BONUS – 5 POINTS

Find the points on the curve $x^2 + xy + y^2 = 7$ where (a) the tangent is parallel to the x-axis and (b) where the tangent is parallel to the y-axis.
SOLUTION

With all this talk about tangent lines, we need the derivative of the implicit curve. So….

$$x^2 + xy + y^2 = 7$$
$$2x + y + xy' + 2yy' = 0$$
$$y'(x + 2y) = -y - 2x$$
$$y' = \frac{-y - 2x}{x + 2y}$$

(a) NOW, where is the tangent parallel to the x-axis? We need to find the points on the curve where the derivative is equal to zero. For this to be true, the numerator of the derivative must be zero, so:

$$-y - 2x = 0$$
$$y = -2x$$

We can use this and plug it back into the original equation to eliminate the variable *y*.

$$x^2 + x(-2x) + (-2x)^2 = 7$$
$$3x^2 = 7$$
$$x = \pm\sqrt{\frac{7}{3}}$$

Now use the relationship $y = -2x$ to find the y coordinates which turn out to be, $y = \mp 2\sqrt{\frac{7}{3}}$ so the points on the curve we desire are $\left(\sqrt{\frac{7}{3}}, -2\sqrt{\frac{7}{3}}\right)$ and $\left(-\sqrt{\frac{7}{3}}, 2\sqrt{\frac{7}{3}}\right)$.

(b) This part is very similar, but now we need to find where the tangent is parallel to the y-axis, that is, where it is vertical. Vertical tangents yield undefined slopes, so we set the denominator of the derivative equal to zero.

$$x + 2y = 0$$
$$x = -2y$$

We could repeat the process from (a) to find the points, but if we look closely we notice a similar relationship between x and y, this time the x-coordinate is the opposite of twice the y-coordinate, so this time our points are $\left(2\sqrt{\frac{7}{3}}, -\sqrt{\frac{7}{3}}\right)$ and $\left(-2\sqrt{\frac{7}{3}}, \sqrt{\frac{7}{3}}\right)$.