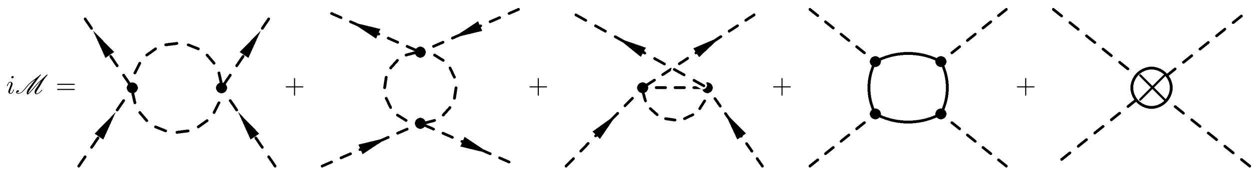
Again, using the 'canonical procedure' and dropping all but divergent pieces (linear in ϵ^{-1}) we see that

$$\begin{aligned}
\delta\Gamma^5(q=0) &= -ig^2\int\frac{d^dk}{(2\pi)^d}\frac{\gamma^5(\not{k}+m_e)\gamma^5(\not{k}+m_e)\gamma^5}{((p-k)^2-m_\phi^2)(k^2-m_e^2)(k^2-m^2)}+\delta_1\gamma^5,\\
&= ig^2\gamma^5\int\frac{d^dk}{(2\pi)^d}\frac{(\not{k}+m_e)(\not{k}-m_e)}{((p-k)^2-m_\phi^2)(k^2-m_e^2)(k^2-m^2)}+\delta_1\gamma^5,\\
&= ig^2\gamma^5\int_0^1 dz\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^2+(z^2-1)m_e^2}{(\ell^2-\Delta)^3}+\delta_1\gamma^5,\\
&= ig^2\gamma^5\int_0^1 dz(1-z)\left[\frac{i}{(4\pi)^2}\frac{d}{2}\frac{2}{\epsilon}\right]+\delta_1\gamma^5,\\
&= -\gamma^5\frac{g^2}{8\pi^2}\frac{1}{\epsilon}+\delta_1\gamma^5.
\end{aligned}$$

Therefore, applying our renormalization conditions, we see that

$$\boxed{\therefore\ \delta_1=\left(\frac{g^2}{8\pi^2}\right)\frac{1}{\epsilon}.} \qquad \text{(b.3)}$$

Let us now compute the δ_λ counterterm by computing the one-loop correction to the standard ϕ^4 vertex. The five contributing diagrams are:



We may save a bit of sweat by noting that the sum of the first four diagrams is identical to the analogous diagrams in ϕ^4-theory. The sum was computed fully both in class and in the text and give a divergent contribution of $\frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon}$ to δ_λ. Therefore, we are only burdened with the calculation of the remaining two. We see that, (note the combinatorial factor of 6)

$$\begin{aligned}
i\mathscr{M} &= i\frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon}-6g^4\int\frac{d^dk}{(2\pi)^d}\text{Tr}\left[\frac{\gamma^5(\not{k}+m_e)\gamma^5(\not{k}-\not{p}_1+m_e)\gamma^5(\not{k}-\not{p}_1-\not{p}_2+m_e)\gamma^5(\not{k}-\not{p}_1-\not{p}_2+\not{p}_3+m_e)}{(k^2-m_e^2)((k-p_1)^2-m_e^2)((k-p_1-p_2)^2-m_e^2)((k-p_1-p_2+p_3)^2-m_e^2)}\right]-i\delta_\lambda,\\
&\underset{k\to\infty}{\simeq} i\frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon}-6g^4\int\frac{d^dk}{(2\pi)^d}\frac{\text{Tr}\left[\gamma^5\not{k}\gamma^5\not{k}\gamma^5\not{k}\gamma^5\not{k}\right]}{(k^2-m_e^2)^4}-i\delta_\lambda,\\
&= i\frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon}-6g^4\int\frac{d^dk}{(2\pi)^d}\frac{4k^4}{(k^2-m_e^2)^4}-i\delta_\lambda,\\
&= i\frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon}-24g^2\frac{i}{(4\pi)^{d/2}}\frac{d(d+2)}{4}\frac{\Gamma\left(2-\frac{d}{2}\right)}{6\Delta^{2-d/2}}-i\delta_\lambda,\\
&= i\frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon}-i\frac{3g^4}{\pi^2}\frac{1}{\epsilon}-i\delta_\lambda.
\end{aligned}$$

Therefore, applying our renormalization conditions, we see that

$$\boxed{\therefore\ \delta_\lambda=\left(\frac{3\lambda^2}{16\pi^2}-\frac{3g^4}{\pi^2}\right)\frac{1}{\epsilon}.} \qquad \text{(b.4)}$$

APPENDIX

Feynman Rules and Renormalization Conditions

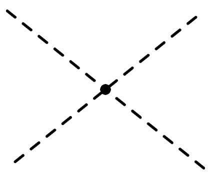
Given the Lagrangian for pseudo-scalar Yukawa theory,

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m_\phi^2\phi^2 + \overline{\psi}(i\partial\!\!\!/ - m_e)\psi - ig\overline{\psi}\gamma^5\psi\phi - \frac{\lambda}{4!}\phi^4$$
$$+\frac{1}{2}\delta_\phi(\partial_\mu\phi)^2 - \frac{1}{2}\delta_{m_\phi}\phi^2 + \overline{\psi}(i\delta_2\,\partial\!\!\!/ - \delta_{m_e})\psi - ig\delta_1\overline{\psi}\gamma^5\psi\phi - \frac{\delta_\lambda}{4!}\phi^4,$$

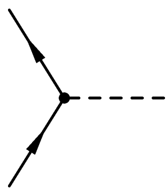
we can derive the renormalized Feynman rules.

$= \frac{i}{p^2 - m_{\phi^2} + i\epsilon}$

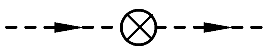
$= \frac{i}{p\!\!\!/ - m_e + i\epsilon}$



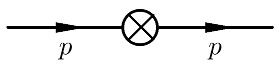
$= -i\lambda$



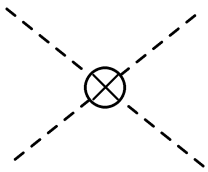
$= g\gamma^5$



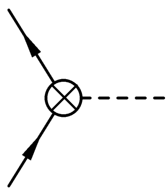
$= i(p^2\delta_\phi - \delta_{m_\phi})$



$= i(p\!\!\!/\delta_2 - \delta m_e)$

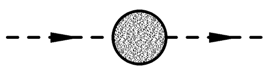


$= -i\delta_\lambda$

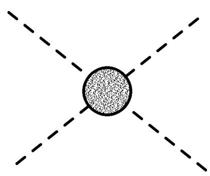


$= g\delta_1\gamma^5$

To derive the counter terms explicitly, it is necessary to offer a convention of renormalization conditions. Above, we have used the conditions:



$= \frac{i}{p^2 - m_{\phi^2} + i\epsilon}$ with pole = 1.



$= -i\lambda$ at $s = 4m^2, t = u = 0.$

$$\Sigma(p\!\!\!/ = m) = 0.$$
$$\left.\frac{d\Sigma(p\!\!\!/)}{dp\!\!\!/}\right|_{p\!\!\!/ = m} = 0.$$
$$g\Gamma^5(q = 0) = g\gamma^5.$$