Tilings, Dimers, and Quiver Gauge Theories

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Thanks to:

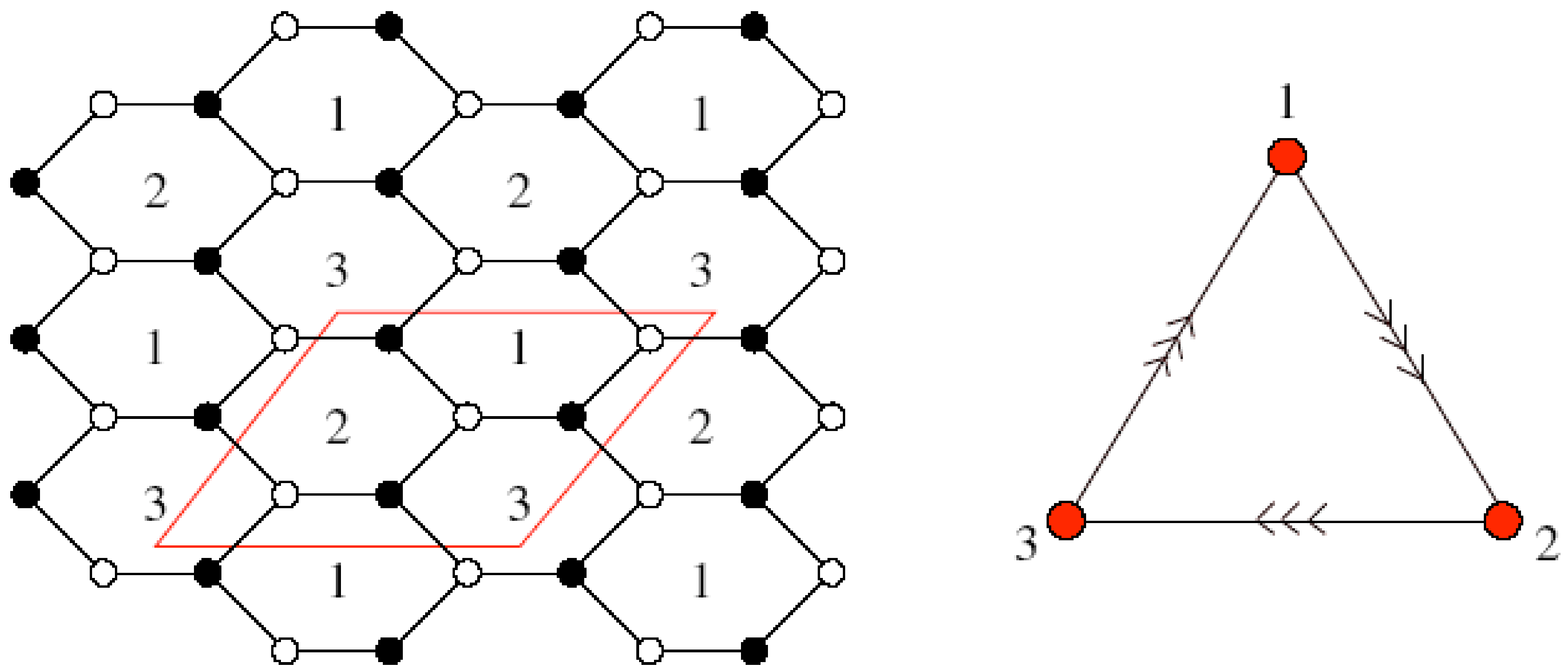
Benvenuti, Franco, Kazakopoulos, Kennaway, Martelli, Sparks, Uranga, Vegh, Wecht

Central Question

- What is the gauge theory living on a D3 brane that probes a non-compact singular CY manifold?
- The construction we will see today solves this long standing problem for the case of toric CY singularities

Motivation for study

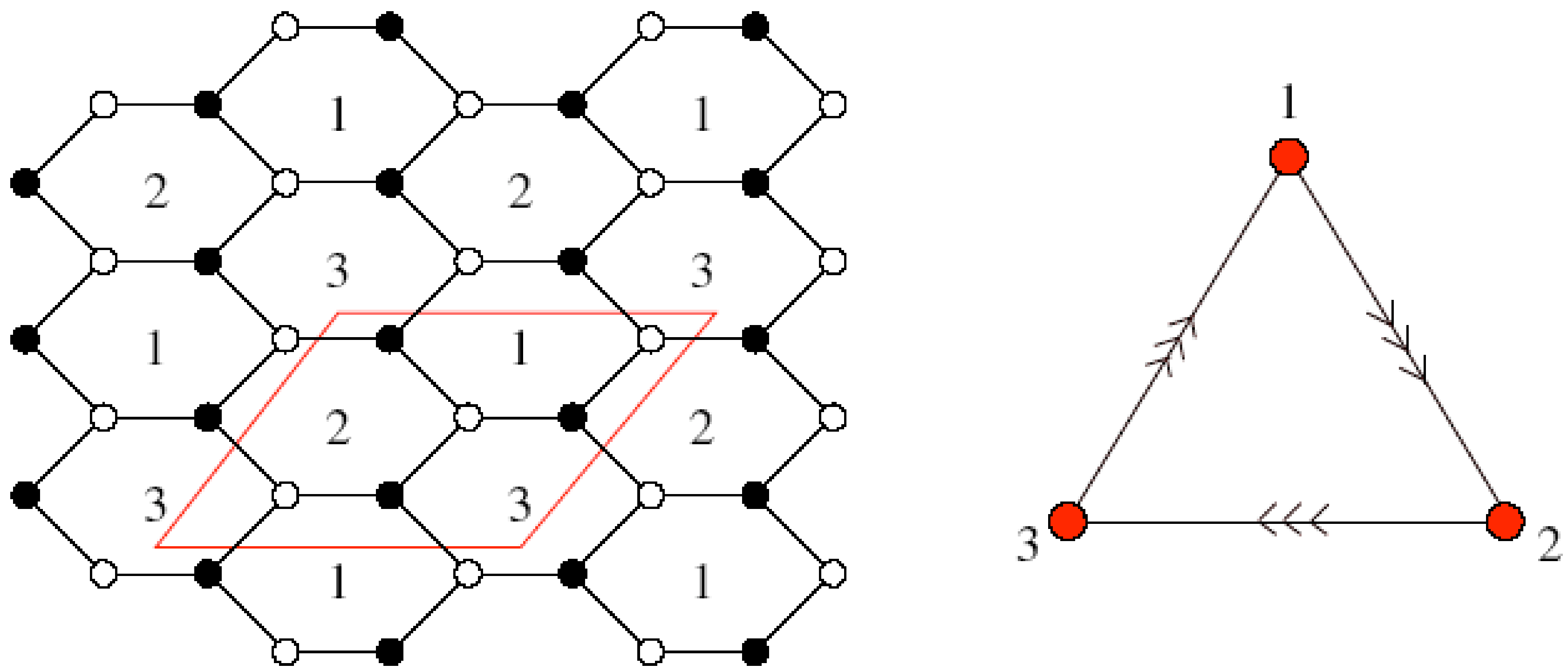
- Get information on $\mathcal{N}$=1 supersymmetric gauge theories - look close to real world
- Generically theories one studies are chiral - as in real world..
- More examples of SCFTs in 4 dimensions
- Get information on string backgrounds using D brane probes - what is a D brane?



Periodic bipartite tiling

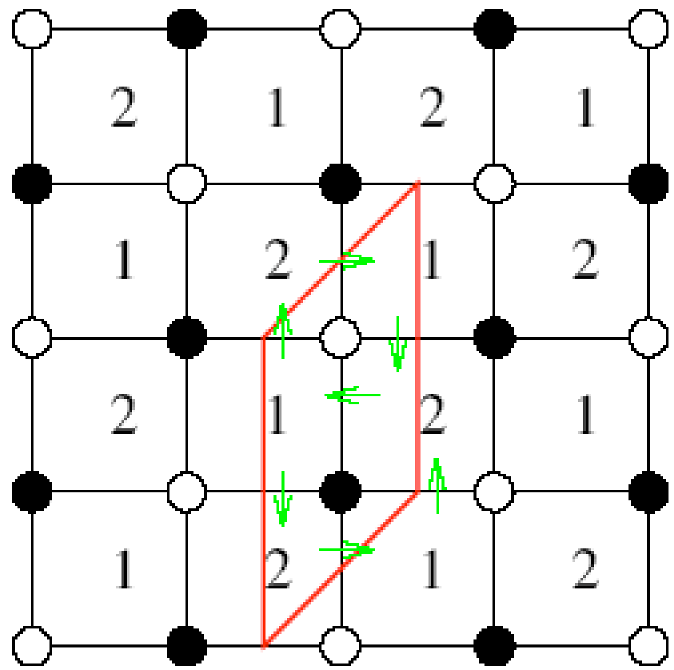
Tiling - Quiver dictionary

- 2n sided face - U(N) Gauge group with nN flavors
- Edge - A bi-fundamental chiral multiplet charged under the two gauge groups corresponding to the faces it separates.
- k valent node - A k-th order interaction term in the superpotential

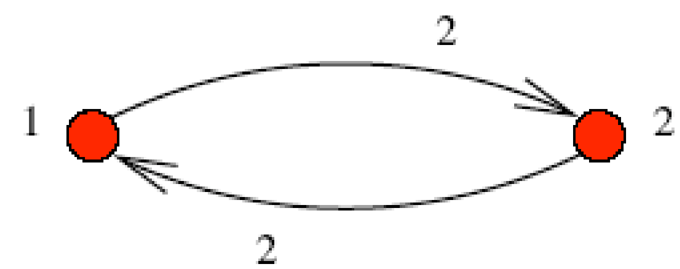


Z_3 orbifold of C^3

$$CY_6 = conifold$$



brane tiling



quiver

$$W = X_{12}^{(1)} X_{21}^{(1)} X_{12}^{(2)} X_{21}^{(2)} - X_{12}^{(1)} X_{21}^{(2)} X_{12}^{(2)} X_{21}^{(1)}$$

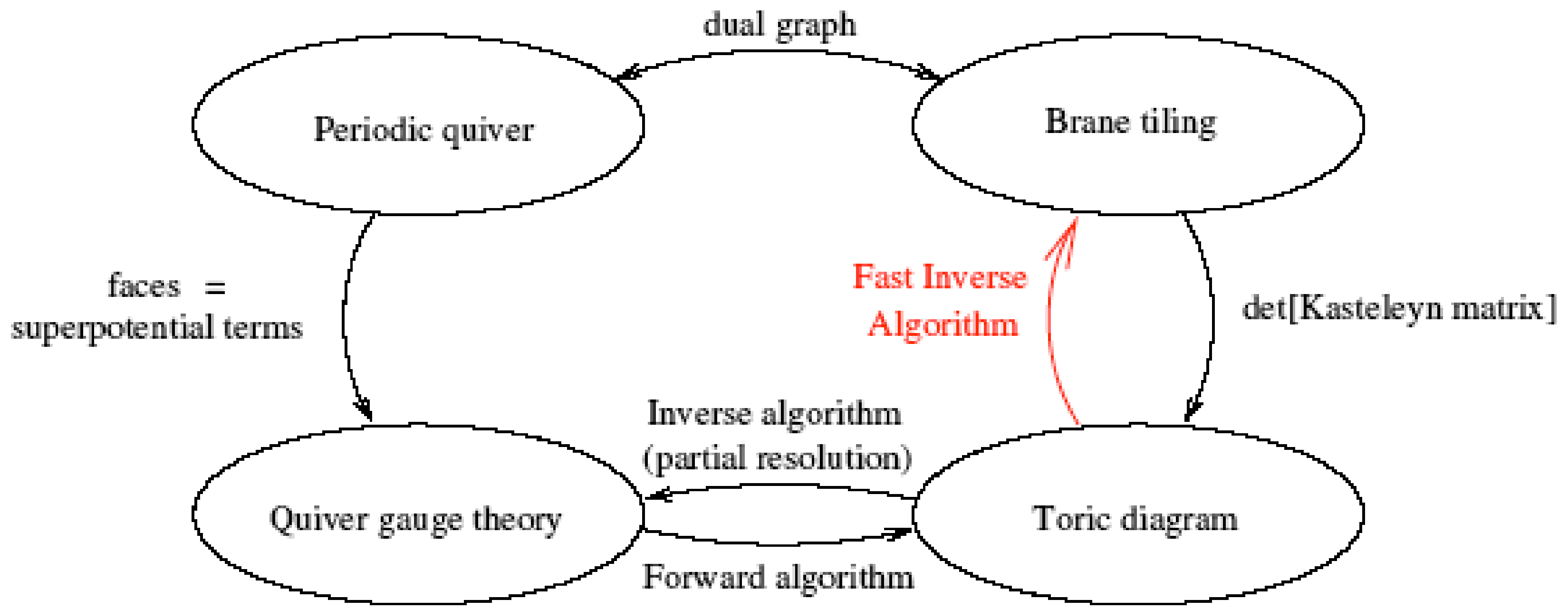
Example: Conifold

Comments

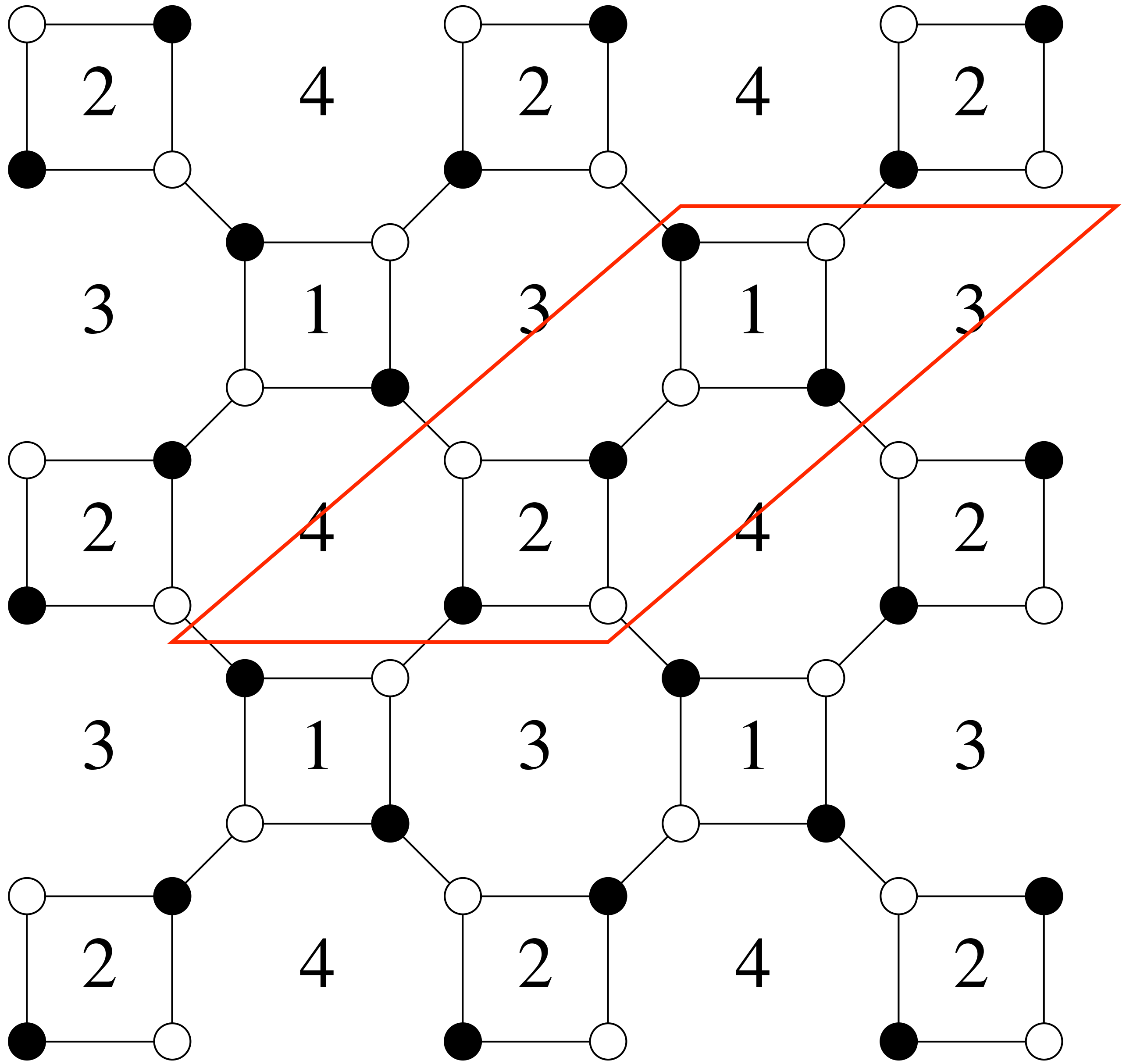
- Arrows are oriented in an alternating fashion
- Graph is bi-partite: Nodes alternate between clockwise (white) and counterclockwise (black) orientations of arrows
- black (white) nodes connected to white (black) only

Comments

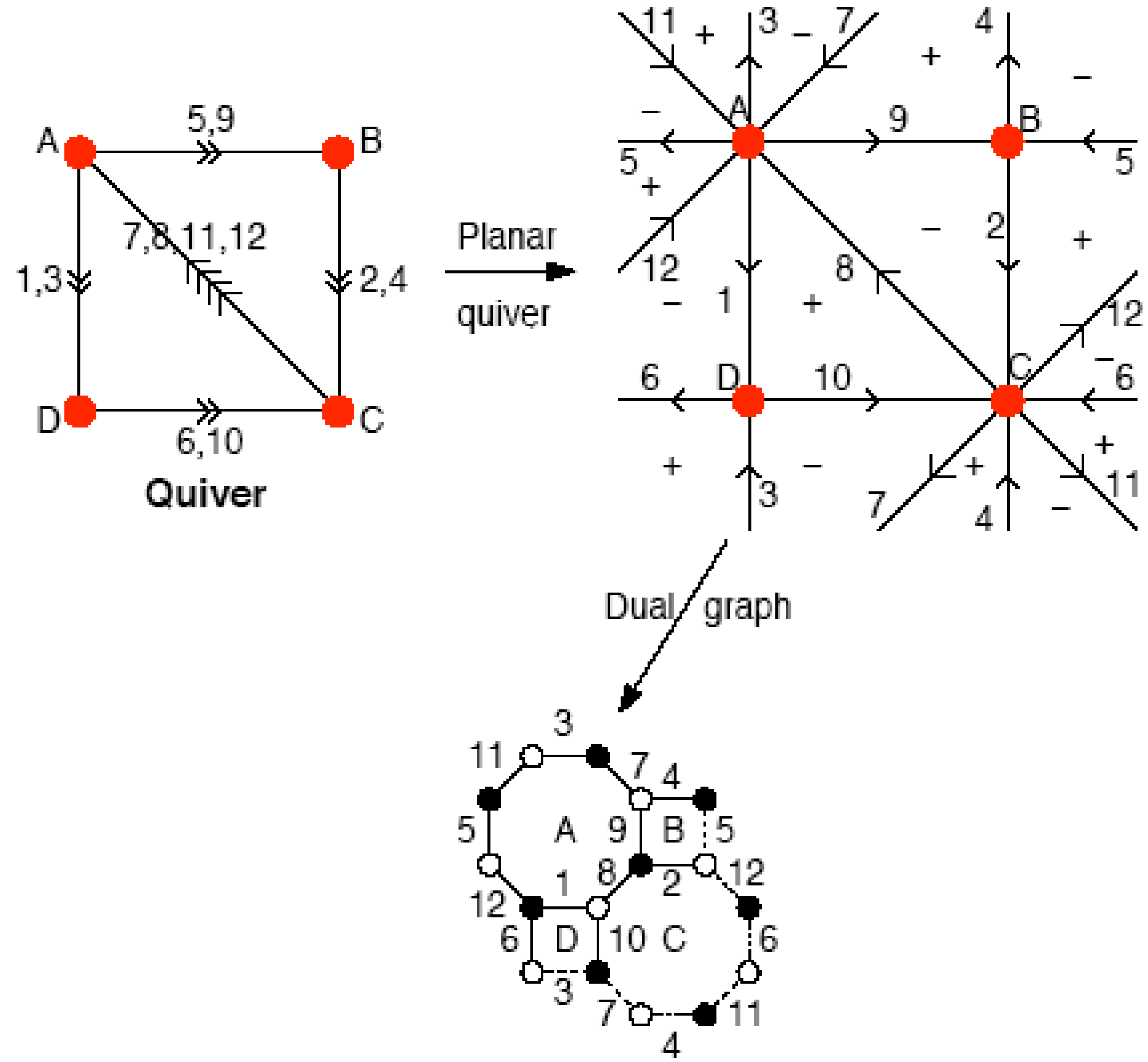
- odd sided faces are forbidden by anomaly cancellation condition
- white nodes with + sign in the superpotential
- black nodes with - sign in the superpotential
- These rules define a unique Lagrangian



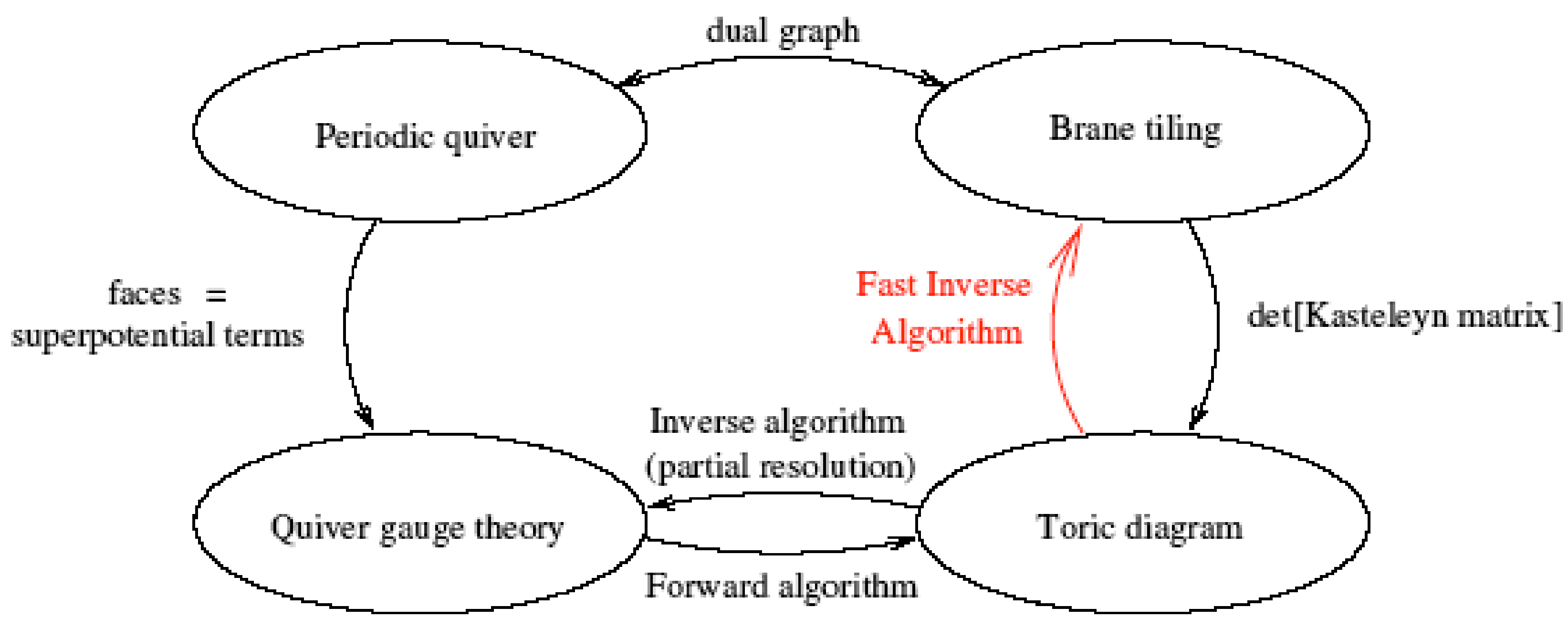
Logical flow chart



Tiling for F_0 ($P^1 \times P^1$)



Periodic Quiver for F_0



Logical flow chart

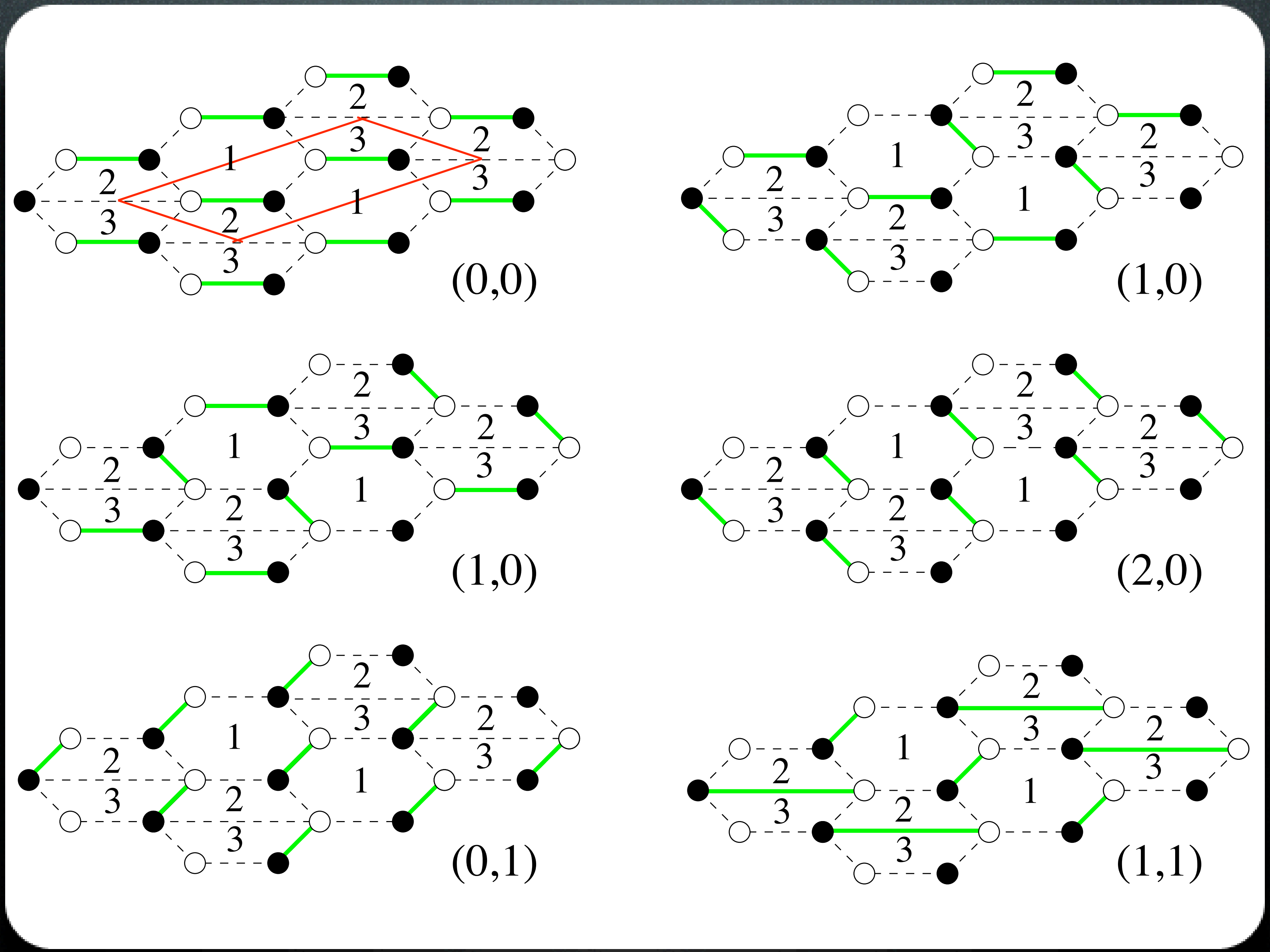
Brane tiling	Gauge theory
F: number of faces	N_g: number of gauge groups
E: number of edges	N_f: number of fields
N: number of nodes	N_W: number of superpotential terms

$$N_g + N_W - N_f = 0.$$

Euler's formula

Dimers

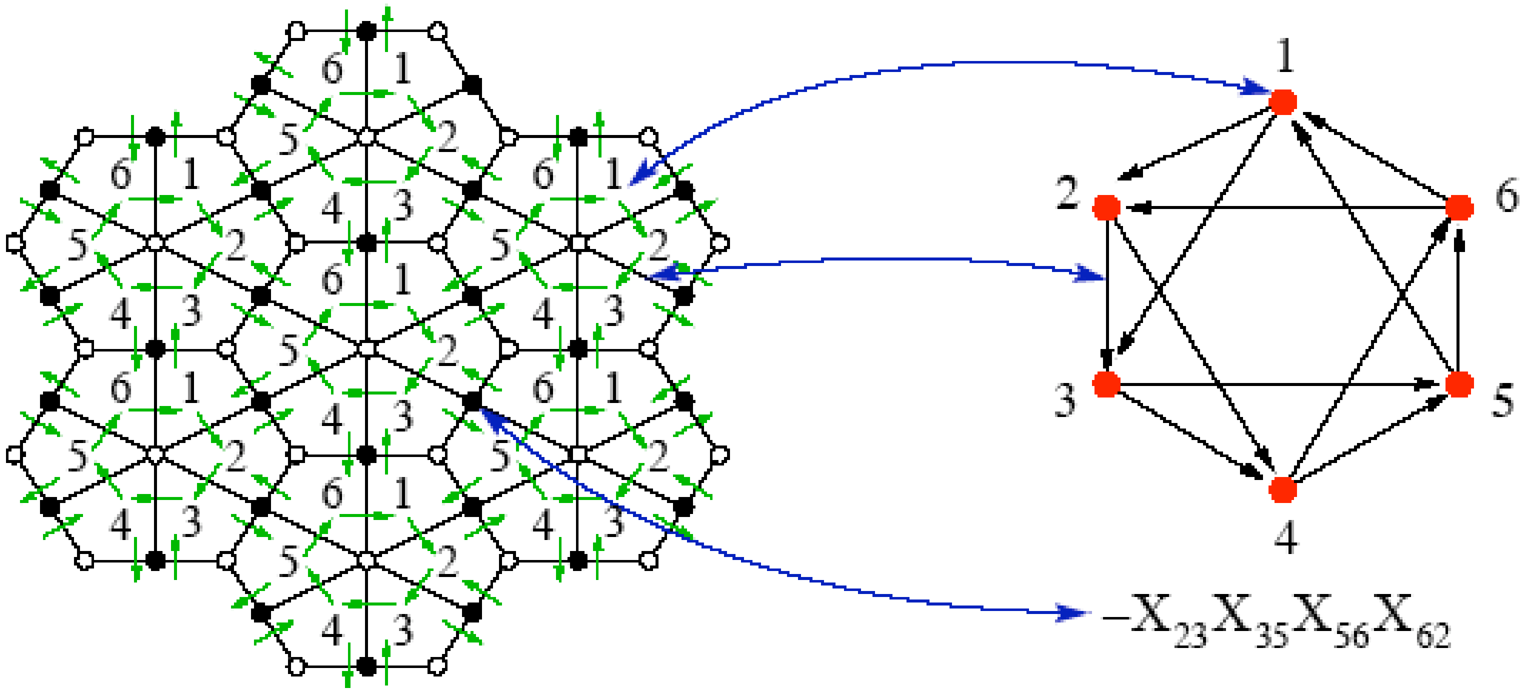
- and now for “Dimer” techniques
- Dimer - a line connecting 2 nodes
- Perfect matching - a collection of dimers such that every node is covered precisely once
- Adjacency matrix between white & black nodes - Kasteleyn matrix



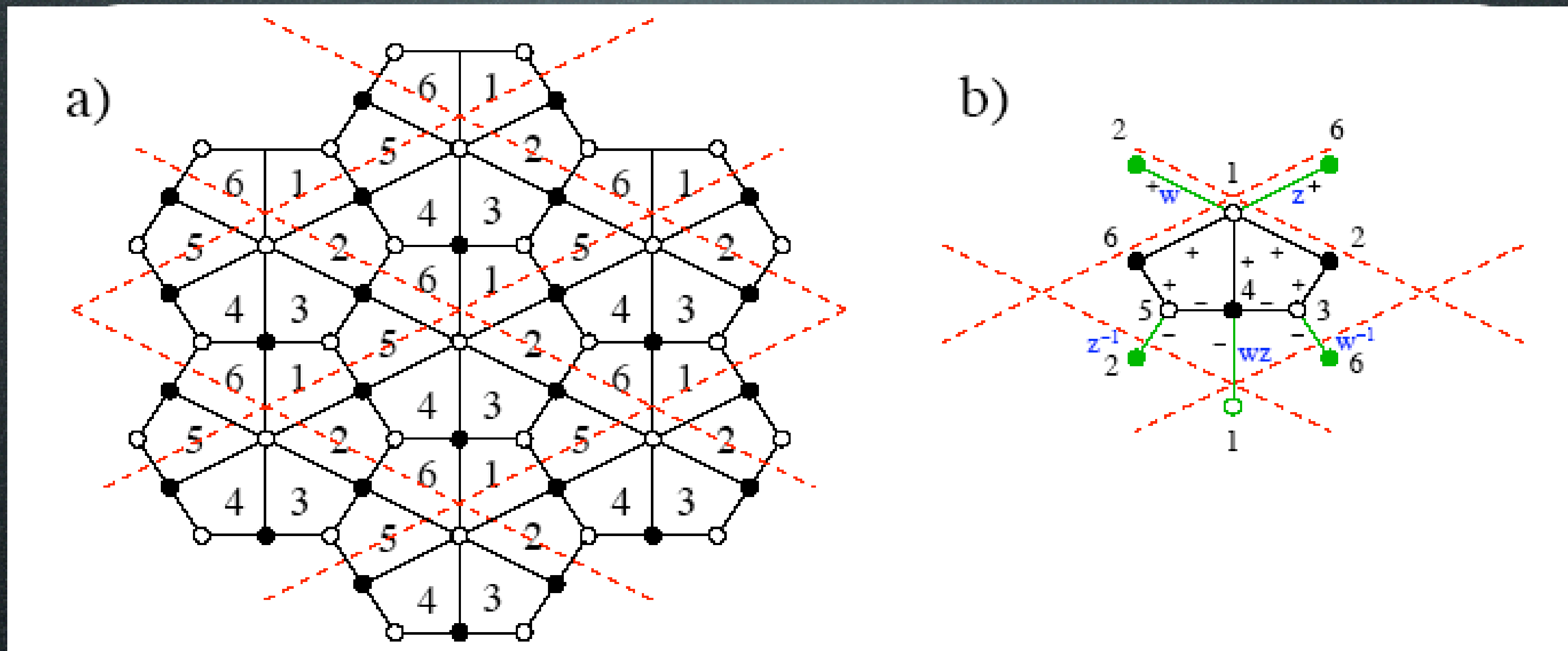
Perfect matchings SPP

Combinatorial Problem

- Given a Tiling, how many perfect matchings can one write down?
- Solved by writing the Adjacency (Kasteleyn) Matrix



Example: Del Pezzo 3, Model I



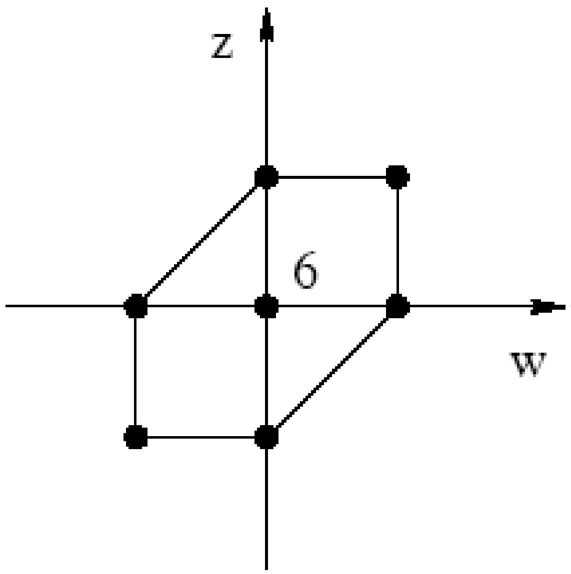
$$K = \left(\begin{array}{c|ccc} & 2 & 4 & 6 \\ \hline 1 & 1+w & 1-zw & 1+z \\ 3 & 1 & -1 & -w^{-1} \\ 5 & -z^{-1} & -1 & 1 \end{array}\right)$$

Kasteleyn matrix

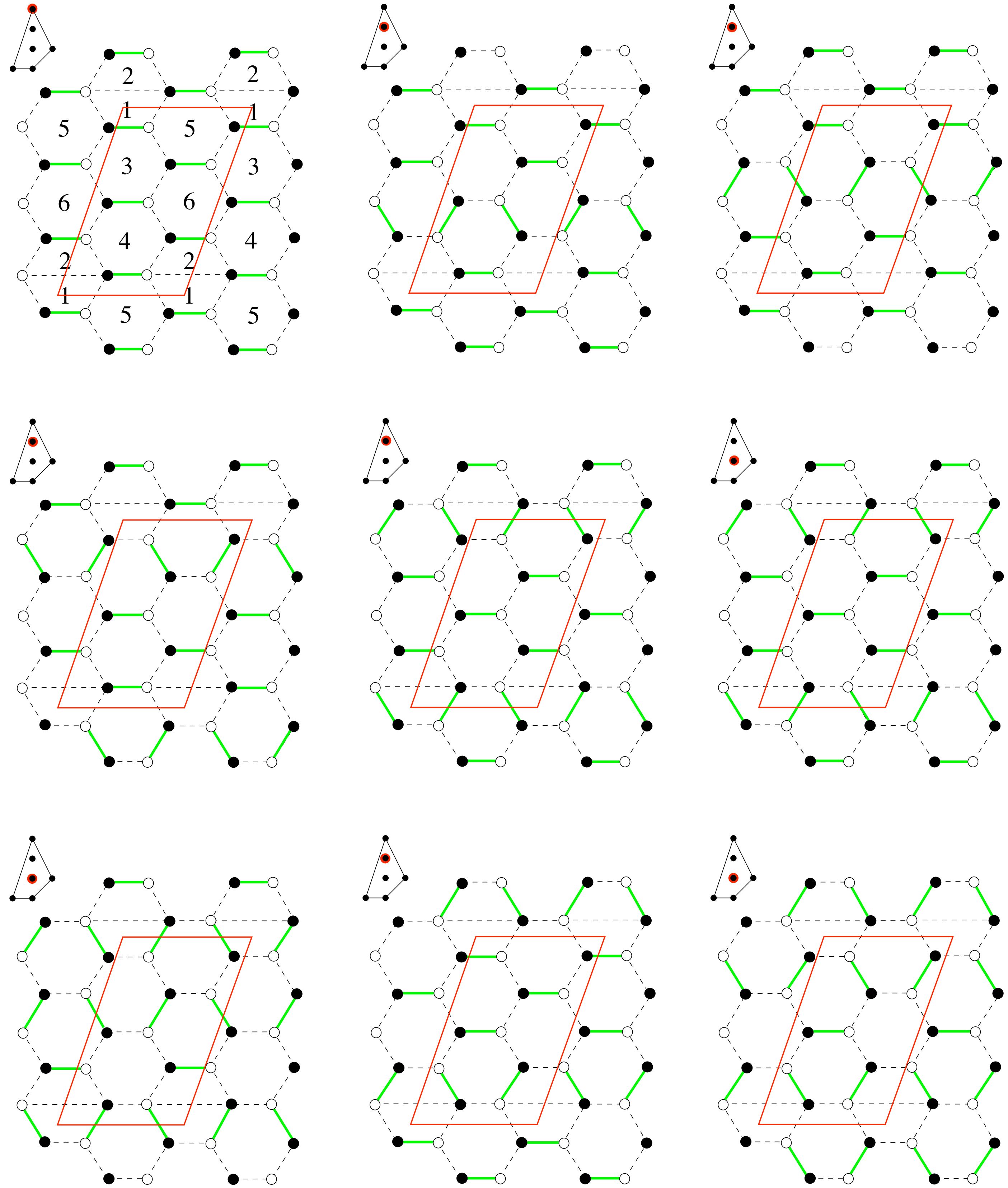
R. Kenyon

$$P(z,w) = \det K$$

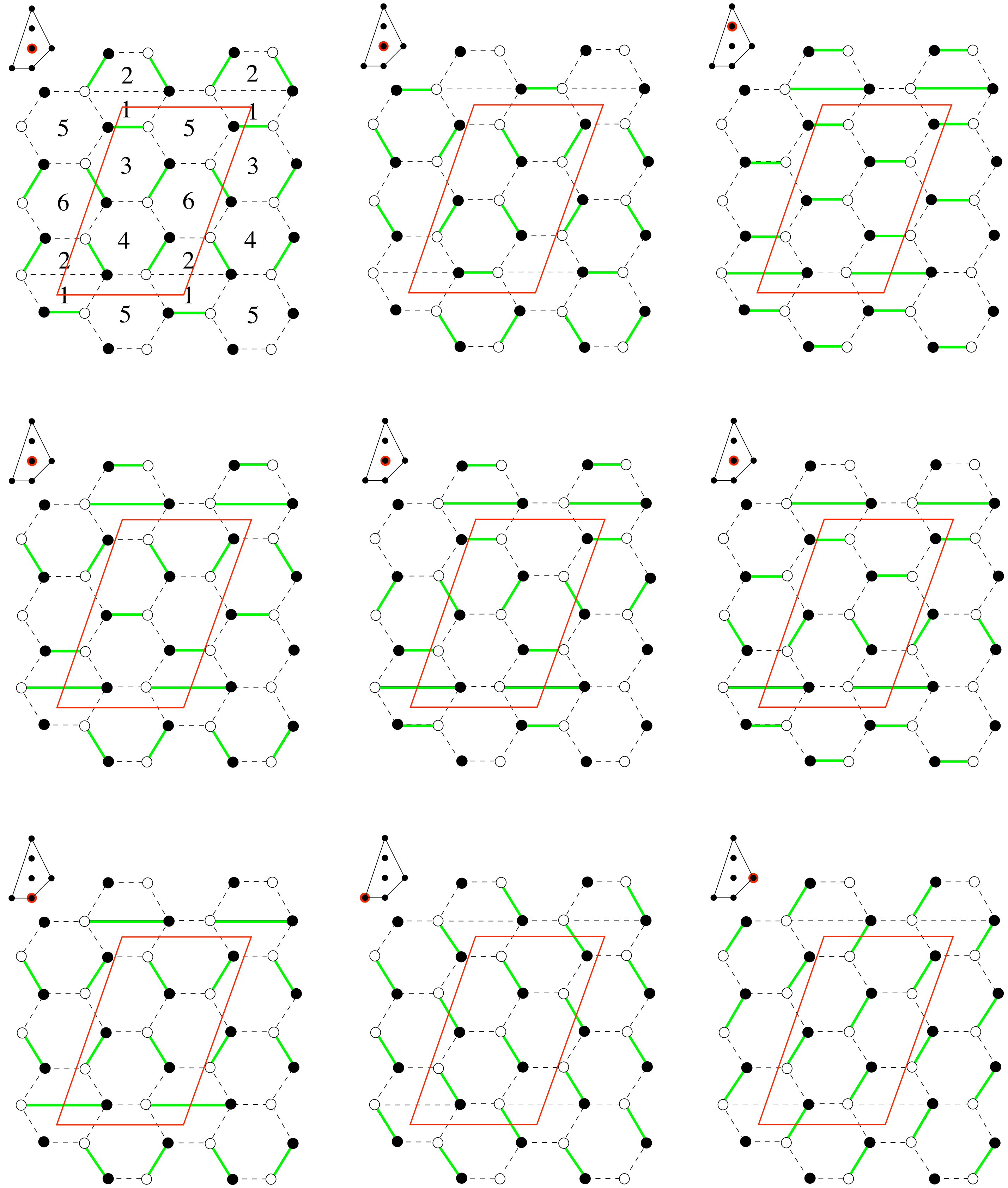
$$P(z,w) = w^{-1}z^{-1} - z^{-1} - w^{-1} - 6 - w - z + wz.$$



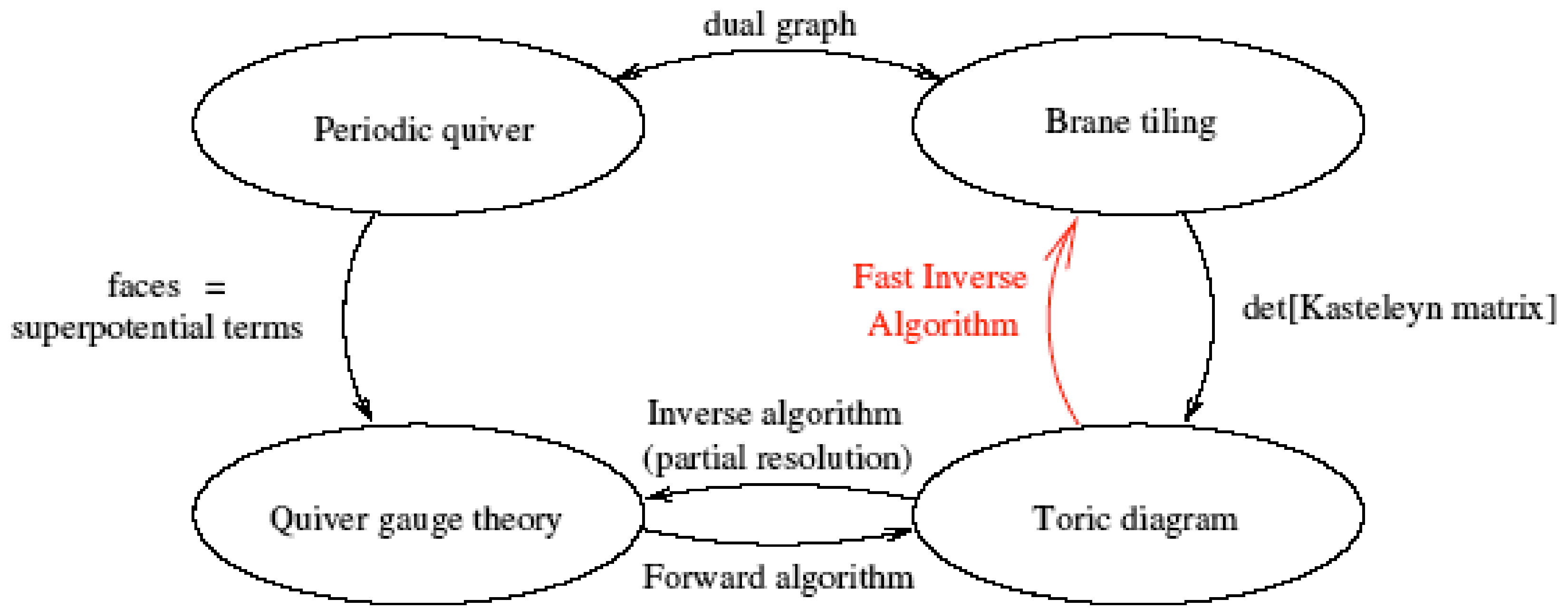
Toric diagram of dP3I



Perfect matching Y^{32} 1



Perfect matching Y^{32} II



Logical flow chart

Moduli Space of Vacua

- All quiver theories arising from periodic bipartite tilings have toric noncompact CY as their moduli space of vacua
- Computed using the Kasteleyn matrix:
- Adjacency matrix between white and black nodes
- det K gives a convex polygon on 2d lattice

homology from toric diagram

- Given toric diagram set
- I = # internal nodes, E = # external nodes
- #4 cycles = I
- #2 cycles = I + E - 3
- 2 Area = 2I + E - 2 (Pick's Theorem)
- # Gauge Groups = #4 + #2 + #0 = 2 Area

Bonus: multiplicities

- The coefficients of P(z,w) are integers and are the multiplicities of the linear sigma model fields used to define the CY as a toric variety

Orbifolds

- Here we report on some interesting aspect for orbifolds of the type
- C^3/Z_n or $C^3/(Z_n * Z_m)$
- Multiplicities of toric diagrams give a very rich combinatorial structure and sheds new light on orbifolds
- interesting from a mathematical point of view and physical point of view.

Multiplicities of Toric Diagrams

$$\mathbb{C}^3/(\mathbb{Z}_3\times\mathbb{Z}_3):\begin{bmatrix}1 & -3 & 3 & -1\\ -3 & -21 & -3 & \\ 3 & -3 & & \\ -1 & & & \end{bmatrix}\quad \mathbb{C}^3/(\mathbb{Z}_4\times\mathbb{Z}_4):\begin{bmatrix}1 & -4 & 6 & -4 & 1\\ -4 & -124 & -124 & -4 & \\ 6 & -124 & 6 & & \\ -4 & -4 & & & \\ 1 & & & & \end{bmatrix}$$

$$\mathbb{C}^3/(\mathbb{Z}_2\times\mathbb{Z}_4):\begin{bmatrix}1 & -2 & 1\\ -4 & -12 & \\ 6 & -2 & \\ -4 & & \\ 1 & & \end{bmatrix}\quad \mathbb{C}^3/(\mathbb{Z}_5\times\mathbb{Z}_5):\begin{bmatrix}1 & -5 & 10 & -10 & 5 & -1\\ -5 & -605 & -1905 & -605 & -5 & \\ 10 & -1905 & 1905 & -10 & & \\ -10 & -605 & -10 & & & \\ 5 & -5 & & & & \\ -1 & & & & & \end{bmatrix}$$

$$\mathcal{C}/(\mathbb{Z}_1\times\mathbb{Z}_2):\begin{bmatrix}1 & -1\\ -2 & -2\\ 1 & -1\end{bmatrix}\quad \mathcal{C}/(\mathbb{Z}_3\times\mathbb{Z}_3):\begin{bmatrix}1 & -3 & 3 & -1\\ -3 & -105 & -105 & -3\\ 3 & -105 & 105 & -3\\ -1 & -3 & -3 & -1\end{bmatrix}$$

orbifold examples

$\mathbb{C}^3/\mathbb{Z}_7$ with action $(1,3,3)$ [27, 29]

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 14 & 0 & 0 \\ 0 & 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & \\ & 3 & 1 \\ & 1 & \end{bmatrix}$$

$\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_4)$ with point $(0,4)$ removed.

$$\begin{bmatrix} 1 & -2 & 1 \\ -3 & -9 & \\ 3 & -1 & \\ -1 & & \end{bmatrix}$$

$\mathbb{C}^3/\mathbb{Z}_7$ orbifold with action $(1,2,4)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 7 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 16 & 0 \\ 0 & 20 & 0 \\ 0 & 8 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$C^3/Z_4 \times Z_4$ with 2 points removed

$$\begin{bmatrix} 1 & -3 & 3 & -1 \\ -3 & -65 & -43 & -1 \\ 3 & -51 & 2 & \\ -1 & -1 & & \end{bmatrix} \quad \begin{bmatrix} 1 & -3 & 3 & -1 \\ -3 & -71 & -53 & -1 \\ 3 & -45 & 2 & \\ -1 & -1 & & \end{bmatrix} \quad \begin{bmatrix} 1 & -3 & 3 & -1 \\ -3 & -81 & -59 & -1 \\ 3 & -59 & 2 & \\ -1 & -1 & & \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 3 & -1 \\ -3 & -65 & -51 & -1 \\ 3 & -43 & 2 & \\ -1 & -1 & & \end{bmatrix} \quad \begin{bmatrix} 1 & -3 & 3 & -1 \\ -3 & -71 & -45 & -1 \\ 3 & -53 & 2 & \\ -1 & -1 & & \end{bmatrix} \quad \begin{bmatrix} 1 & -3 & 3 & -1 \\ -3 & -73 & -43 & -1 \\ 3 & -43 & 2 & \\ -1 & -1 & & \end{bmatrix}$$

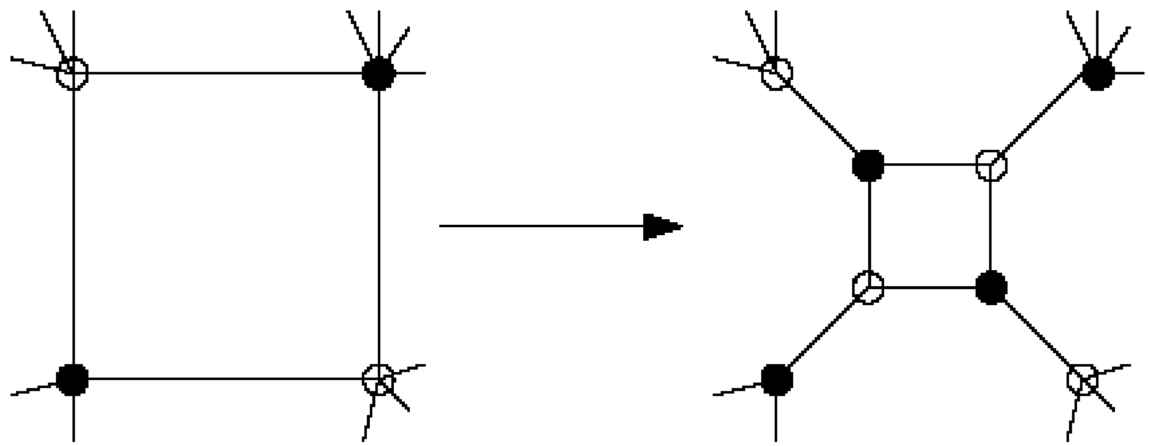
orbifold formulae

$$(z_1, z_2, z_3) \mapsto (\lambda z_1, z_2, \lambda^{-1} z_3), \quad \lambda^n = 1$$
$$(z_1, z_2, z_3) \mapsto (z_1, \omega z_2, \omega^{-1} z_3), \quad \omega^m = 1$$

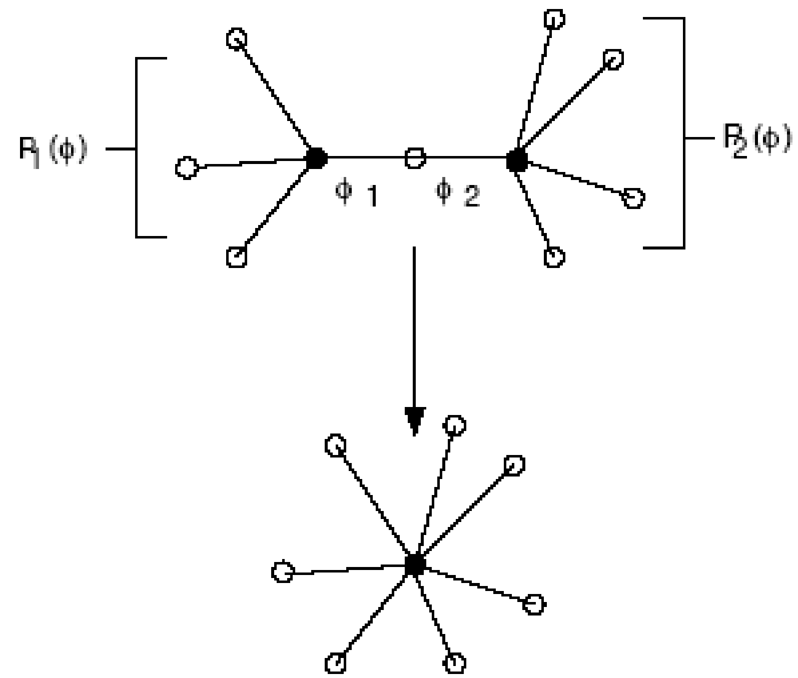
$$P_{n,m}(z,w) = \prod_{i=0}^{n-1} \prod_{j=0}^{m-1} P(\lambda^i z^{1/n}, \omega^j w^{1/m})$$

- P(z,w)=1+z+w
- Z_n action: (1,a,-1-a)
- P_n (z,w) is Resultant_x of x^n+z & $x^{a+1}+x^a+w$

$$P_n(z,w) = \prod_{i=0}^{n-1} P(\lambda^{ai} w^{a/n} z^{(n-a)/n}, \lambda^{(a+1)i} w^{(a+1)/n} z^{(n-a-1)/n})$$



Seiberg Duality



Integrating out massive fields

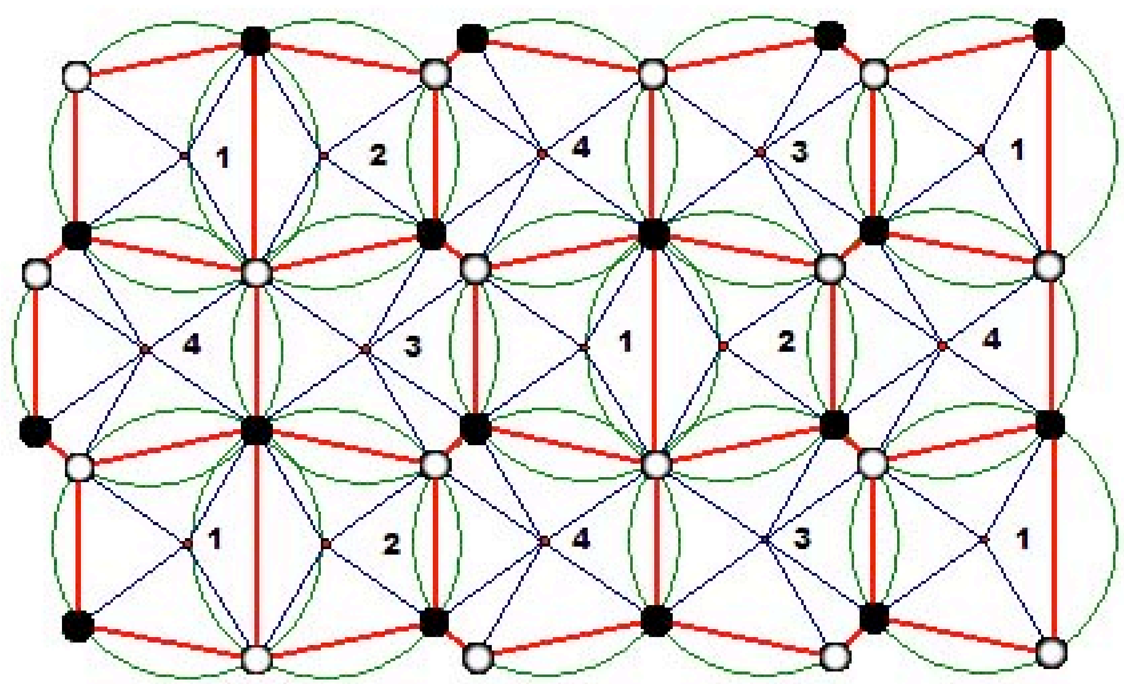
$$\sum_{i \in edges\ around\ node} R_i = 2 \qquad \text{for each node}$$

$$\sum_{i \in edges\ around\ face} (1 - R_i) = 2 \qquad \text{for each face}$$

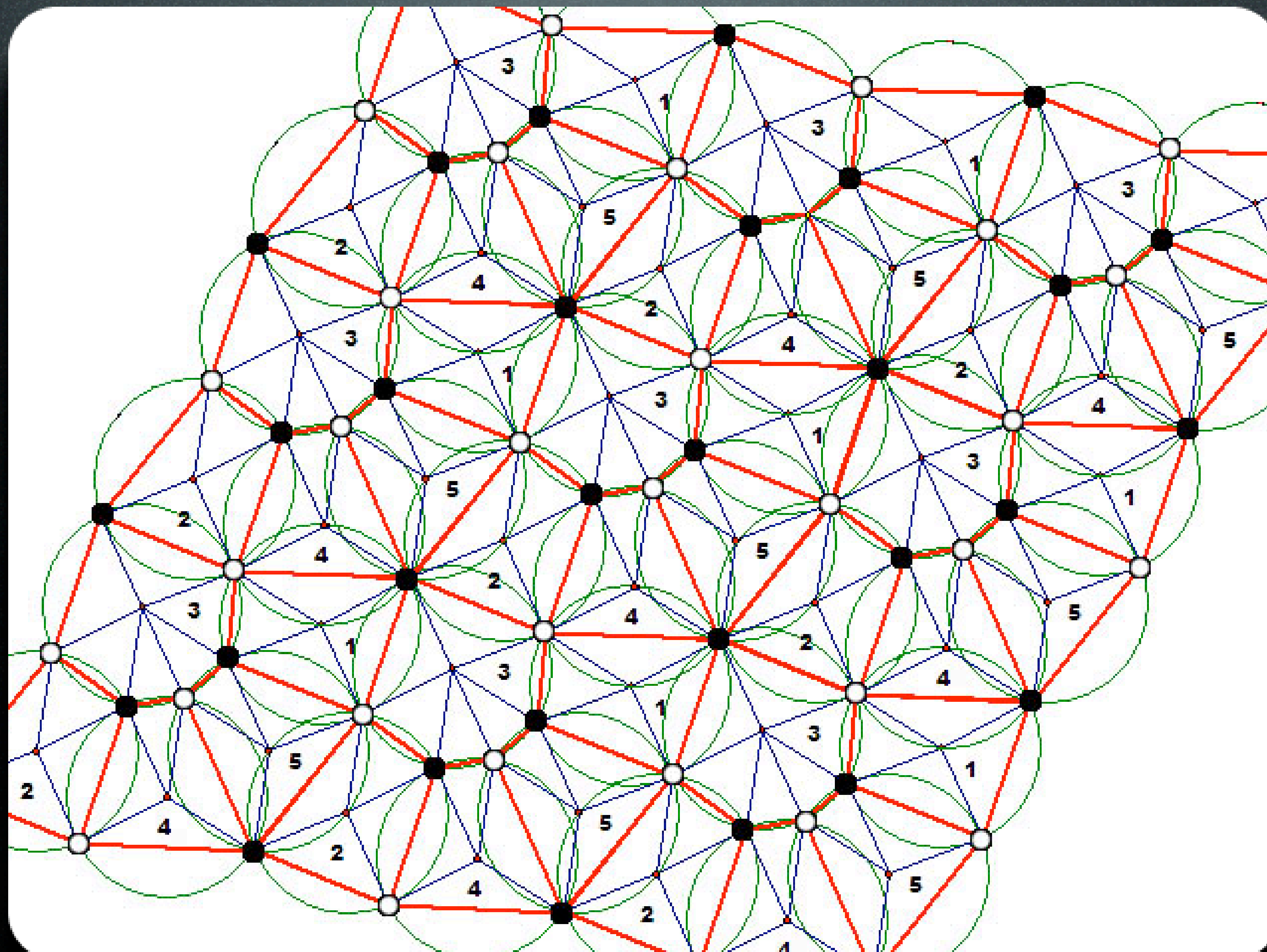
$$\sum_{i \in edges\ around\ node} (\pi R_i) = 2\pi \qquad \text{for each node}$$

$$\sum_{i \in edges\ around\ face} (\pi R_i) = (\#edges - 2)\pi \qquad \text{for each face}$$

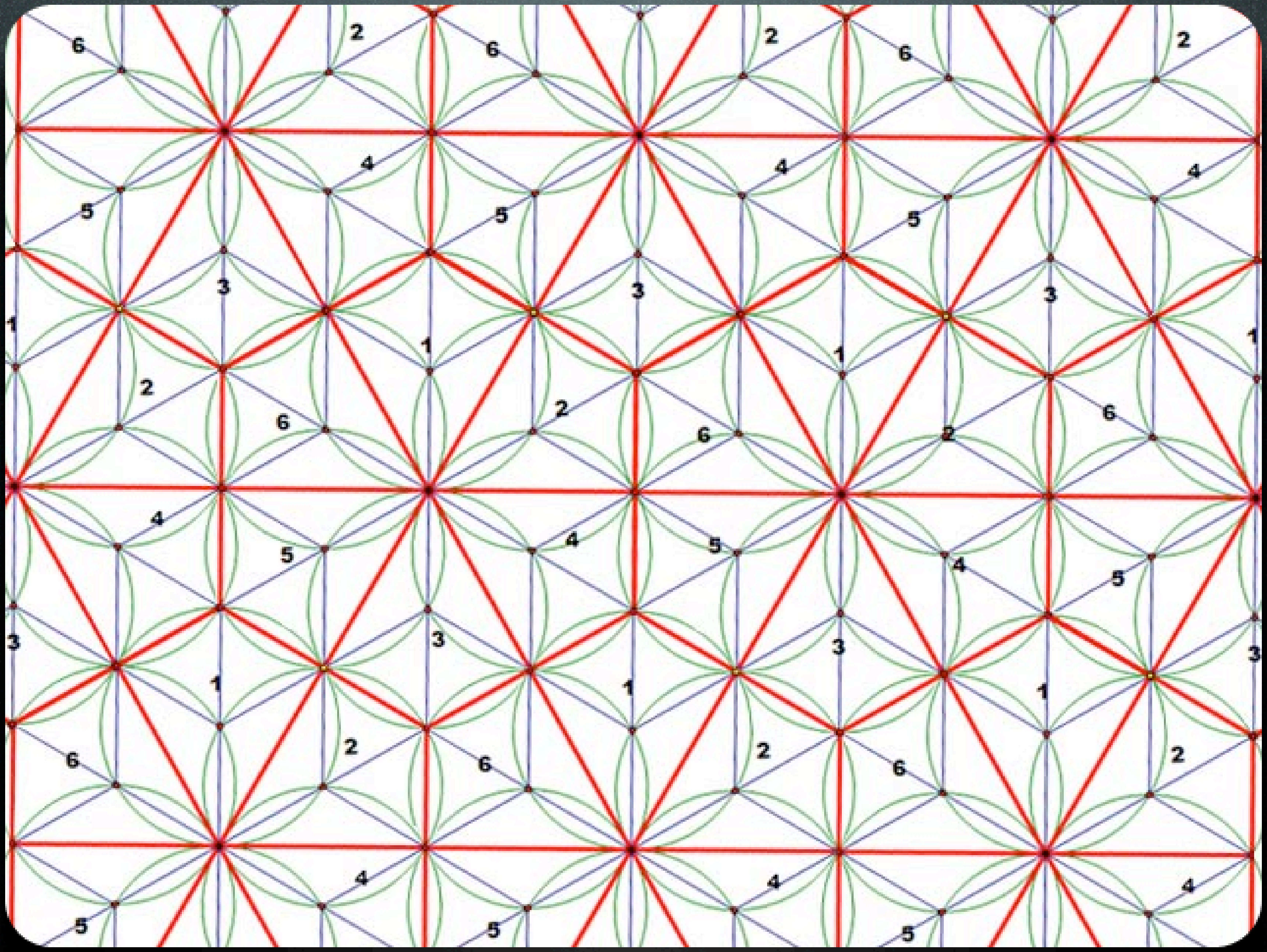
IR fixed point



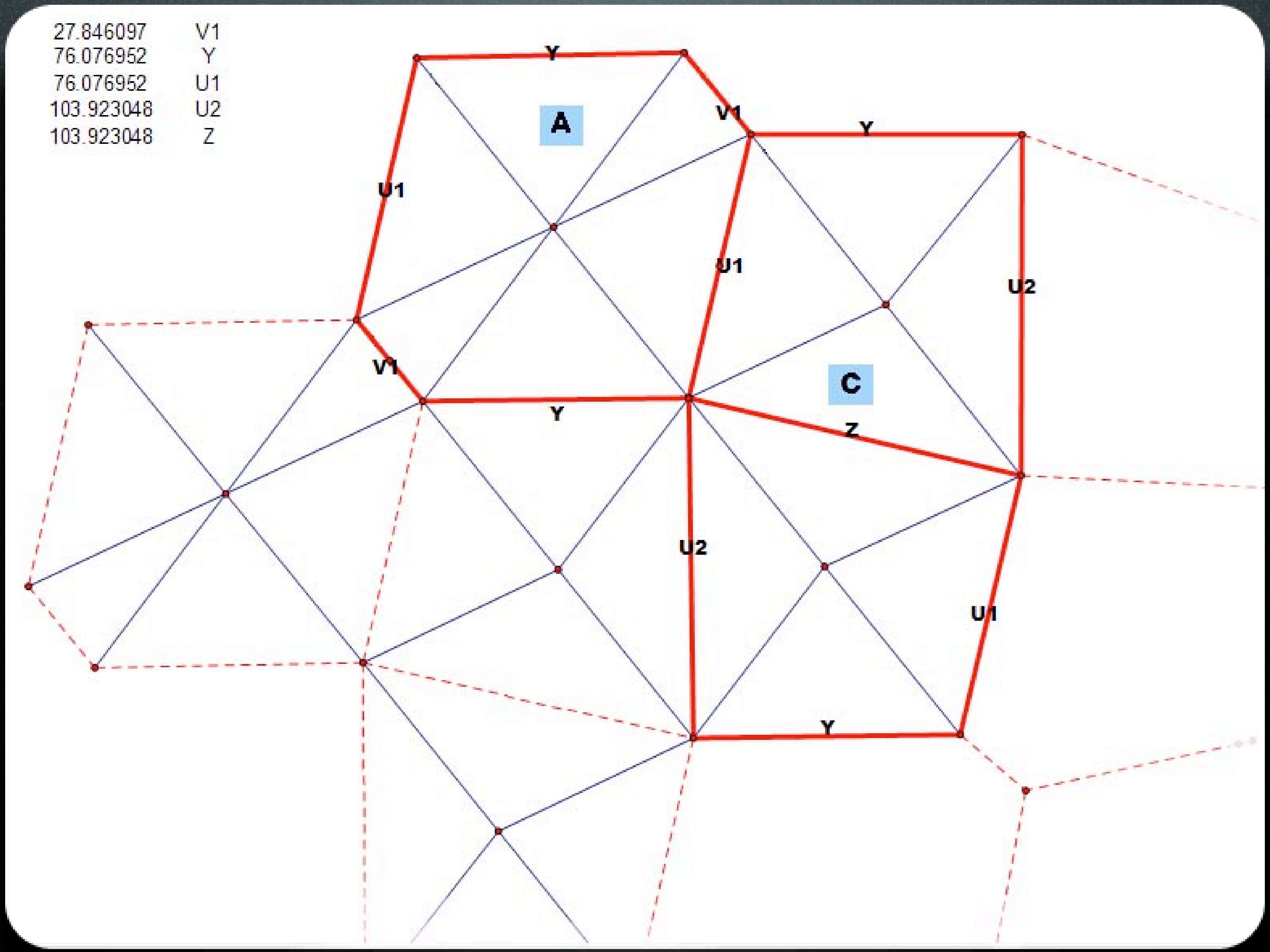
Isoradial embed. dP1



dP2 II

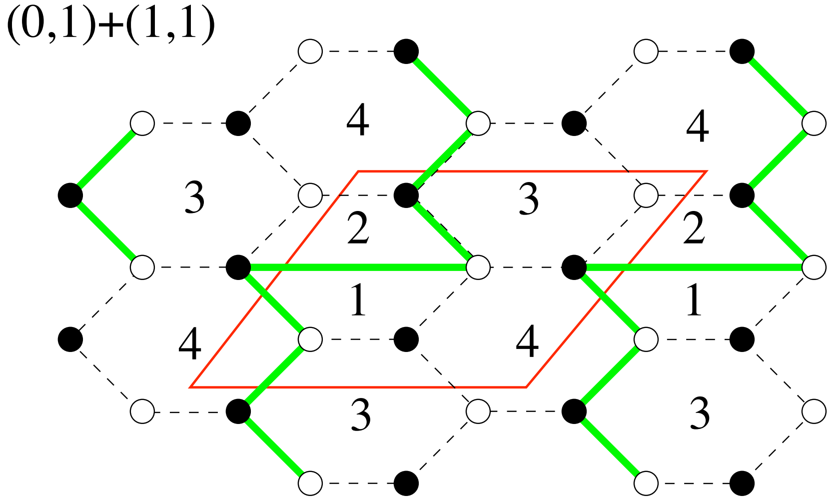


dP3 I



27.846097 V1
76.076952 Y
76.076952 U1
103.923048 U2
103.923048 Z
A
C
Y
V1
U1
U2
Z

SPP tiling



Zig-Zag path (dP1)

properties of zig-zag paths

- each edge has precisely two paths going through it
- each path corresponds to an external leg in the (p,q) web dual to the toric diagram
- important for computing R charges & a-maximization

Conclusions

- Periodic tilings of 2d plane - N=1 SCFT's
- compute properties of quiver gauge theories using dimer techniques
- Solved a long standing problem - computing superpotetials for D3 brans probing singular CY's
- Construct infinite families of quiver gauge theories ($Y^{p,q}$ $L^{a,b,c}$ ...)