The stochastic heat equation driven by a Gaussian noise: Markov property

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Outline

Germ Markov property

The germ σ-field

Let S be a subset in $[0, T] \times \mathbb{R}^d$.
$\mathcal{F}_S$: the σ-field generated by $\{u(t,x) : (t,x) \in S\}$

$$\mathcal{G}_S = \bigcap_{O \text{ open}: O \supset S} \mathcal{F}_S.$$

Definition

The process $\{u(t,x) : (t,x) \in [0, T] \times \mathbb{R}^d\}$ is **germ Markov** if for every precompact open set $A \subset [0, T] \times \mathbb{R}^d$,

$$\mathcal{G}_{\bar{A}} \perp \mathcal{G}_{A^c} \mid \mathcal{G}_{\partial A},$$

where $\partial A = \bar{A} \cap A^c$.

Some cases investigated

- Donati-Martin and Nualart, 1994

$$\begin{cases} -\Delta u + f(u) = \dot{W}, & x \in D \\ u|_{\partial D} = 0 \end{cases},$$

where D is a bounded domain in $\mathbb{R}^d$, $d = 1, 2, 3$. f is an affine function.

- Nualart and Pardoux, 1994

$$\begin{cases} u_t = u_{xx} + f(u) + \dot{W}, & (t,x) \in [0,1]^2 \\ u(0,x) = u_0(x), & 0 \le x \le 1;\ u(t,0) = u(t,1) = 0,\ 0 \le t \le 1. \end{cases}$$

- Dalang and Hou, 1997

$$u_{tt} = \Delta u + \dot{L},$$

where L is locally finite Lévy process.

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The noise

Gaussian noise with spatial correlation (Dalang, 1999)

M=$\{M(\varphi), \varphi \in \mathcal{D}((0,T)\times\mathbb{R}^d)\}$ Gaussian process with covariance

$$\mathbb{E}(M(\varphi)M(\psi)) = \int_0^\infty \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(t,x)f(x-y)\psi(t,y)\,dx\,dy\,dt$$

$$= \int_0^T \int_{\mathbb{R}^d} \mathcal{F}\varphi(t,\xi)\overline{\mathcal{F}\psi(t,\xi)}\,\mu(d\xi)dt := \langle\varphi,\psi\rangle_0$$

Here $f = \mathcal{F}\mu$, where μ is a tempered measure on $\mathbb{R}^d$

- Riesz kernel $f(x) = c_{\alpha,d}|x|^{-\alpha}$
- Bessel kernel $f(x) = c_\alpha \int_0^\infty s^{(\alpha-d)/2-1}e^{-s-|x|^2/(4s)}ds$
- Heat kernel $f(x) = c_{\alpha,d}e^{-|x|^2/(4\alpha)}$
- Poisson kernel $f(x) = c_{\alpha,d}(|x|^2+\alpha^2)^{-(d+1)/2}$

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Outline

Stochastic integral with respect to M

Space of deterministic integrands

$\mathcal{P}_0^{(d)}$ is the completion of $\mathcal{D}((0,T)\times\mathbb{R}^d)$ with respect to $\langle\cdot,\cdot\rangle_0$
(This is a space of *distributions* in x !)

Stochastic integral

$$M(\varphi)=\int_0^T\int_{\mathbb{R}^d}\varphi(s,x)M(ds,dx)$$

is defined an an isometry $\varphi\mapsto M(\varphi)$ between $\mathcal{P}_0^{(d)}$ and the Gaussian space H^M:

$$\mathbb{E}M(\varphi)M(\psi)=\langle\varphi,\psi\rangle_0,\quad\forall\varphi,\psi\in\mathcal{P}_0^{(d)}$$

Outline

Stochastic heat equation

Stochastic heat equation driven by $\dot{M}$

$$\begin{cases} u_t = \Delta u + \dot{M} \quad \text{in} \quad [0,T] \times \mathbb{R}^d \\ u(0,x) = 0 \end{cases}.$$

Mild solution

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) M(ds, dy),$$

where

$$G(t,x) = (4\pi t)^{-d/2} e^{-|x|^2/(4t)}, \quad t > 0, x \in \mathbb{R}^d$$

Remark:

$G(t-\cdot, x-\cdot) \in \mathcal{P}_0^{(d)}$ if and only if $\int_{\mathbb{R}^d} (1+|\xi|^2)^{-1} \mu(d\xi) < \infty$.

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The Reproducing kernel Hilbert space $\mathcal{H}^u$

Row of isometries

$$\mathcal{P}_0^{(d)} \to H^M = H^u \to \mathcal{H}^u$$
$$\varphi \mapsto M(\varphi) = Y \mapsto h_Y(t,x) = \mathbb{E}(Yu(t,x))$$

H^u = span of $\{u(t,x) : (t,x) \in [0,T] \times \mathbb{R}^d\}$ in $L_2(\Omega)$
H^M = $\{M(\varphi); \varphi \in \mathcal{P}_0^{(d)}\}$

Definition of $\mathcal{H}^u$:

$$\mathcal{H}^u = \{h(t,x) = \mathbb{E}(M(\varphi)u(t,x)) : \varphi \in \mathcal{P}_0^{(d)}\}$$

and

$$\langle h, g\rangle_{\mathcal{H}^u} = \mathbb{E}(M(\varphi)M(\psi)) = \langle \varphi, \psi\rangle_0,$$

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Attempt to characterize the elements of $\mathcal{H}^u$

Formal calculation

$$\begin{aligned} h(t,x) &= \mathbb{E}M(\varphi)u(t,x) = \mathbb{E}M(\varphi)M(G(t-\cdot,x-\cdot)) \\ &= \langle \varphi, G(t-\cdot,x-\cdot)\rangle_0 \\ &= \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(t-s,x-y)f(y-z)\varphi(s,z)\,dy\,dz\,ds \\ &= \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)\varphi_1(s,y)\,dy, \end{aligned}$$

where

$$\varphi_1(s,y) = \int_{\mathbb{R}^d} \varphi(s,z)f(y-z)\,dz.$$

Intuitively, h should be a solution of:

$$\begin{cases} h_t = \Delta h + \varphi_1 & \text{in} \quad (0,T)\times\mathbb{R}^d \\ h(0,x) = 0 \end{cases}.$$

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Outline

Characterization of $\mathcal{H}^u$ (Bessel kernel of order α)

Spaces of Bessel potentials

$$H_2^{\gamma}(\mathbb{R}^d) = \{(1-\Delta)^{-\gamma/2} g;\ g \in L_2(\mathbb{R}^d)\}, \quad \gamma \in \mathbb{R}$$

Isometry

$$\mathcal{P}_0^{(d)} \subset L_2((0,T), H_2^{-\alpha/2}(\mathbb{R}^d)) \to L_2((0,T), H_2^{\alpha/2}(\mathbb{R}^d))$$
$$\varphi \mapsto \varphi_1 = (1-\Delta)^{-\alpha/2}\varphi$$

Theorem A

Let $h(t,x) = \mathbb{E}M(\varphi)u(t,x)$, $\varphi \in \mathcal{P}_0^{(d)}$. Then h is the unique solution in $L_2((0,T), H_2^{\alpha/2+2}(\mathbb{R}^d))$ of

$$h_t = \Delta h + \varphi_1, \qquad h(0,x) = 0.$$

Outline

Characterization of $\mathcal{H}^u$ (Riesz kernel of order $\alpha = 4k$)

Spaces of Riesz potentials

$$K^{\alpha/2}(\mathbb{R}^d) = \{(-\Delta)^{-\alpha/4} g;\ g \in L_2(\mathbb{R}^d)\} \subset L_q(\mathbb{R}^d)$$

where $1/q = 1/2 - \alpha/(2d)$.

$$K_2^{-\alpha/2}(\mathbb{R}^d) := \{\varphi \in \mathcal{S}'(\mathbb{R}^d); \mathcal{F}\varphi \text{ is a function, } \int_{\mathbb{R}^d} |\mathcal{F}\varphi(\xi)|^2 |\xi|^{-\alpha} d\xi < \infty\}$$
$$= \text{completion of } \mathcal{S}(\mathbb{R}^d) \text{ w.r.t. } \|\cdot\|_{K_2^{-\alpha/2}(\mathbb{R}^d)}$$

Maps:

$$\mathcal{P}_0^{(d)} \to L_2((0,T) \times \mathbb{R}^d) \to L_2((0,T), K_2^{\alpha/2}(\mathbb{R}^d))$$
$$\varphi \mapsto \varphi_0 = (-\Delta)^{-\alpha/4}\varphi \mapsto \varphi_1 = (-\Delta)^{-\alpha/4}\varphi_0$$

Characterization of $\mathcal{H}^u$ (Riesz kernel of order α)

Theorem B

Let $h(t,x) = \mathbb{E}M(\varphi)u(t,x)$, $\varphi \in \mathcal{P}_0^{(d)}$. Then h is the unique solution in $W_{2,q}^{1,2}((0,T) \times \mathbb{R}^d)$ of

$$h_t = \Delta h + \varphi_1, \qquad h(0,x) = 0,$$

where $W_{2,q}^{1,2}((0,T) \times \mathbb{R}^d)$ is the space of functions u such that $u, u_t, u_{x_i}, u_{x_i x_j}$ are in $L_2((0,T), L_q(\mathbb{R}^d))$.

Outline

A fundamental result

H. Künsch, 1979

The Gaussian process u is germ Markov if and only if the following two conditions are satisfied:

- If $h, g \in \mathcal{H}^u$ are such that $(\operatorname{supp} h) \cap (\operatorname{supp} g) = \emptyset$ and $\operatorname{supp} h$ is compact, then
$$\langle h, g \rangle_{\mathcal{H}^u} = 0.$$
- If $\zeta = h + g \in \mathcal{H}^u$, where h and g are such that $(\operatorname{supp} h) \cap (\operatorname{supp} g) = \emptyset$ and $\operatorname{supp} h$ is compact, then
$$h, g \in \mathcal{H}^u.$$

Outline

Theorem

B. and Kim, 2006

The process solution u is germ Markov if:
(i) f is the Bessel kernel of order $\alpha = 2k, k \in \mathbb{Z}_+$; or
(ii) f is the Riesz kernel of order $\alpha = 4k, k \in \mathbb{Z}_+$

Idea of the Proof:

$h(t,x) = \mathbb{E}(M(\varphi)u(t,x))$, $g(t,x) = \mathbb{E}(M(\psi)u(t,x))$, and $(\operatorname{supp} h) \cap (\operatorname{supp} g) = \emptyset$.

$$\begin{cases} h_t = \Delta h + \varphi_1 \\ h(0,x) = 0 \end{cases}, \qquad \begin{cases} g_t = \Delta g + \psi_1 \\ g(0,x) = 0 \end{cases}.$$

We need to prove that $\langle h, g \rangle_{\mathcal{H}^u} = 0$.

Idea of the Proof: (cont'd)

$$\begin{aligned}\langle h, g\rangle_{\mathcal{H}^u} &= \langle \varphi, \psi\rangle_0 \\ &= \int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \psi(s,x) f(x-y)\varphi(s,y)\,dx\,dy\,ds \\ &= \int_0^T \int_{\mathbb{R}^d} \psi(s,x)\varphi_1(s,x)\,dx\,ds.\end{aligned}$$

Note that

$$\begin{aligned}\operatorname{supp}\varphi_1 &\subset \operatorname{supp} h, \\ \operatorname{supp}\psi \subset \operatorname{supp}\psi_1 &\subset \operatorname{supp} g.\end{aligned}$$

since: (i) if f is the Bessel kernel of order $\alpha = 2k$

$$\psi(t,\cdot) = (1-\Delta)^k \psi_1(t,\cdot)$$

(ii) if f is the Riesz kernel of order $\alpha = 4k$

$$\psi(t,\cdot) = (-\Delta)^{2k} \psi_1(t,\cdot)$$