CRITICAL EXPONENTS IN ACTIVATED ESCAPE OF NONEQUILIBRIUM SYSTEMS

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Motivation

Noise-induced escape in dynamical systems far from thermal equilibrium:

universal features

- onset of soft modes leading to scaling of the escape rate
- large susceptibility where coexisting stable states are almost equally occupied
- …

experimental accessibility + applications

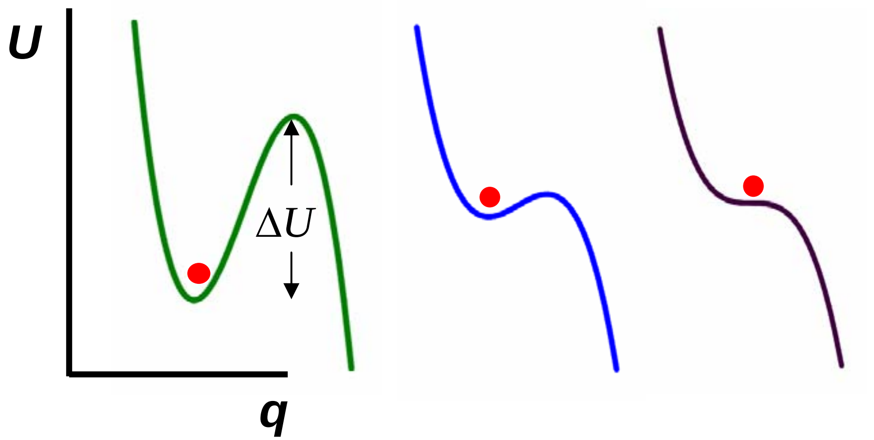
- comparatively large escape probability for weak noise
- high sensitivity to system parameters
- …

bifurcation points

noise-induced frequency mixing, stochastic resonance, …

Saddle-node bifurcation

Near the bifurcation point one of the motions is **slow**, **a soft mode** ➡ **universal behavior of the escape rate**



Stationary **multivariable** systems:

$$\dot{q} = -\partial_q U + f(\tau), \qquad U(q) = -\tfrac{1}{3} q^3 + \eta\, q, \qquad \langle f(\tau) f(\tau') \rangle = 2D\,\delta(\tau - \tau')$$

noise

$D = k_B T$ in thermal equilibrium

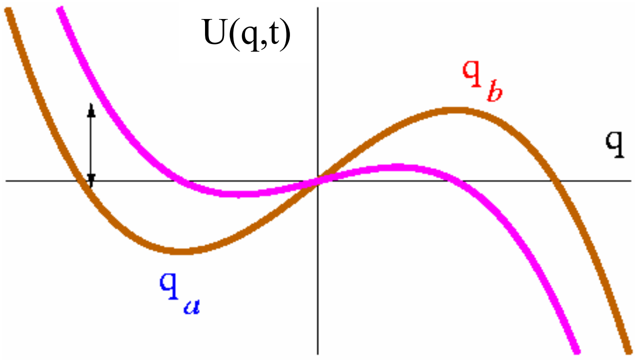
No noise: **the relaxation time** $t_r = -1/\partial_q^2 U(q_a) = \eta^{-1/2}/2$ **is large for small** $|\eta|$

Noise: **delta-correlated in slow time. The escape rate**

$$W = \nu \exp(-R/D), \quad R = \Delta U = \tfrac{4}{3}\eta^{\xi}, \quad \nu \propto \eta^{\zeta} \qquad \xi = 3/2, \quad \zeta = 1/2$$

Kurkijarvi (1972), MD & Krivoglaz (1979, 1980), Graham & Tél (1987), Victora (1989)

Slow periodic modulation

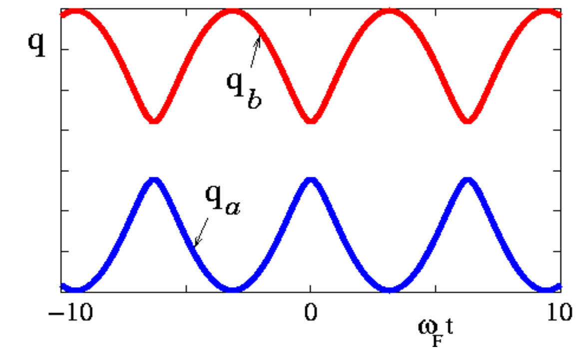


A periodically modulated dynamical system:

$$\dot{\mathbf{q}} = \mathbf{K}(\mathbf{q},t) + \mathbf{f}(t), \quad \mathbf{K}(\mathbf{q},t) = \mathbf{K}(\mathbf{q},t+\tau_F)$$

for example $K = -\partial_q U(q,t) = -\partial_q U(q) + A\cos\omega_F t$

Slow driving: $\omega_F t_r << 1$ or $\tau_F = 2\pi/\omega_F >> t_r$



The system follows the driving field **adiabatically** (?)

Adiabatic periodic states: $\mathbf{K}(\mathbf{q}_{a,b}^{\mathrm{ad}}(t),t) = \mathbf{0}$

[local minimum and maximum of the potential *U(q,t)*]

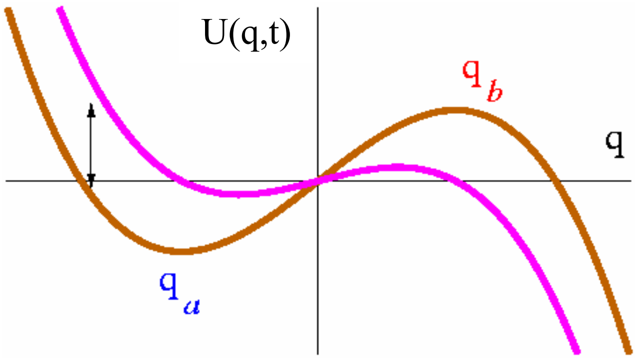
Adiabatic bifurcation amplitude A_c^{ad} : the states $\mathbf{q}_{a,b}^{\mathrm{ad}}(t)$ touch each other, once per period, for $t = n\tau_F$

Small $\delta A^{\mathrm{ad}} = A - A_c^{\mathrm{ad}}, \quad |t - n\tau_F|$**: one-dimensional motion**

Adiabatic scaling

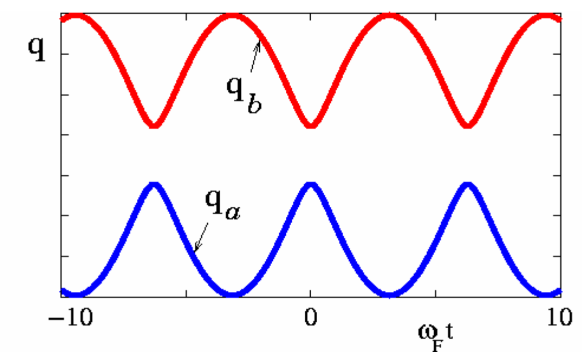
In the fully adiabatic picture, escape rate is determined by the instantaneous barrier height

$$\Delta U(t) = U(\mathbf{q}_b, t) - U(\mathbf{q}_a, t)$$



Period-averaged escape rate

$$\overline{W} = \nu \exp(-R/D), \quad R_{\text{ad}} = \min_t \Delta U(t)$$



The barrier is at its lowest once per period, for $t = n\tau_F$

the effective 1D potential is a cubic parabola,

$$U(q, t = n\tau_F) \approx -\tfrac{1}{3} q^3 - \delta A^{\text{ad}} q, \quad \delta A^{\text{ad}} = A - A_c^{\text{ad}}$$

Adiabatic scaling:

$$R_{\text{ad}} = \Delta U(t=0) = \tfrac{4}{3}(-\delta A^{\text{ad}})^{\xi}, \quad \nu_{\text{ad}} \propto (-\delta A^{\text{ad}})^{\zeta}$$

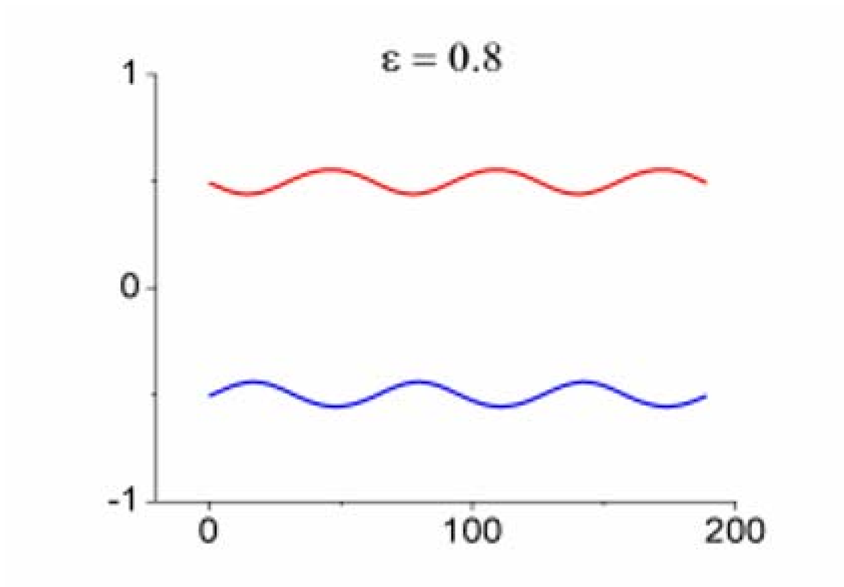
$$\boxed{\xi = 3/2, \quad \zeta = 1/4}$$

Relaxation time $t_r \to \infty$ for $\delta A^{\text{ad}} \to 0$. The adiabaticity condition $\omega_F t_r << 1$ **breaks down at the bifurcation point**.

A step back…

The true saddle-node bifurcation occurs for $A = A_c$ where the stable and the unstable states merge for all times,

$$\mathbf{q}_c(t) = \mathbf{q}_a(t) = \mathbf{q}_b(t)$$



$\omega_F t$

$$\varepsilon = (A_c - A)/A_c$$

Avoided crossing of stable and unstable states where adiabaticity is broken

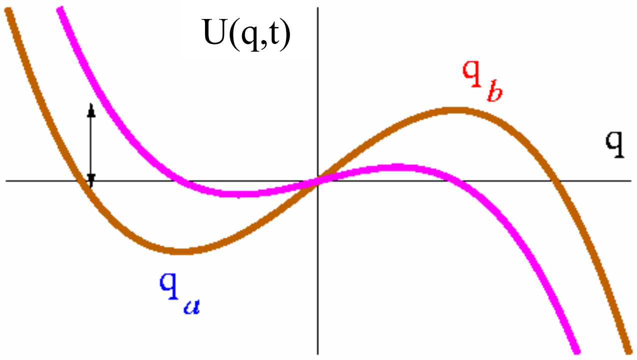
Beyond the adiabatic approximation

Expand $\mathbf{K}(\mathbf{q}, t)$ around the adiabatic bifurcation point

$$q = q_c^{ad}\ , \ t = n\tau_F\ , \ A = A_c^{ad}$$

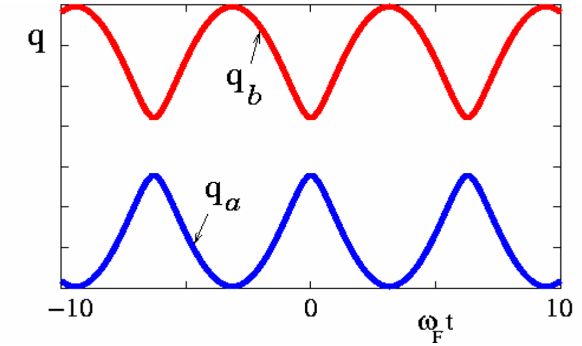
eliminate "fast" modes, rescale:

$$\dot{q} = q^2 + \delta A^{ad} - \gamma^2(\omega_F t)^2 \quad , \quad \gamma \sim 1$$



The adiabatic relaxation time

$$t_r^{ad}(t) = |2q_a(t)|^{-1} = [(\gamma\omega_F t)^2 - \delta A^{ad}]^{-1/2}/2$$



The adiabaticity conditions:

1) $t_r^{ad}\omega_F << 1$

2) $\left|\frac{\partial t_r^{ad}}{\partial t}\right| << 1 \ \Rightarrow \ t_r^{ad} << t_l = (\gamma\omega_F)^{-1/2}$ ← **new dynamical time scale**

Locally nonadiabatic regime

Shift of the bifurcation point due to crossing avoidance

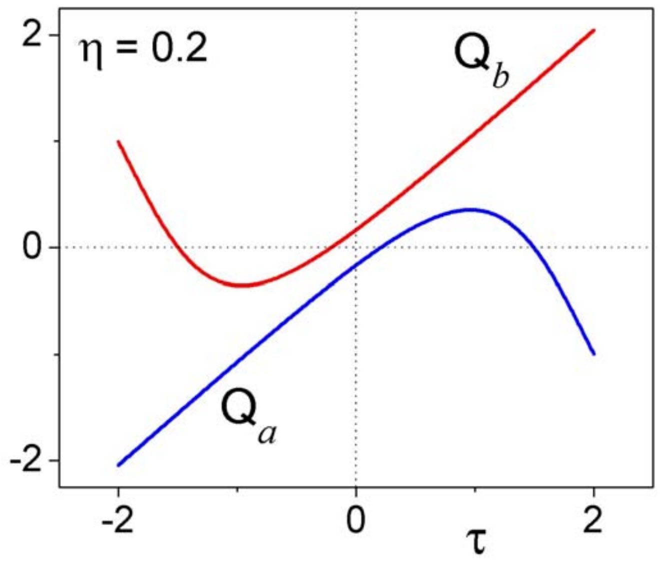
$$A_c^{sl} = A_c^{ad} + \gamma\omega_F$$

Control parameter $\eta = (A_c^{sl} - A)/(\gamma\omega_F)$

Scaled coordinate and time $Q = t_l q, \quad \tau = t/t_l$

$\eta \sim 1:$ **locally nonadiabatic regime,**

$\omega_F t_r << 1$ **but** $t_r \sim t_l$ **and decay is nonexponential**



Locally nonadiabatic regime

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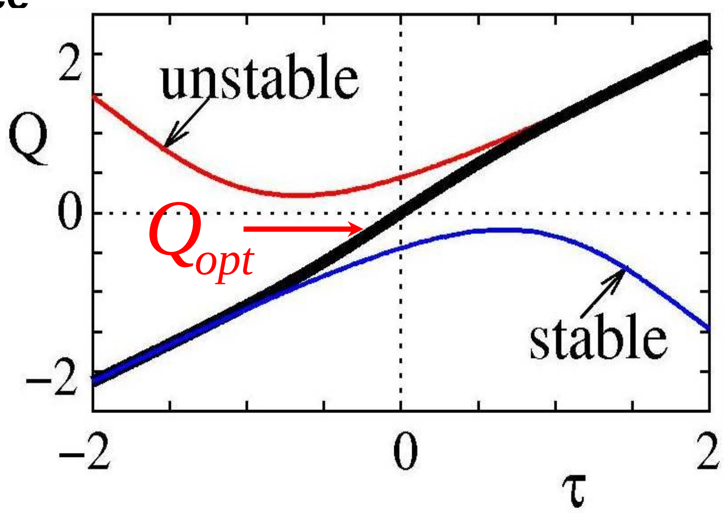


Control parameter $\eta = (A_c^{sl} - A)/(\gamma\omega_F)$

Scaled coordinate and time $Q = t_l q, \quad \tau = t/t_l$

1D time-dependent local Langevin equation

$$\frac{dQ}{d\tau} = Q^2 - \tau^2 + 1 - \eta + \tilde{f}(\tau)$$

Close to the bifurcation point, $\eta << 1$ **, the variational problem for the optimal escape path can be *linearized* and solved:**

$$Q_{opt}(\tau) = \tau - \eta\int_{-\infty}^{\tau} d\tau_1 e^{\tau^2-\tau_1^2}\left[1-\sqrt{2}e^{-\tau_1^2}\right]$$

The escape activation energy and the prefactor in the escape rate are

$$R = (\pi/8)^{1/2}\eta^{\xi}, \quad \nu \propto \eta^{\zeta}$$

$$\boxed{\xi = 2, \quad \zeta = -1}$$

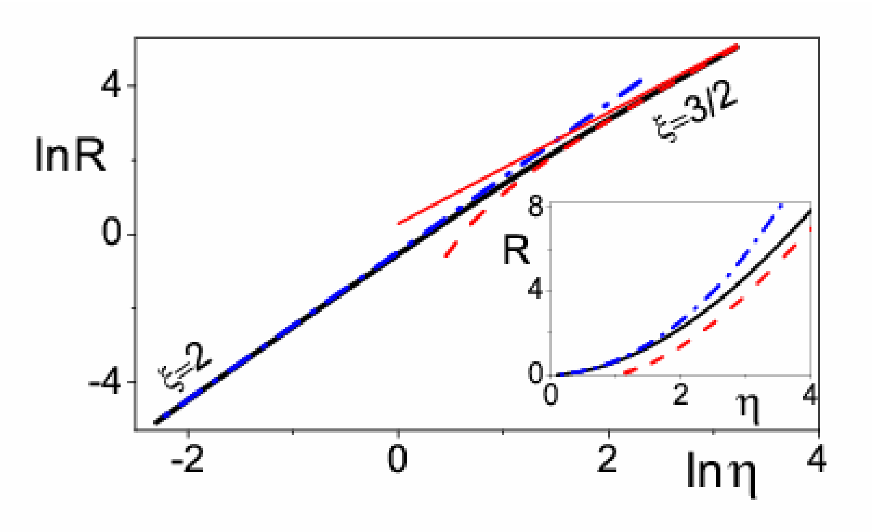
Scaling crossover

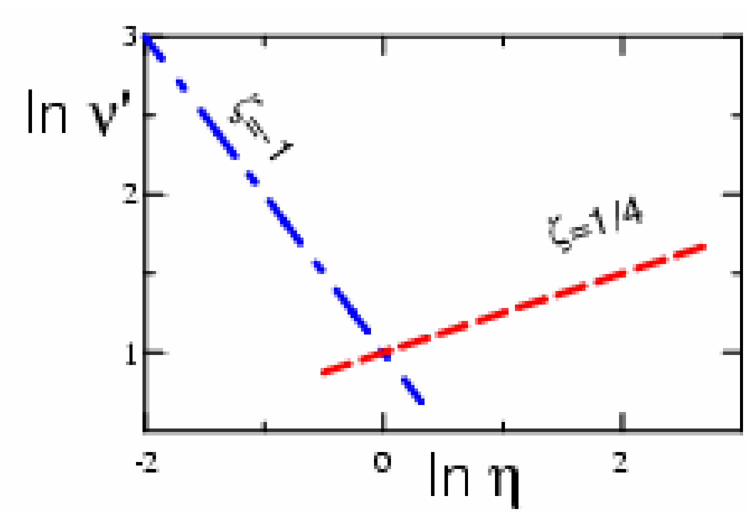
Slow driving, $\omega_F t_r \ll 1$: **locally nonadiabatic ⟷ adiabatic crossover:**

$$\overline{W} = \nu \exp(-R/D)$$

$$R \propto (A_c - A)^{\xi}, \quad \xi = 2 \Leftrightarrow 3/2$$

$$\nu \propto (A_c - A)^{\zeta}, \quad \zeta = -1 \Leftrightarrow 1/4$$

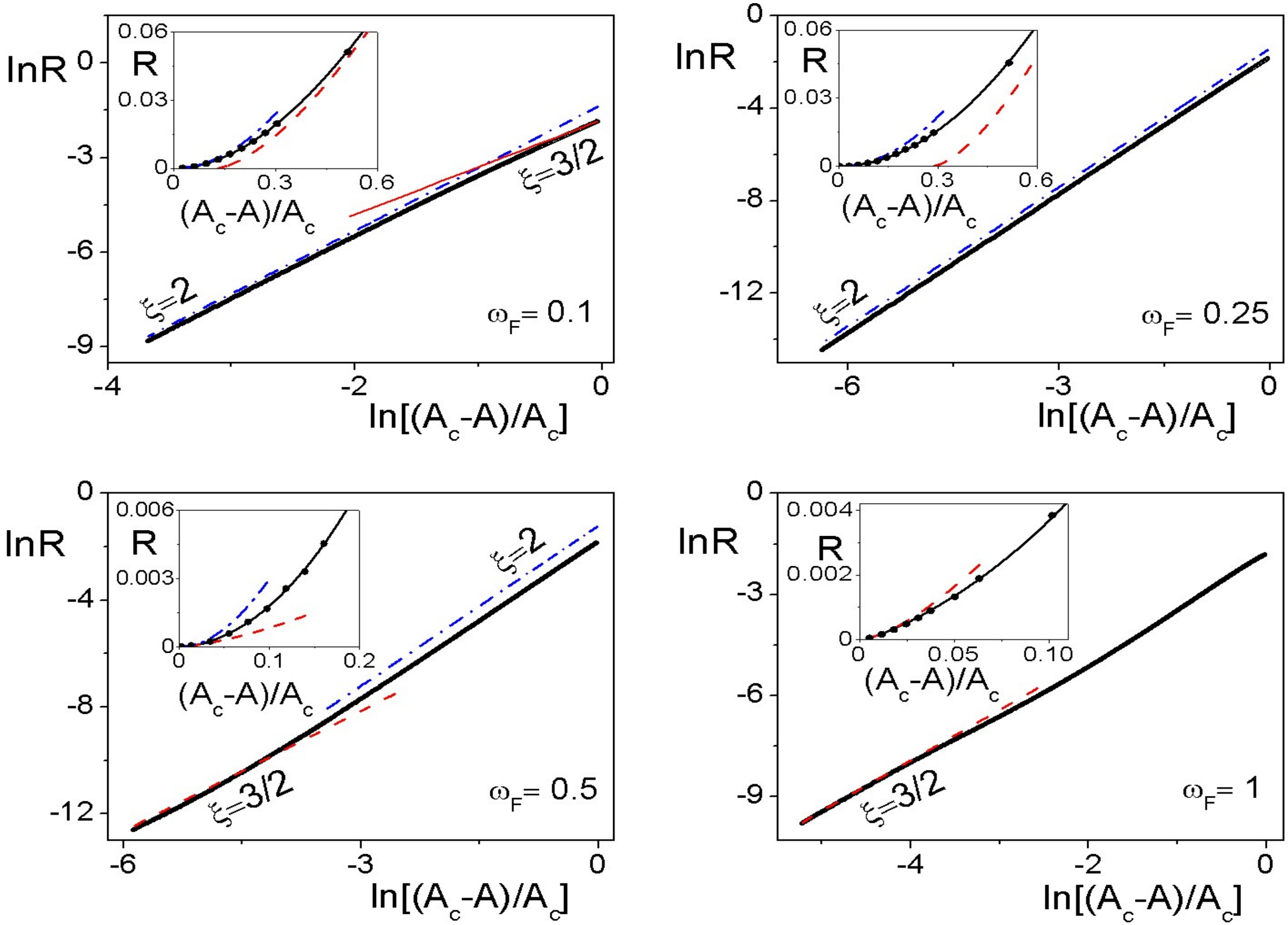




$$\eta \propto A_c - A$$

Results for a model system

$$U(q,t) = -\tfrac{1}{3}q^3 + \tfrac{1}{4}q - Aq\cos\omega_F t, \quad t_{\mathrm{r}}^{(0)} = 1$$



Underdamped systems close to a bifurcation point

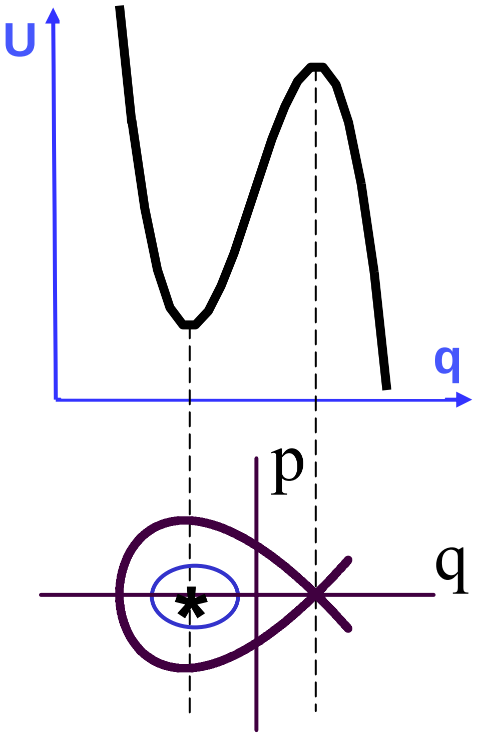
A Hamiltonian system with weak damping and noise, close but not too close to the saddle-node bifurcation point:

$$\frac{dq}{dt} = \partial_p H(p,q;\eta) - \varepsilon \upsilon^{(q)}(p,q) + f^{(q)}(t)$$

$$\frac{dp}{dt} = -\partial_q H(p,q;\eta) - \varepsilon \upsilon^{(p)}(p,q) + f^{(p)}(t)$$

Hamiltonian — friction — noise

$$\left\langle f^{(i)}(t) f^{(j)}(t') \right\rangle = 2D_{ij}(p,q)\delta(t-t')$$



Frequency of vibrations about the center ω_c **is small, but friction is still smaller,** $\omega_c t_r >> 1, \quad t_r \propto \varepsilon^{-1}$

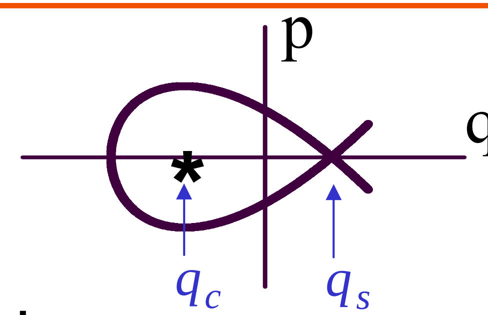
Assume:

- $\omega_c \to 0$ **where the control parameter** $\eta \to 0$ **("Hamiltonian bifurcation point")**
- $H(p,q;\eta)$ **is analytic in** η **for small** $|\eta|$
- $\partial_\eta H \neq 0$ **for** $\eta = 0$

The scenarios of approaching the bifurcation

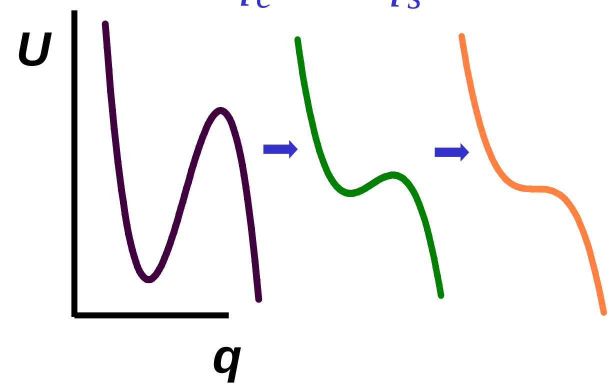
Local scenario: shrinking of the homoclinic orbit as $\eta \to 0$ **,**

with $q_c \to q_s$



Generic Hamiltonian, to leading order in η

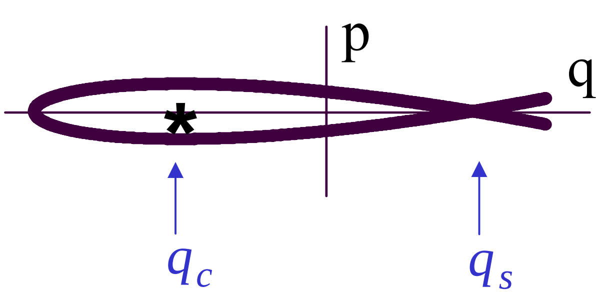
$$H = \tfrac{1}{2} p^2 - \tfrac{1}{3} q^3 + \eta q$$

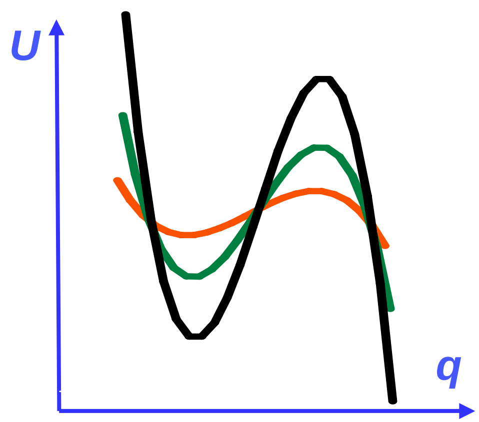


The energy barrier $\Delta E = H(p_s, q_s; \eta) - H(p_c, q_c; \eta)$

$$\Delta E = \tfrac{4}{3} \eta^{3/2}$$

Nonlocal scenario: squeezing of the homoclinic orbit as $\eta \to 0$**, with no change in** q_c, q_s





$$H = \tfrac{1}{2} p^2 + \eta U(q)$$

$$\Delta E = C\eta, \quad C = U(q_s) - U(q_c)$$

Escape rate

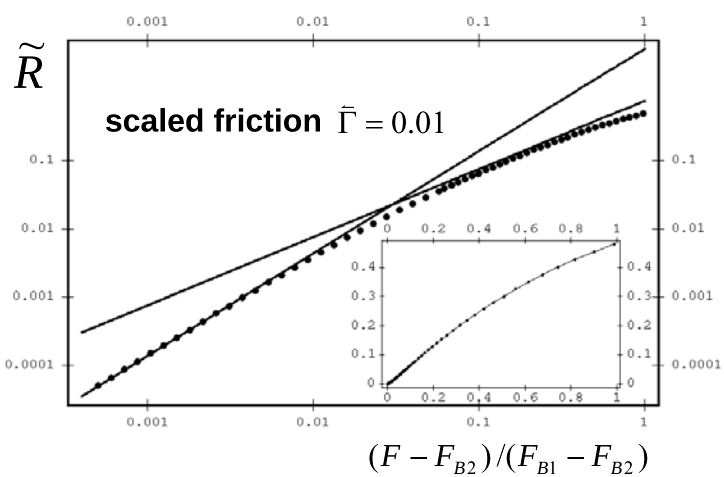
Friction: **drift over energy towards a metastable state**
Noise: **energy diffusion, ultimately leading to escape**

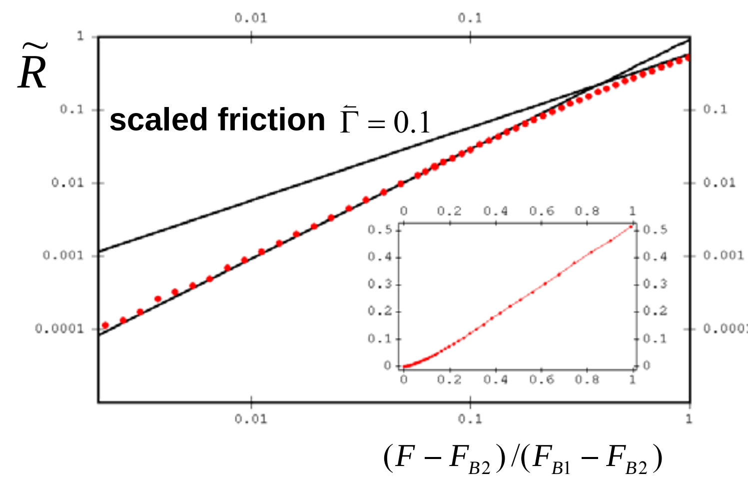
$$W \propto \exp(-R/D)$$

Local scenario: $R = C_{loc}\varepsilon\eta^{3/2}$, **the same scaling as in the overdamped region**

Non-local scenario: $R = C_{nl}\varepsilon\eta,$ ***crossover*** **to** $R \propto \eta^{3/2}$ **with decreasing** η

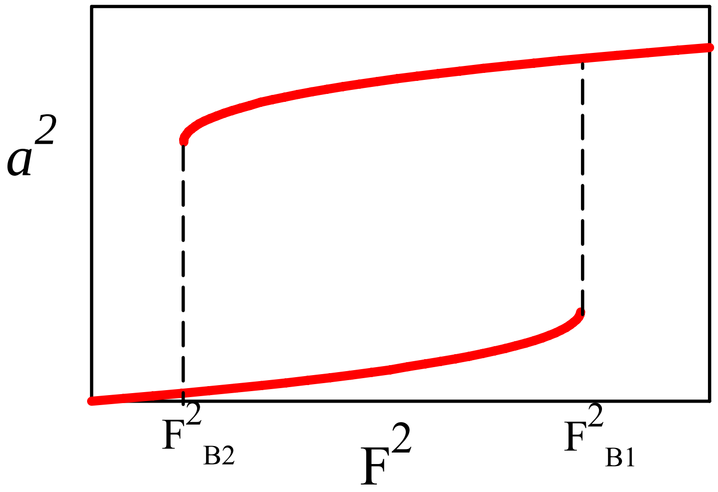
Crossover for a resonantly driven oscillator





(Kogan et al., 2006)

Modulated nonlinear oscillator



$$\ddot{q} + 2\Gamma\dot{q} + \omega_0^2 q + \gamma q^3 = F\cos\omega_F t$$

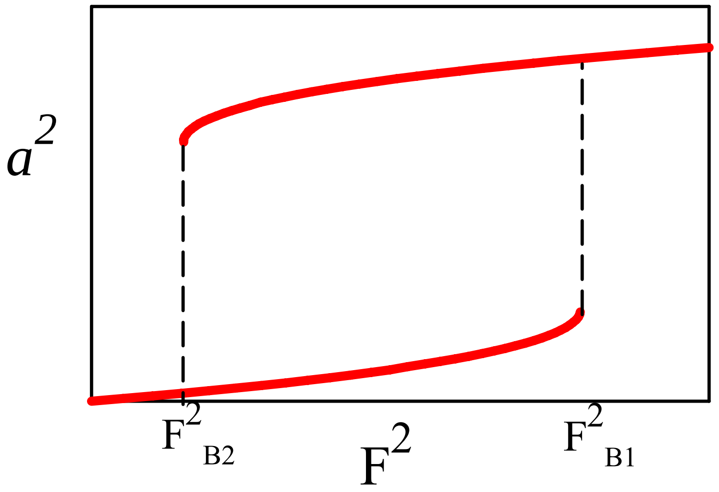
Periodic state: $q = a\cos(\omega_F t + \phi)$

Both a and ϕ display hysteresis

Eigenfrequency depends on vibration amplitude, $\omega_0 \to \omega_0 + \tilde{\gamma} a^2$

Cubic equation for the amplitude of forced vibrations, $a^2 = \dfrac{F^2/4\omega_F^2}{(\omega_F - \omega_0 \qquad)^2 + \Gamma^2}$

Hysteresis in a modulated Josephson junction



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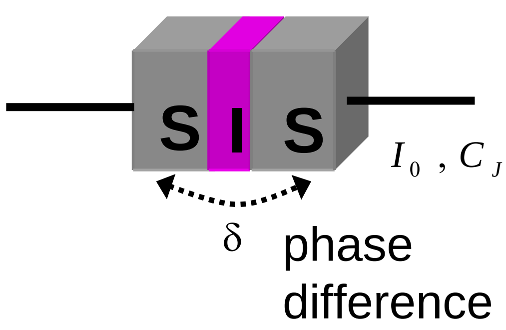
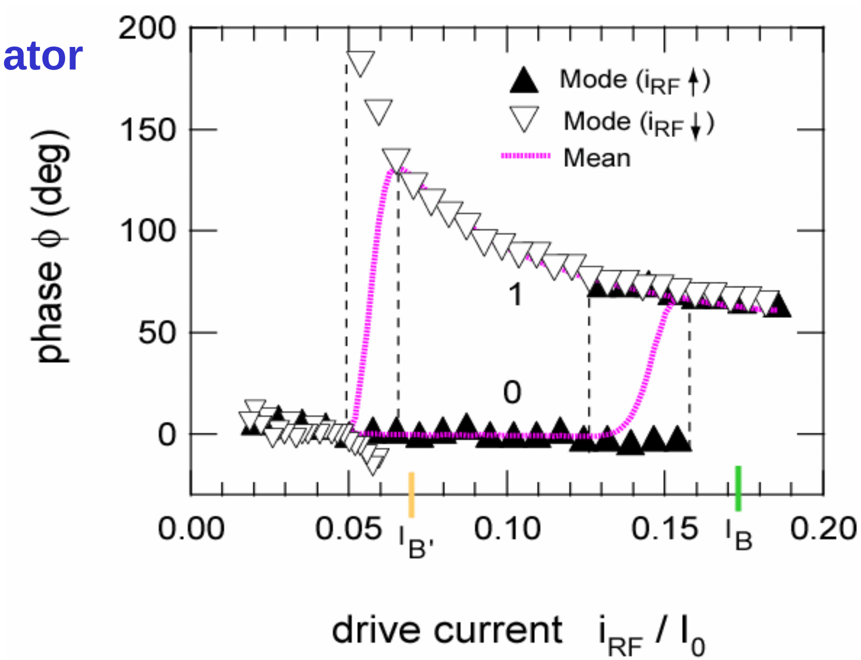
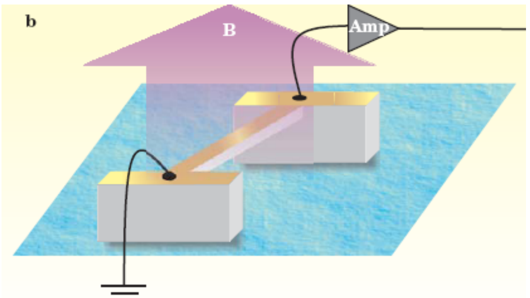
A Josephson junction based nonlinear oscillator
(Siddiqi *et al*. PRL 2004, 2005)



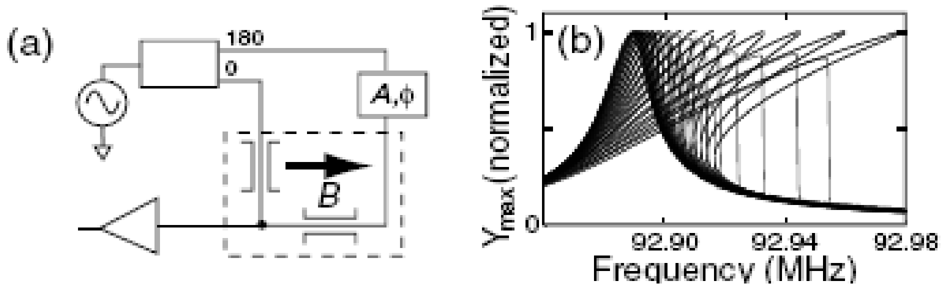


Hysteresis and switching in nanooscillators

Resonantly modulated nanoelectromechanical resonators



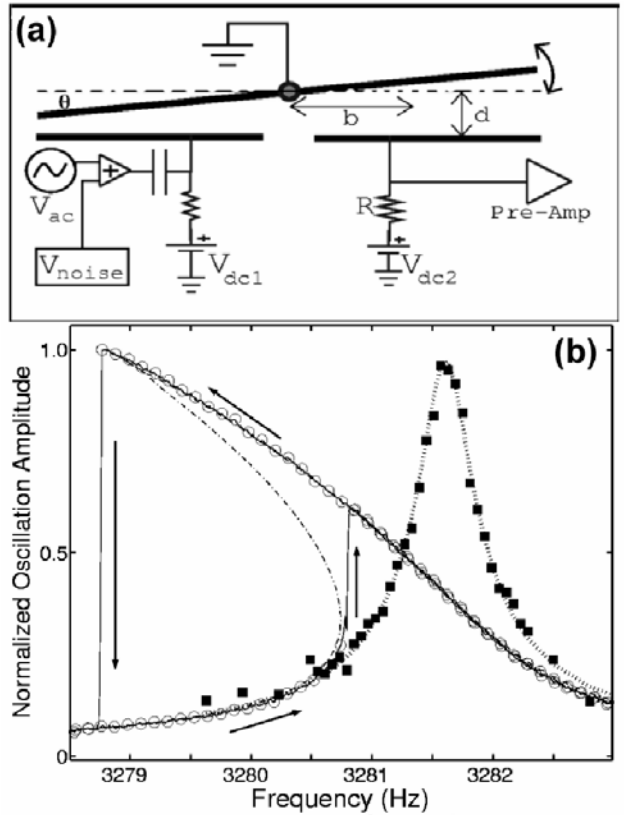
(Schwab & Roukes, Phys. Today 2005)



$3\times0.2\times0.14\mu m^3$

(Aldridge & Cleland, PRL 2005)

Resonantly modulated MEMS



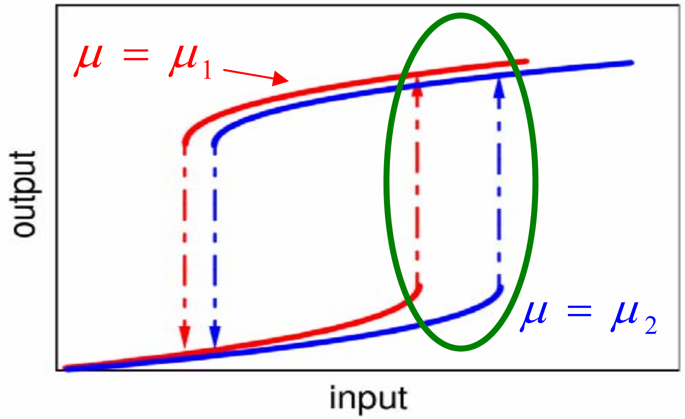
(Stambaugh & Chan, PRL, PRB 2006)

Quantum readout with a bifurcation amplifier

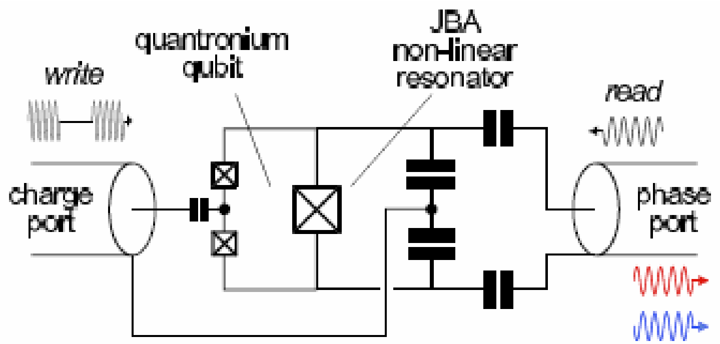
Strong signal amplification at a bifurcation point

$$\frac{d(\text{output})}{d(\text{input})} \to \infty$$

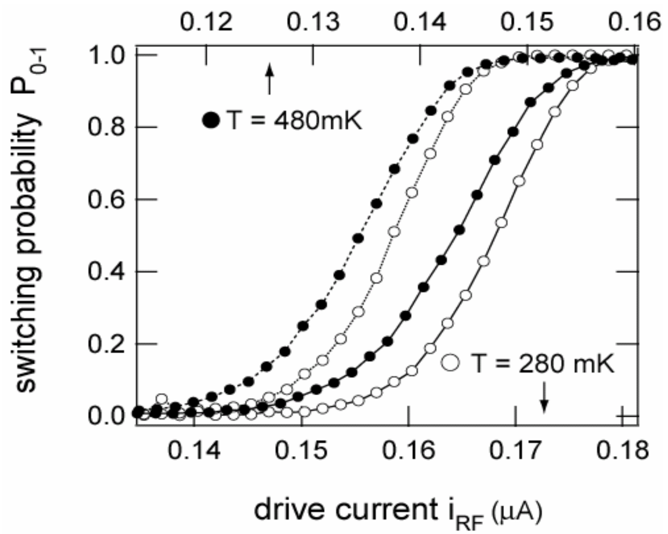
High sensitivity to a system parameter (e.g., oscillator eigenfrequency): a small change in the parameter leads to a shift of the bifurcation point



Real switching:
fluctuation-induced smearing



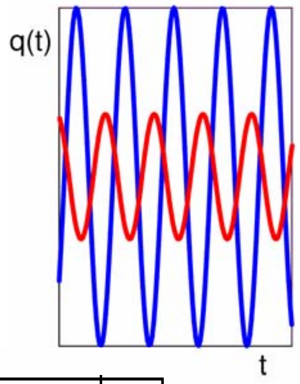
(Siddiqi et al., 2004-2006)

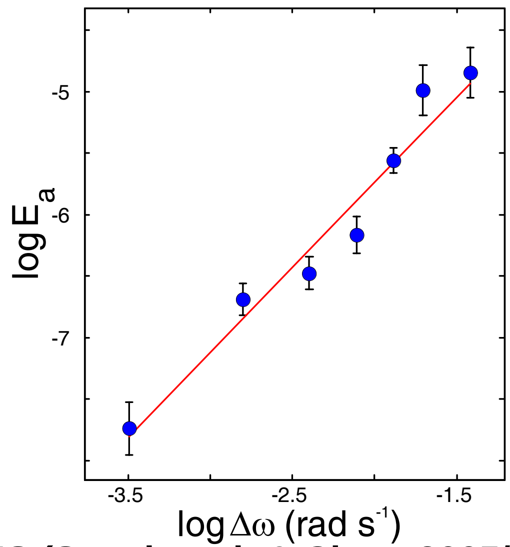


Scaling for a classical oscillator

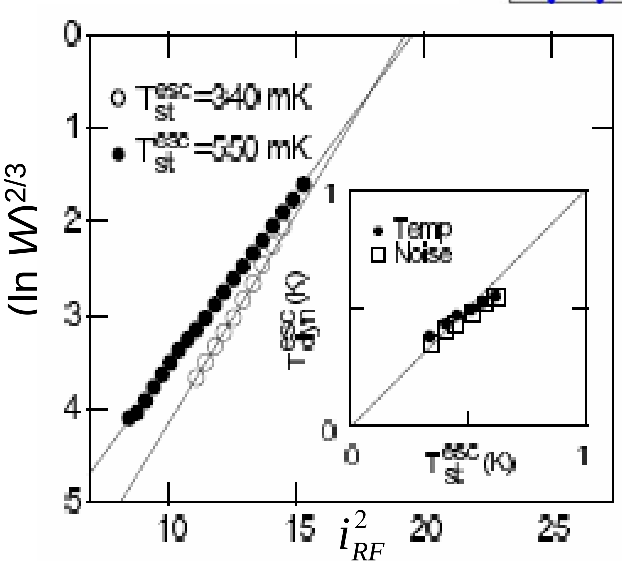
A modulated oscillator does not have detailed balance. Switching occurs between periodic states. For resonant modulation, the activation energy scales as $R \propto \eta^{3/2}$ close to bifurcation points. For small damping it may also display $R \propto \eta$ scaling

(MD and Krivoglaz, 1979, 1980)





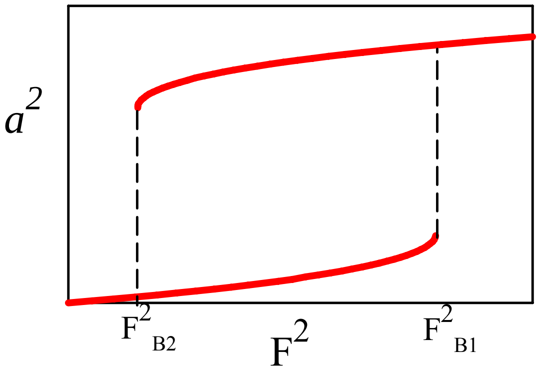
MEMS (Stambaugh & Chan, 2005/2006)

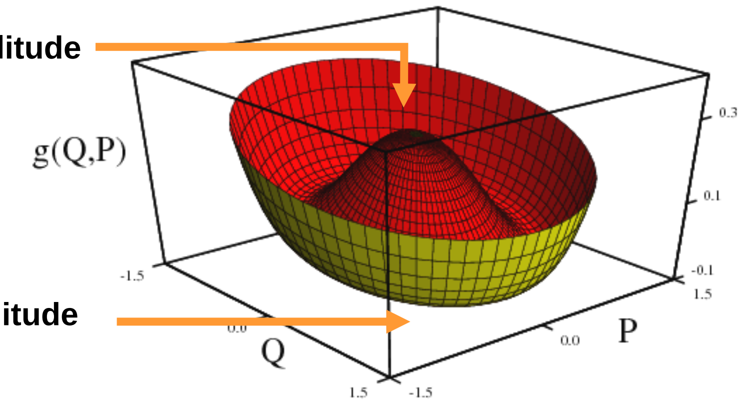


Josephson junctions (Siddiqi et al., 2005)

The predicted scaling $R \propto \eta^2$ has been seen near a cusp on the bifurcation curve (Aldridge & Cleland, 2005) and for a parametrically modulated oscillator (Stambaugh & Chan, 2006)

Quantum scaling for resonant driving





Underdamped regime:

small-amplitude $R_A \propto (2\bar{n}+1)^{-1}\eta^{3/2}, \quad \eta = F - F_{B1}$

large-amplitude $R_A \propto (2\bar{n}+1)^{-1}\eta, \quad \eta = F - F_{B2}$

small-amplitude: tunneling decay, $S_{\rm tun} \propto \eta^{5/4}$ **(Dmitriev & Dyakonov, 1986)**

Still closer to $F_{B1,2}$ **motion becomes overdamped and**

$$R_A \propto \eta^{3/2}/(2\bar{n}+1+\kappa_{\rm ph})$$

scaled dephasing rate **(MD, 2006)**

Conclusions

- Activation energy and the prefactor in the escape rate scale as powers of the distance to the bifurcation point

- Systems lacking detailed balance display scaling behavior that does not arise in systems in thermal equilibrium, with new scaling exponents.

- Scaling crossovers: $\xi = 3/2 \to 2 \to 3/2$ in slowly modulated systems and $\xi = 1 \to 3/2$ in underdamped systems

- Critical exponents near bifurcation points have been observed in modulated nonlinear oscillators.