Foundations: The 4 Schools

Set Theory

Improving on Cantor

Zermelo-Fraenkel Axioms for Set Theory (1908 & 1922):

- Two sets are identical if they have the same members
- The empty set exists
- Infinite sets exist
- If x and y are sets, then the unordered pair $\{x,y\}$ is a set
- The union of a set of sets is a set
- Any property that can be formalized in the language of the theory can be used to define a set
- The set of all subsets of any given set is a set
- x does not belong to x
- The axiom of choice

Formalism

Founded by David Hilbert

Completely axiomatic

Used formal system of theorem statement and proof.

Used a restricted metalanguage for the logic used to prove theorems.

Devastated by Gödel's Proofs, but for many still the "official" theory.

Logicism

Precursors: de Morgan and Boole

1910 Russell & Whitehead, *Principia Mathematica*

Problems: The Axioms of Reducibility, Infinity, and Choice

Avoid antinomies by resort to Theory of Types

Intuitionism

Precursors:

Kronecker 'God made the natural numbers; all the rest is the work of man'

Poincaré on logistic attempts to define number:

' "Zero is the number of objects that satisfy a condition that is never satisfied." But as *never* means *in no case* I do not see that any great progress has been made.'

' "1 is the number of a class in which any 2 elements are identical." This seems to be a definition admirably suited to give an idea of the number *1* to people who have never heard of it before; but I am afraid if we asked what *2* is, they would be obliged to use the word *one*.'

1907 Brouwer's dissertation

Accepted:

- "Intuitively obvious" set of natural numbers.
- Addition and Multiplication
- Rational numbers, and certain irrationals that can be constructed.

Rejected:

- Infinite sets and cardinal numbers
- Law of Excluded Middle in logic
- Proof by Mathematical Induction
- Existence Proofs that are not constructive
- Dependence on logic or axiomatic method