The expectation value for the number created is simply,

$$E(n) = \sum_{n=0}^{\infty} \frac{n\lambda^n}{n!} e^{-\lambda} = \lambda e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} = \lambda e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} = \lambda e^{-\lambda} e^{\lambda} = \lambda.$$

To compute the variance, we will use the relation $Var(n) = E(n^2) - E(n)^2$. Let us compute $E(n^2)$.

$$\begin{aligned}
E(n^2) &= \sum_{k=0}^{\infty} n^2 \frac{\lambda^n}{n!} e^{-\lambda}, \\
&= \lambda e^{-\lambda} \sum_{n=1}^{\infty} n \frac{\lambda^{n-1}}{(n-1)!}, \\
&= \lambda e^{-\lambda} \sum_{n=1}^{\infty} ((n-1)+1) \frac{\lambda^{n-1}}{(n-1)!}, \\
&= \lambda e^{-\lambda} \left[\sum_{n=1}^{\infty} (n-1) \frac{\lambda^{n-1}}{(n-1)!} + \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} \right], \\
&= \lambda e^{-\lambda} e^{\lambda} + \lambda e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-2)!}, \\
&= \lambda^{+} \lambda^2 e^{-\lambda} \sum_{n=2}^{\infty} \frac{\lambda^{n-2}}{(n-2)!}, \\
&= \lambda^2 + \lambda.
\end{aligned}$$

Knowing this, it is clear that

$$Var(n) = \lambda^2 + \lambda - \lambda = \lambda.$$

Problem 4.4

The cross section for scattering of an electron by the Coulomb field of a nucleus can be computed, to lowest order, without quantizing the electromagnetic field. We will treat the field as a given. classical potential $A_\mu(x)$. The interaction Hamiltonian is then

$$H_I = \int d^3x \; e\bar{\psi}\gamma^\mu\psi A_\mu,$$

where $\psi(x)$ is the usual quantized Dirac field.

a) We must show that the T-matrix element for an electron scatter to off a localized classical potential is given to the lowest order by

$$\langle p_f | iT | p_i \rangle = -ie\bar{u}(p_f)\gamma^\mu u(p_i) \cdot \tilde{A}_\mu(p_f - p_i).$$

where $\tilde{A}_\mu$ is the Fourier transform of A_μ.

We may compute this contribution directly.

$$\begin{aligned}
\langle p_f | iT | p \rangle &= -i \int d^4x \langle p_f | T\{H_I(x)\} | p_i \rangle, \\
&= -ie \int d^4x \; A_\mu \langle p_f | T\{\bar{\psi}(x)\gamma^\mu \psi(x)\} | p_i \rangle, \\
&= -ie \int d^4x \; A_\mu \langle \overline{p_f | \bar{\psi}}(x)\gamma^\mu \overline{\psi(x) | p_i} \rangle, \\
&= -ie \int d^4x \; A_\mu(x) \overline{u}^{s'}(p_f)\gamma^\mu u^s(p_i) e^{ix(p_f - p_i)}, \\
&= -ie\overline{u}^{s'}(p_f)\gamma^\mu u^s(p_i) \int d^4x \; A_\mu(x) e^{ix(p_f - p_i)}, \\
&= -ie\overline{u}^{s'}(p_f)\gamma^\mu u^s(p_i) \tilde{A}_\mu(p_f - p_i).
\end{aligned}$$

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b) If $A_\mu(x)$ is time independent, its Fourier transform contains a delta function of energy. We therefore define

$$\langle p_f|iT|p_i\rangle \equiv i\mathcal{M}\cdot(2\pi)\delta(E_f-E_i).$$

Given this definition of $\mathcal{M}$, we must show that the cross section for scattering off a time-independent localized potential is given by

$$d\sigma = \frac{1}{v_i}\frac{1}{2E_i}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}(2\pi)\delta(E_f-E_i)|\mathcal{M}(p_i\to p_f)|^2.$$

From class we know that we can represent an incoming wave packet with momentum p_i in the z-direction and impact parameter b by the relation

$$|\psi_b\rangle = \int\frac{d^3p_i}{(2\pi)^3}\frac{1}{\sqrt{2E_{p_i}}}e^{-ibp_i}\psi(p_i)|p_i\rangle.$$

The probability of interaction given an impact parameter is then

$$\begin{aligned}
P(b) &= \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}|\langle p_f|iT|\psi_b\rangle|^2,\\
&= \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\int\frac{d^3p_id^3k}{(2\pi)^6}\frac{1}{\sqrt{2E_{p_i}2E_k}}e^{-ib(p_i-k)}\psi(p_i)\psi^*(k)\langle p_f|iT|p_i\rangle\langle p_f|iT|k\rangle^*,\\
&= \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\int\frac{d^3p_id^3k}{(2\pi)^6}\frac{e^{-ib(p_i-k)}}{\sqrt{2E_{p_i}2E_k}}\psi(p_i)\psi^*(k)(2\pi)^2\delta(E_f-E_{p_i})\delta(E_f-E_k)\mathcal{M}(p_i\to p_f)\mathcal{M}(k\to p_f)^*.
\end{aligned}$$

Therefore,

$$\begin{aligned}
d\sigma &= \int d^2b\ P(b),\\
&= \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\int d^2b\frac{d^3pd^3k}{(2\pi)^6}\frac{e^{-ib(p-k)}}{\sqrt{2E_p2E_k}}\psi(p)\psi^*(k)(2\pi)^2\delta(E_f-E_p)\delta(E_f-E_k)\mathcal{M}(p\to p_f)\mathcal{M}(k\to p_f)^*,\\
&= \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\int\frac{d^3pd^3k}{(2\pi)^6}\frac{\psi(p)\psi^*(k)}{\sqrt{2E_p2E_k}}(2\pi)^2\delta^{(2)}(p_\perp-k_\perp)\delta(E_f-E_p)\delta(E_f-E_k)\mathcal{M}(p\to p_f)\mathcal{M}(k\to p_f)^*,\\
&= \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\frac{1}{|v_i|}(2\pi)\int\frac{d^3pd^3k}{(2\pi)^3}\frac{\psi(p)\psi^*(k)}{\sqrt{2E_p2E_k}}\delta^{(2)}(p_\perp-k_\perp)\delta(p_z-k_z)\delta(E_f-E_p)\mathcal{M}(p\to p_f)\mathcal{M}(k\to p_f)^*,\\
&= \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\frac{1}{|v_i|}(2\pi)\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}|\psi(p)|^2\delta(E_f-E_p)|\mathcal{M}(p\to p_f)|^2,
\end{aligned}$$

With a properly normalized wave function, this reduces directly to (allow me to apologize for the inconsistency with notation. It is hard to keep track of. The incoming momentum p has energy E_i.)

$$d\sigma = \frac{1}{v_i}\frac{1}{2E_i}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}(2\pi)\delta(E_f-E_i)|\mathcal{M}(p_i\to p_f)|^2.$$

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Now, let us try to write an expression for $d\sigma/d\Omega$.

$$\begin{aligned}
\int d\sigma &= \int\frac{d^3p_f}{(2\pi)^3}\frac{1}{v_i}\frac{1}{2E_i}\frac{1}{2E_f}(2\pi)\delta(E_f-E_i)|\mathcal{M}|^2,\\
&= \int\frac{p_f^2dp_fd\Omega}{(2\pi)^2}\frac{1}{v_i}\frac{1}{2E_f2E_i}\frac{1}{v_f}\delta(p'-p)|\mathcal{M}|^2,\\
&= \int\frac{d\Omega}{(2\pi)^2}\frac{p^2}{4v_i^2E_i^2}|\mathcal{M}|^2,\\
&= \int d\Omega\frac{1}{16\pi^2}|\mathcal{M}|^2.
\end{aligned}$$

Therefore, we have that

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2}|\mathcal{M}|^2.$$

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c) We will now specialize to the non-relativistic scattering of a Coulomb potential ($A^0 = Ze/4\pi r$). We must show that in this limit

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 Z^2}{4m^2 v^4 \sin^4(\theta/2)}.$$

Let us first take the Fourier transform of the Coulomb potential.

$$\tilde{A}_\mu(\mathbf{k}) = \frac{Ze}{4\pi}\int d^3r \frac{e^{i\mathbf{kr}}}{\mathbf{r}},$$
$$= \frac{Ze}{4\pi}\frac{4\pi}{\mathbf{k}^2},$$
$$\therefore \tilde{A}_\mu(\mathbf{k}) = \frac{Ze}{\mathbf{k}^2}.$$

From part (a) above, we calculated that

$$\mathcal{M} = -ie\overline{u}^{s'}(p_f)\gamma^\mu u^s(p)\tilde{A}_\mu(p_f - p),$$
$$= \frac{-ie^2 Z}{(p_f - p)^2}\overline{u}^{s'}(p_f)\gamma^0 u^s(p).$$

In the nonrelativistic limit, $E >> p$ so we may approximate that

$$\overline{u}^{s'}(p_f)\gamma^0 u^s(p) = u^{s'\dagger}(p_f)u^s(p) = 2E\delta^{s's}.$$

Therefore, our amplitude becomes

$$\mathcal{M} = \frac{-ie^2 Z}{(p_f - p)^2}2E\delta^{s's}.$$

From part (b), we may compute $d\sigma/d\Omega$ directly.

$$\frac{d\sigma}{d\Omega} = \frac{4Z^2 e^4 E62}{16\pi^2 (p_f - p)^4},$$
$$= \frac{Z^2\alpha^2 E^2}{p^4(1 - cos\theta)^2},$$
$$= \frac{Z^2\alpha^2 E^2}{4p^4 \sin^4(\theta/2)},$$
$$= \frac{Z^2\alpha^2}{4E^2 v^4 \sin^4(\theta/2)}.$$

In the nonrelativistic limit, we have that $E^2 \sim m^2$. Therefore we may conclude as desired that

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 Z^2}{4m^2 v^4 \sin^4(\theta/2)}.$$

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