November 14th, 2003

ANSWER KEY

(1) Given that $Q(x)=x^{\frac{1}{3}}(x+3)^{\frac{2}{3}}$:

(a) Find the intervals of increase or decrease.

$$Q'(x)=\frac{1}{3}x^{-\frac{2}{3}}(x+3)^{\frac{2}{3}}+\frac{2}{3}(x+3)^{-\frac{1}{3}}x^{\frac{1}{3}}$$ (1) Product Rule

$$0=\frac{(x+3)^{\frac{2}{3}}}{3x^{\frac{2}{3}}}+\frac{2x^{\frac{1}{3}}}{3(x+3)^{\frac{1}{3}}}$$ (2) Simplify and set to zero

$$0=\left(\frac{(x+3)^{\frac{2}{3}}}{3x^{\frac{2}{3}}}\right)\left(\frac{(x+3)^{\frac{1}{3}}}{(x+3)^{\frac{1}{3}}}\right)+\left(\frac{2x^{\frac{1}{3}}}{3(x+3)^{\frac{1}{3}}}\right)\left(\frac{x^{\frac{2}{3}}}{x^{\frac{2}{3}}}\right)$$ (3) Common denominator. If you subtracted, that is OK too.

$$0=\frac{x+3+2x}{3x^{\frac{2}{3}}(x+3)^{\frac{1}{3}}}=\frac{3(x+1)}{3x^{\frac{2}{3}}(x+3)^{\frac{1}{3}}}$$ (4) Final expression, can now use the zero product property to solve.

The critical values are $x=0,-3,-1$

$Q'(x)$		$x=-3$		$x = -1$		$x=0$	
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Therefore, $Q(x)$ is increasing on the interval $(-\infty,-3)\cup(-1,\infty)$ and decreasing on the interval $(-3,-1)$.

(b) Find the local maximum and minimum vales.

From above, the point $(-3,0)$ is a local maximum and the point $(-1,-\sqrt[3]{4})$ is a local minimum.

(c) Find the intervals of concavity and the inflection points.

$$Q'(x)=\frac{(x+1)}{x^{\frac{2}{3}}(x+3)^{\frac{1}{3}}}$$

$$Q''(x)=\frac{x^{\frac{2}{3}}(x+3)^{\frac{1}{3}}(1)-(x+1)\left[\frac{2}{3}x^{-\frac{1}{3}}(x+3)^{\frac{1}{3}}+\frac{1}{3}x^{\frac{2}{3}}(x+3)^{-\frac{2}{3}}\right]}{\left(x^{\frac{2}{3}}(x+3)^{\frac{1}{3}}\right)^{2}}$$

$$0=\frac{x^{\frac{2}{3}}(x+3)^{\frac{1}{3}}-(x+1)\left[\frac{2(x+3)^{\frac{1}{3}}}{3x^{\frac{1}{3}}}+\frac{x^{\frac{2}{3}}}{3(x+3)^{\frac{2}{3}}}\right]}{x^{\frac{4}{3}}(x+3)^{\frac{2}{3}}}$$

$$0=\frac{\left(\frac{3x^{\frac{1}{3}}(x+3)^{\frac{2}{3}}}{3x^{\frac{1}{3}}(x+3)^{\frac{2}{3}}}\right)x^{\frac{2}{3}}(x+3)^{\frac{1}{3}}-(x+1)\left[\left(\frac{2(x+3)^{\frac{1}{3}}}{3x^{\frac{1}{3}}}\right)\frac{(x+3)^{\frac{2}{3}}}{(x+3)^{\frac{2}{3}}}+\left(\frac{x^{\frac{2}{3}}}{3(x+3)^{\frac{2}{3}}}\right)\frac{x^{\frac{1}{3}}}{x^{\frac{1}{3}}}\right]}{x^{\frac{4}{3}}(x+3)^{\frac{2}{3}}}$$

$$0=\frac{3x(x+3)-(x+1)(2)(x+3)-(x+1)(x)}{3x^{\frac{1}{3}}(x+3)^{\frac{2}{3}}x^{\frac{4}{3}}(x+3)^{\frac{2}{3}}}=\frac{3x^2+9x-2x^2-8x-6-x^2-x}{3x^{\frac{5}{3}}(x+3)^{\frac{4}{3}}}$$

$$0=\frac{-2}{x^{\frac{5}{3}}(x+3)^{\frac{4}{3}}}$$

Therefore the possible points of inflection occur at $x=0,-3$.

$Q''(x)$

	$x=-3$		$x=0$	
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So the function is concave up on the interval $(-\infty,0)$ and concave down on the interval $(0,\infty)$. The only point of inflection occurs at $(0,0)$.

(2) Find the equation of the tangent line to $y = x^3 - 6x^2$ at its point of inflection.

$y = x^3 - 6x^2$

$y' = 3x^2 - 12x$

$y'' = 6x - 12$

Thus $x = 2$ is a possible inflection point. Sign pattern analysis shows that concavity goes from concave down to concave up at this point, making it the inflection point of the function.

$f(2) = -16$ This is the ordered pair

$f'(2) = -12$ This is the slope of the curve at the inflection point, therefore;

$-16 = -12(2) + b$

$b = 8$

$y = -12x + 8$

(3) Recall that the Mean Value Theorem states that if f(x) is continuous on [a,b] and differentiable on (a,b) then for some c in (a,b)

$$\frac{f(b)-f(a)}{b-a}=f'(c)$$

Find the value of c that satisfies the Mean Value Theorem on the interval $\left[\frac{1}{2},2\right]$ if $f(x)=x+\frac{1}{x}$.

The average rate of change on the interval is $\dfrac{\frac{5}{2}-\frac{5}{2}}{\frac{3}{2}}=0$. This means that the derivative must take on the value of zero for some number c between a and b.

$$f'(c)=1-\frac{1}{c^2}$$
$$0=1-\frac{1}{c^2}$$
$$c=\pm 1$$
$$c=1$$

We can reject negative one since it is not on our interval.

(4) What information can be deduced from the second derivative of a function regarding the function **and** its first derivative? List all you can, possible bonus points available here.

(Needed four of the following for full credit.)

When the second derivative is positive the function is concave up and the first derivative is increasing.

When the second derivative is negative the function is concave down and the first derivative is decreasing.

Where the concavity changes is called a point of inflection for the function.

If the second derivative is positive when the first derivative is zero, the function has a local minimum. If the second derivative is negative when the first derivative is zero, the function has a local maximum.