Rare Event Statistics of Random Walkers on the Line

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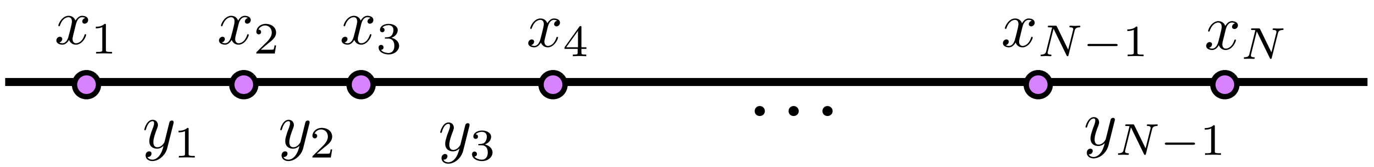
*D. ben-Avraham, B.M. Johnson, C.A. Monaco, P.L. Krapivsky, and S. Redner, *J. Phys. A* **36**, 1789 - 1799 (2003).

Ordering of N walkers on the line:



Vicious walkers: What's the survival probability if *no* two walkers are allowed to switch places?

Leader: What's the survival probability for the walker at x_1 to remain in the lead?

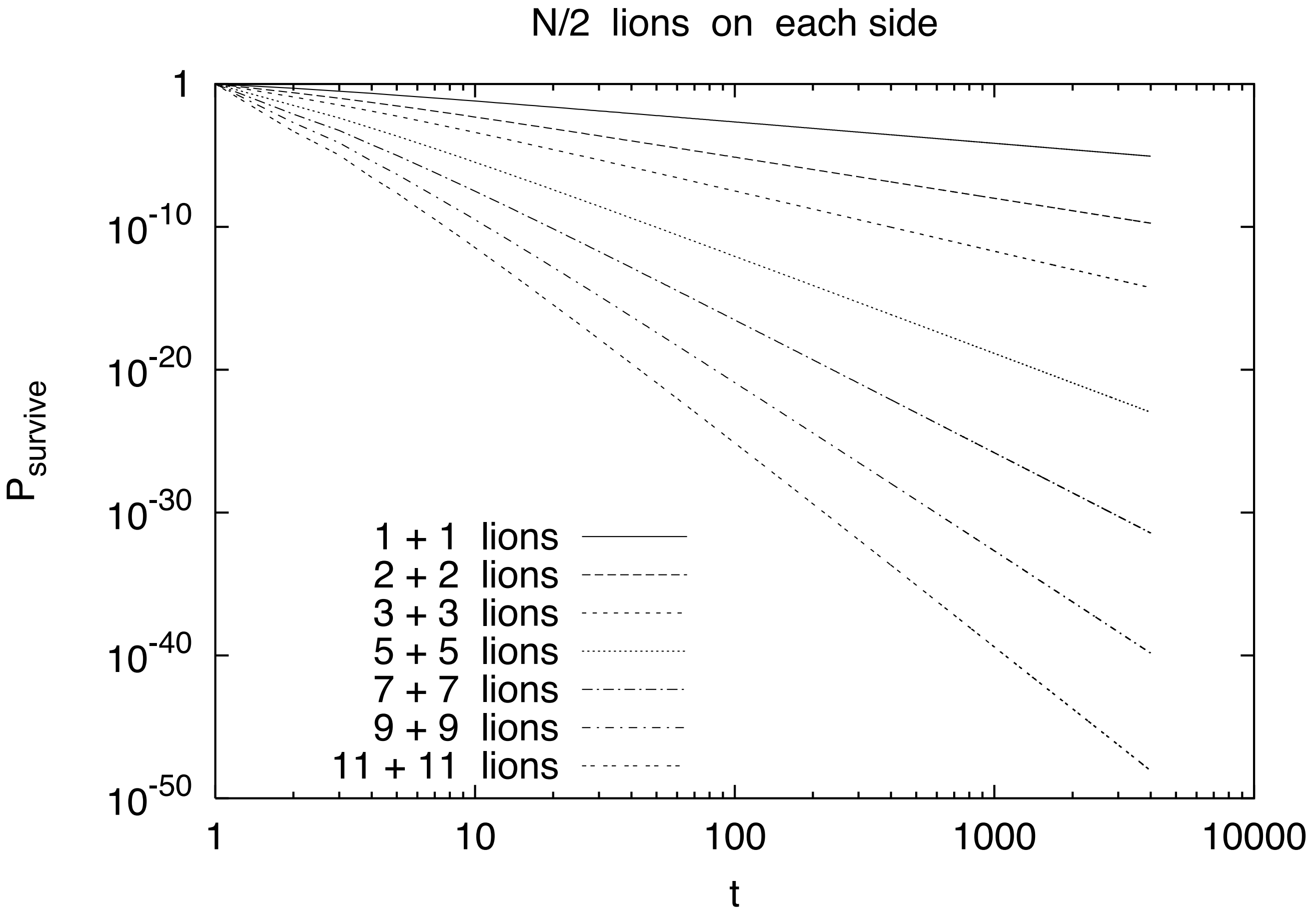
Laggard: What's the survival probability for the "laggard" at x_N to never assume the lead?

Some known results:

Vicious walkers: $S(t) \sim t^{-\alpha_N}, \qquad \alpha_N = N(N-1)/4\,.$

Leader: $S(t) \sim t^{-\beta_N}, \qquad \beta_2 = \frac{1}{2},\ \ \beta_3 = \frac{3}{4},\ \ \beta_N \underset{N\to\infty}{\longrightarrow} \frac{\ln(4N)}{4}\,.$

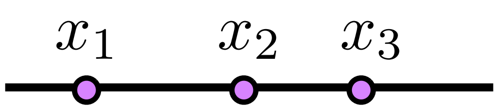
Laggard: $S(t) \sim t^{-\gamma_N}, \qquad \gamma_2 = \frac{1}{2},\ \ \gamma_3 = \frac{3}{8},\ \ \gamma_N \underset{N\to\infty}{\longrightarrow} \frac{\ln N}{N}\,.$



“Go with the winners”

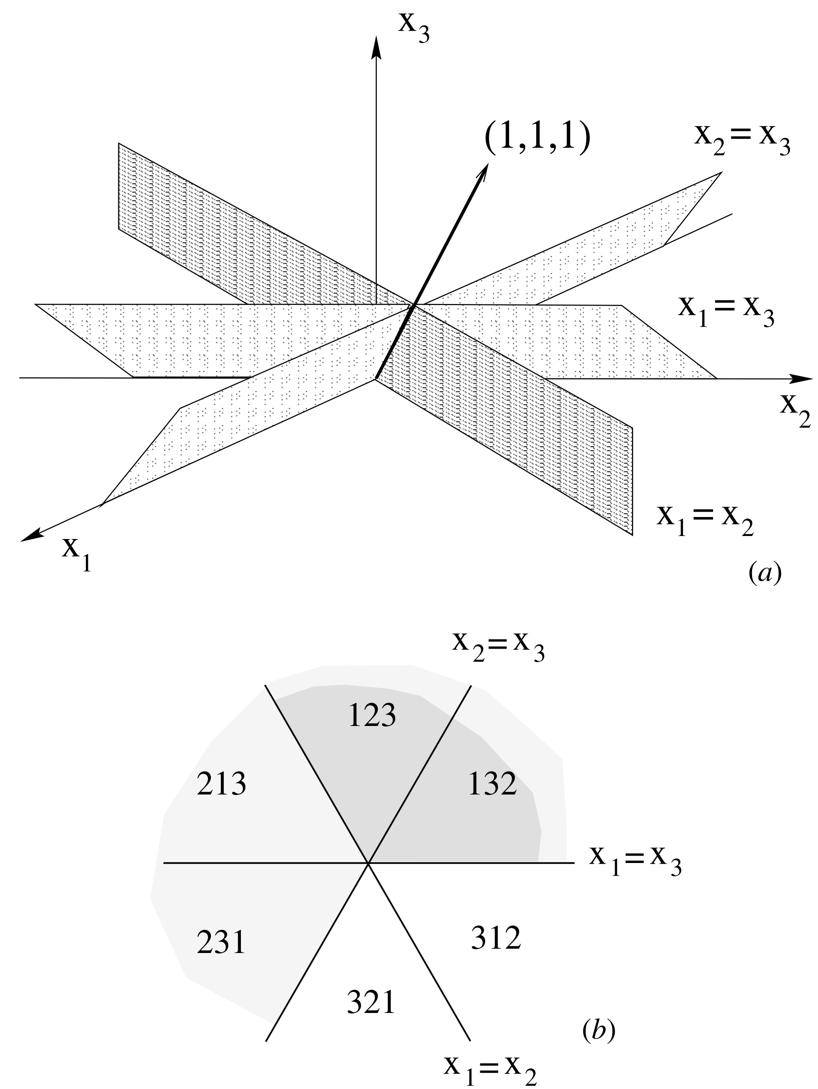
P. Grassberger

N=3 Walkers



$$S(t) \sim t^{-\pi/2\varphi}$$

	φ	$S(t)$
Vicious	$\pi/3$	$\sim t^{-3/2}$
Leader	$2\pi/3$	$\sim t^{-3/4}$
Laggard	$4\pi/3$	$\sim t^{-3/8}$



Relation to Electrostatics

Diffusion

$$\frac{\partial}{\partial t} P(\mathbf{r}, t) = D \nabla^2 P(\mathbf{r}, t)$$

$$S(t) = \int P(\mathbf{r}, t) \, d^{N-1}\mathbf{r}$$

Electrostatics

$$\nabla^2 V(\mathbf{r}) = \frac{1}{\varepsilon_0} \delta(\mathbf{r} - \mathbf{r}_0)$$

$$\int^t S(t) \sim \int^{\sqrt{t}} V(\mathbf{r}) \, d^{N-1}\mathbf{r}$$

$$S(t) \sim t^{-\alpha}$$

$$V(\mathbf{r}) \sim r^{-\mu}$$

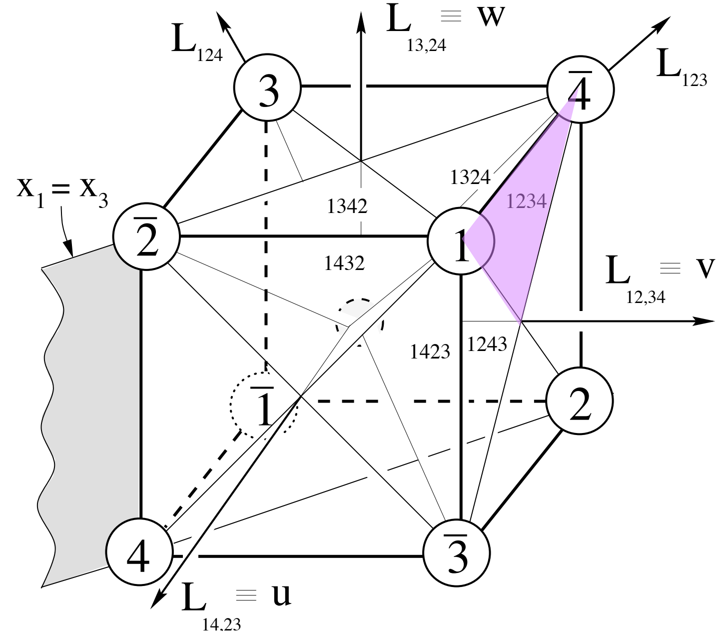
$$\alpha = \frac{\mu - N + 3}{2}$$

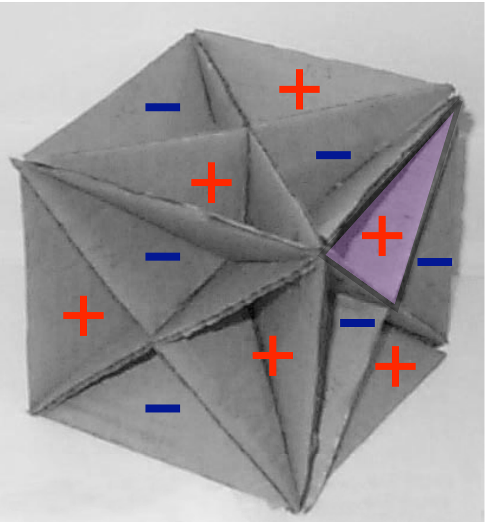
N=4 Walkers

Vicious walkers:

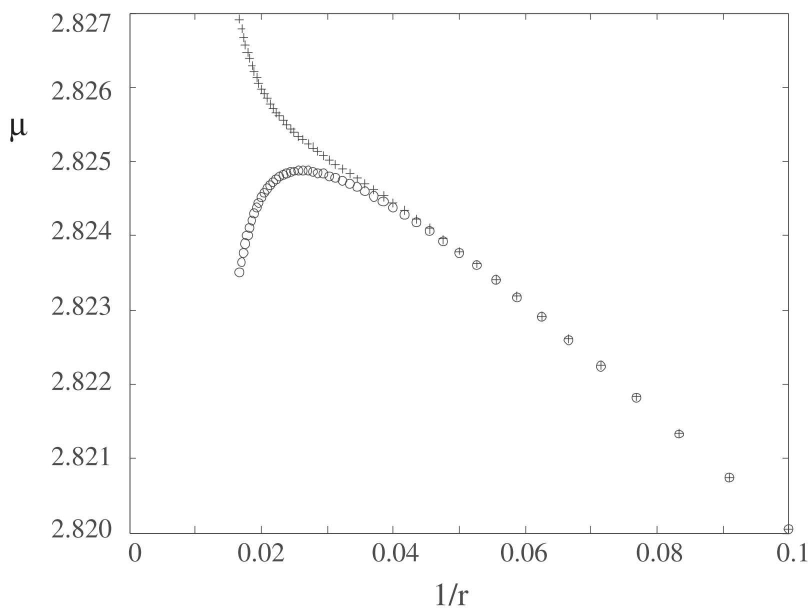
$$V(\mathbf{r}) \sim a_1 r^{-7} + a_2 r^{-11} + a_3 r^{-13} + a_4 r^{-15} + \cdots$$

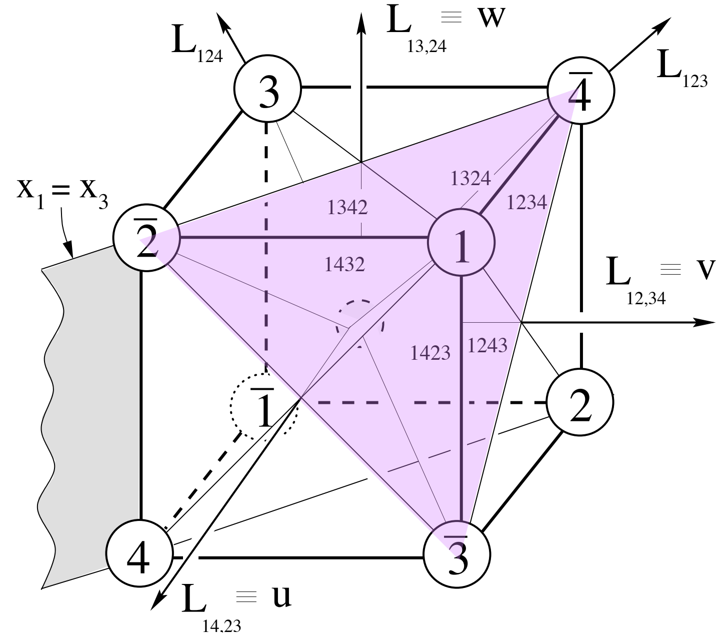
$$S(t) \sim b_1 t^{-3} + b_2 t^{-5} + b_3 t^{-6} + b_4 y^{-7} + \cdots$$

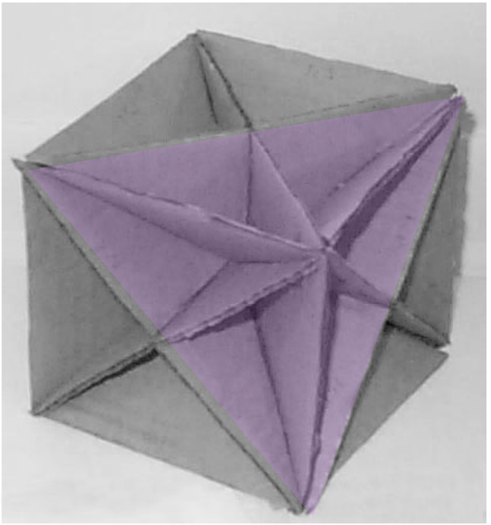




N=4 Walkers   Leader:

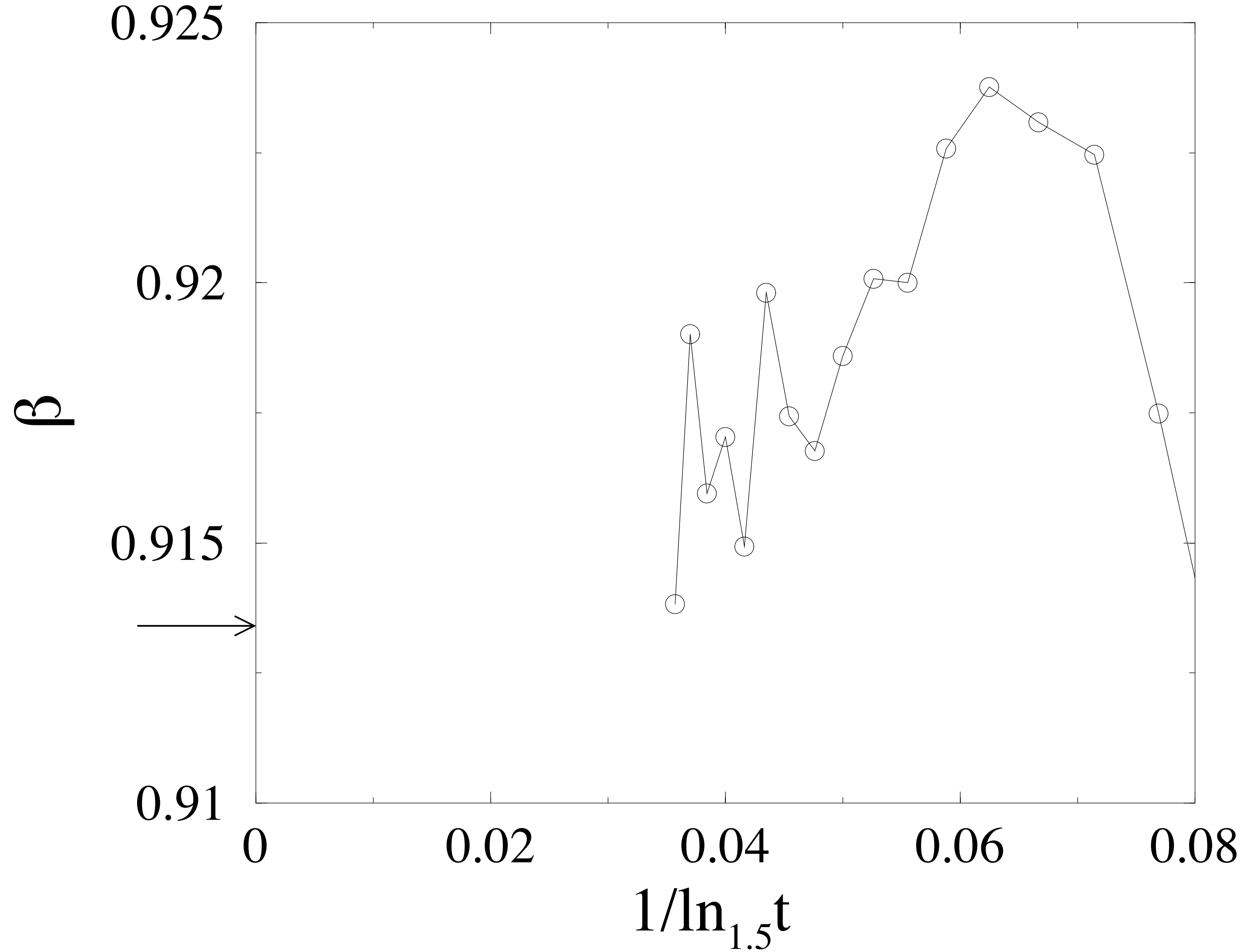






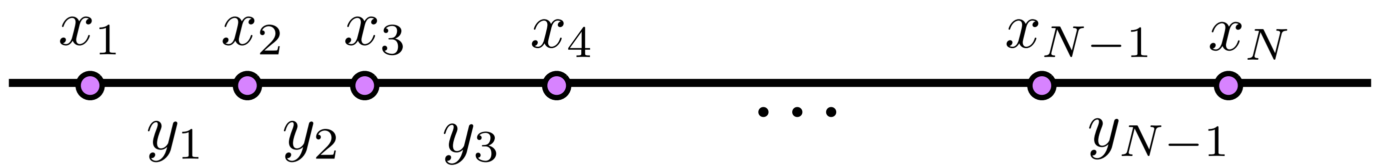
$$V(\mathbf{r}) \sim r^{-\mu} + Ar^{-4}; \quad \mu = 2.82684(16)$$

$$S(t) \sim t^{-\beta_4} + Bt^{-3/2}; \quad \beta_4 = 0.91342(8)$$

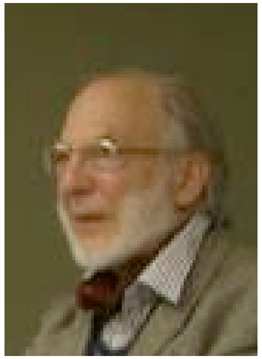


0.925
0.92
0.915
0.91
β
0
0.02
0.04
0.06
0.08
1/ln1.5t

Vicious Walkers



$$P(x_1, \ldots, x_N, t) \sim \prod_{j>i} (x_j - x_i) \exp\left(- \sum_{i=1}^{N} x_i^2 / t \right)$$

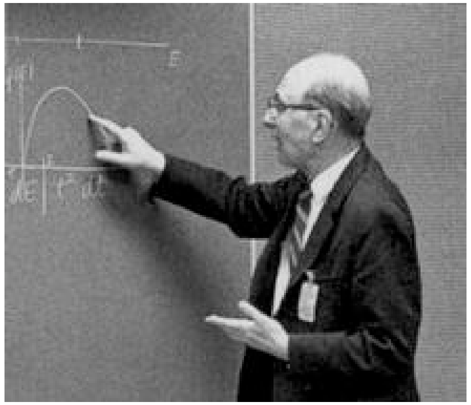


M. Fisher

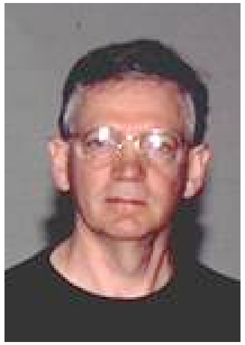
Random Matrices

$$P(\lambda_1, \ldots, \lambda_N, t) \sim \exp\left[-\frac{\beta}{2}\left(\sum_{i=1}^{N}\lambda_i^2 - \sum_{i\neq j}\ln|\lambda_i - \lambda_j|\right)\right]$$

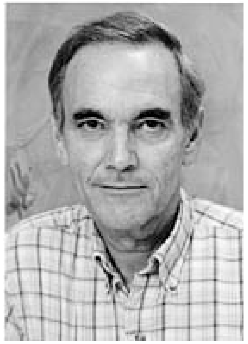
$$\rho(\lambda, N) = \sqrt{\frac{2}{N\pi^2}}\left[1 - \frac{\lambda^2}{2N}\right]^{1/2}$$



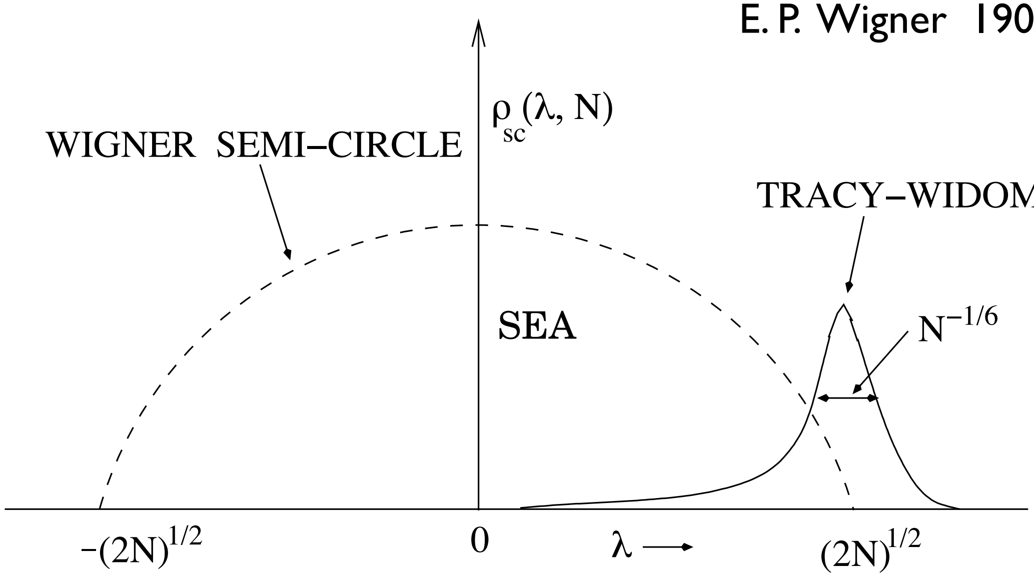
E. P. Wigner 1902-1995



C. Tracy

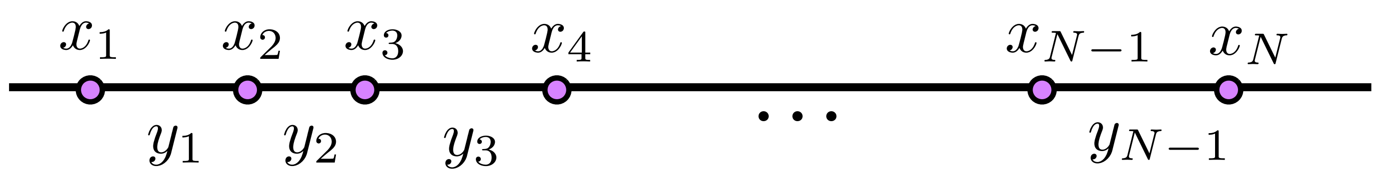


H. Widom



(Source: S. N. Majumdar, cond-mat/0701193)

Vicious Walkers

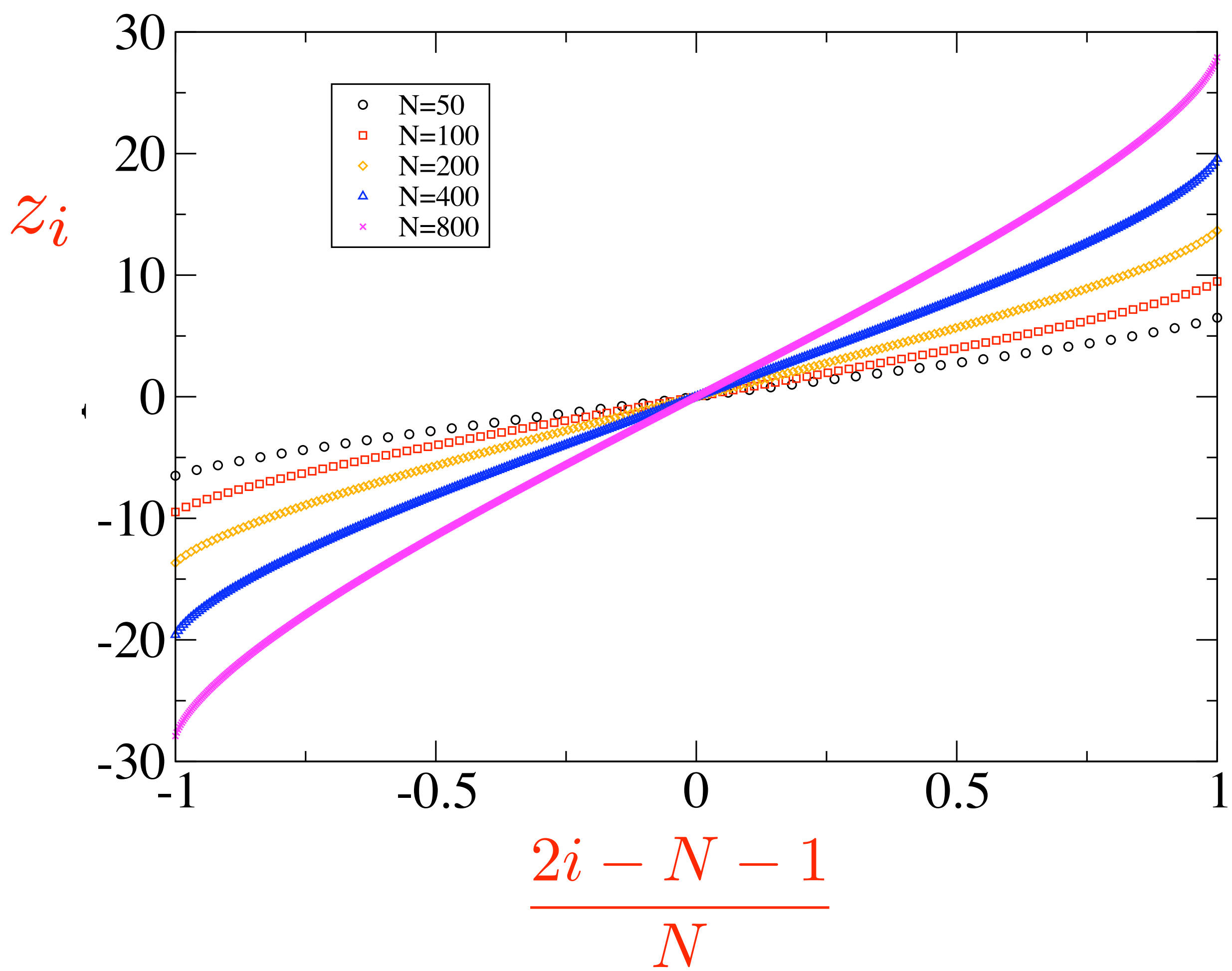


$$P(x_1, \ldots, x_N, t) \sim \prod_{j>i} (x_j - x_i) \exp\left(- \sum_{i=1}^{N} x_i^2 / t \right)$$

Examine the *most probable* location: $\frac{\partial}{\partial x_i} \ln P(x_1, \ldots, x_N, t) = 0$

$$\sum_{i=1}^{N}{}' \frac{1}{x_j - x_i} = 2 \frac{x_j}{t} \,, \qquad j = 1, 2, \ldots, N$$

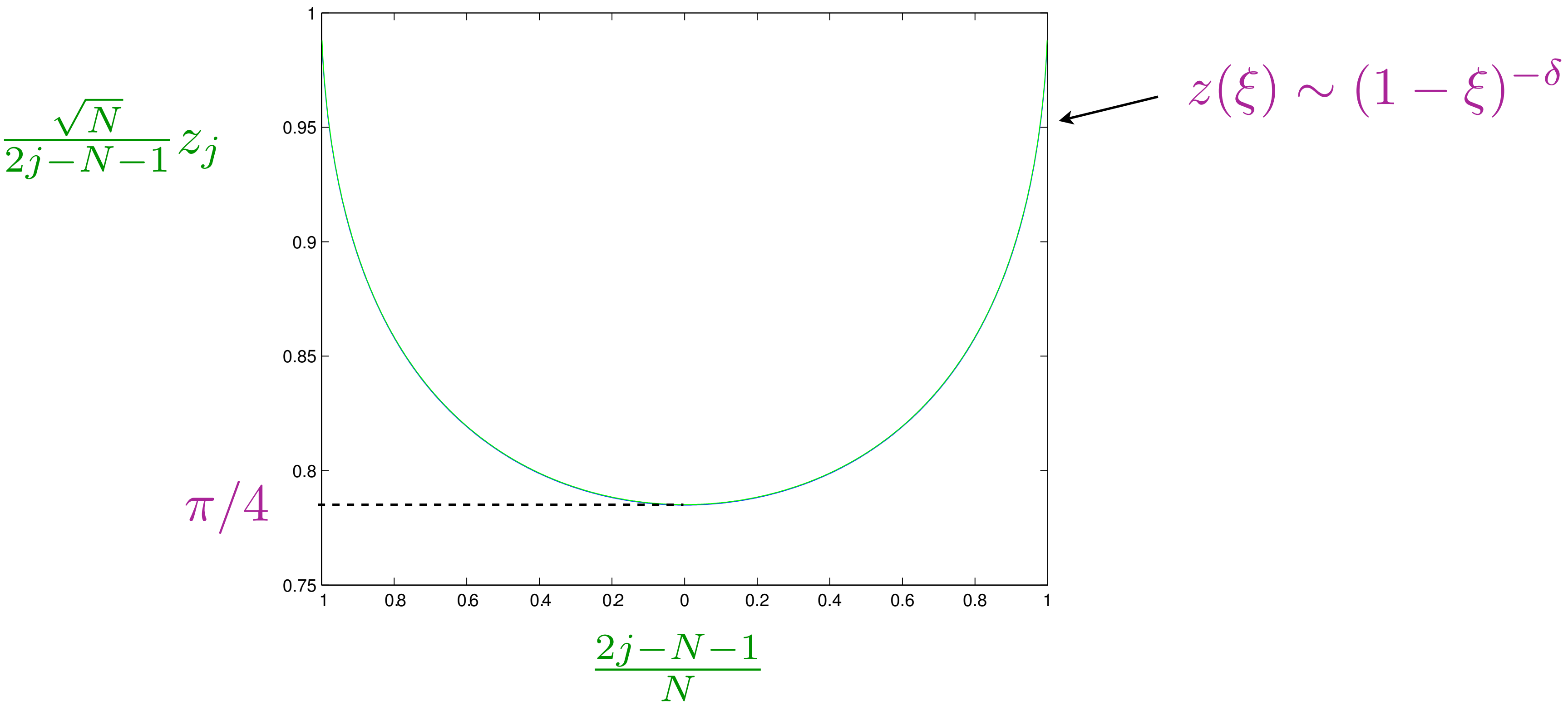
Scale $z_i = \sqrt{\frac{2}{t}}\, x_i$ $\Rightarrow$ $$\sum_{i=1}^{N}{}' \frac{1}{z_j - z_i} = z_j\,, \qquad j = 1, 2, \ldots, N$$



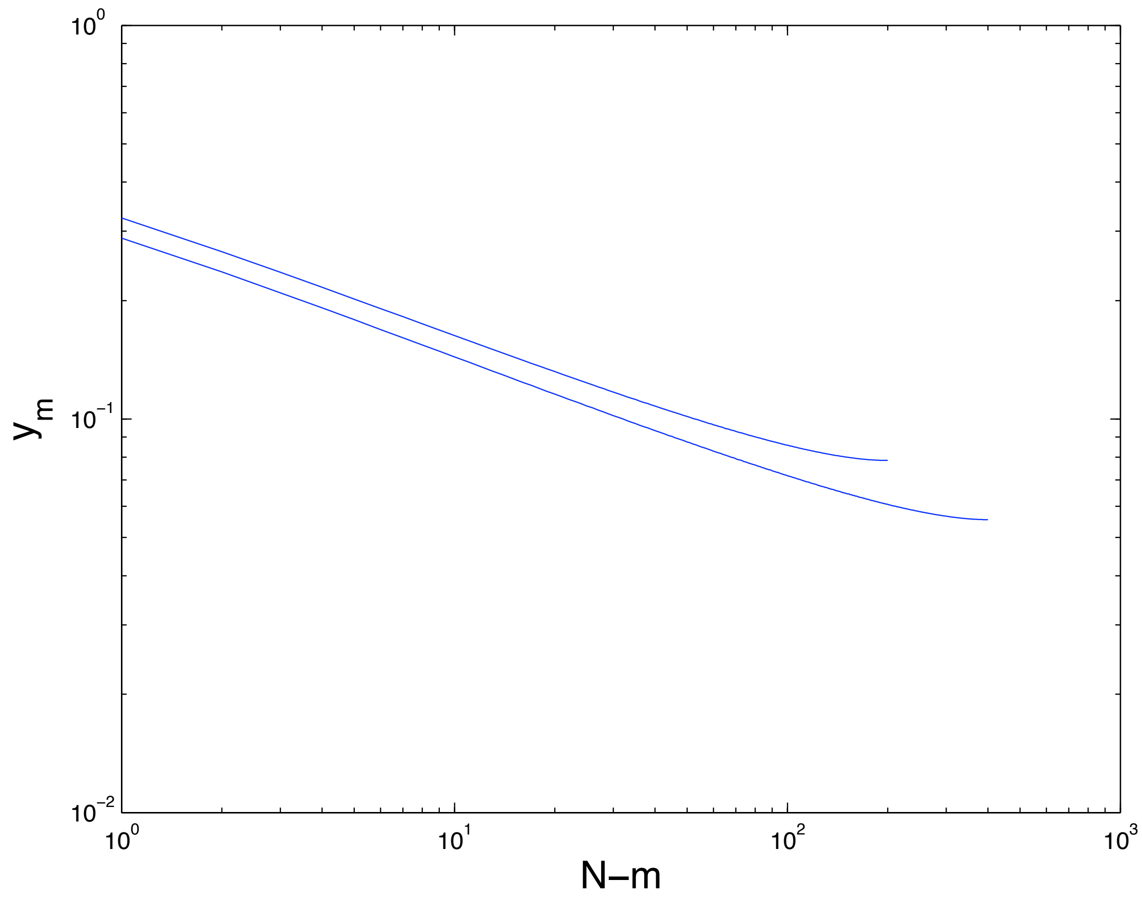
$$z_i \xrightarrow[N\to\infty]{} \sqrt{N} f(\xi)$$

$$f(\xi) \sim \xi\,, \quad \xi \ll 1$$

$$\xi \equiv \frac{2i - N - 1}{N}$$



$\frac{\sqrt{N}}{2j-N-1} z_j$
1
0.95
0.9
0.85
0.8
0.75
$\pi/4$
1
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1
$\frac{2j-N-1}{N}$
$z(\xi) \sim (1-\xi)^{-\delta}$



10^0
10^{-1}
10^{-2}
y_m
10^0
10^1
10^2
10^3
N−m

Compare $$\sum_{i=1}^{N}{}' \frac{1}{z_j - z_i} = z_j\,, \qquad j = 1, 2, \ldots, N$$

to Calogero's relations for the zeros of Bessel functions:

$$J_p(y_j^{1/2})\,, \quad j = 1, 2, 3, \ldots$$

$$\sum_{i=1}^{\infty}{}' \frac{1}{y_j - y_i} = -\frac{p+1}{2y_j}, \quad j = 1, 2, 3, \ldots$$


F. Calogero