Two Applications of Large Deviations to Random Perturbations of Nonlinear Dispersive Waves

Large Deviations Conference - University of Michigan

Eric Gautier
eric.gautier@yale.edu

Yale University & CREST

June 8, 2007

Outline

The stochastic nonlinear Schrödinger equations

- Deterministic equation: generic model for the propagation of wave packets in weakly nonlinear and weakly dispersive media
- The linear equation as an evolution equation:

$$\begin{cases} i\frac{\mathrm{d}}{\mathrm{d}t}u = \Delta u, & t \in \mathbb{R} \\ u(0) = u_0 \end{cases}$$

- $u(t) \in \mathrm{H}^1 = \{f \in \mathrm{L}^2(\mathbb{R}^d, \mathbb{C}) : \nabla f \in \mathrm{L}^2(\mathbb{R}^d, \mathbb{C})\}$ where $\mathrm{L}^2(\mathbb{R}^d, \mathbb{C})$ is equipped w/ $(f, g)_{\mathrm{L}^2(\mathbb{R}^d,\mathbb{C})} = \mathfrak{Re}\int_{\mathbb{R}^d} f(x)\overline{g}(x)dx$.
- $(-i\Delta, \mathrm{H}^3)$ generates a unitary group $(U(t))_{t\in\mathbb{R}}$, $U(t)$ are isometries on the spaces H^s.
- Dispersive property: $\exp\left(i\overrightarrow{k}\cdot\overrightarrow{x} - i\omega t\right)$ solution iff $\omega = |k|^2$.
- $\forall p \geq 2,\ \forall t \neq 0,\ \forall u_0 \in \mathrm{L}^{p'}, \|U(t)u_0\|_{\mathrm{L}^p} \leq (4\pi|t|)^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|u_0\|_{\mathrm{L}^{p'}}$

The stochastic nonlinear Schrödinger equations

- Deterministic equation: generic model for the propagation of wave packets in weakly nonlinear and weakly dispersive media
- The linear equation as an evolution equation:

$$\begin{cases} i\frac{\mathrm{d}}{\mathrm{d}t}u = \Delta u, & t \in \mathbb{R} \\ u(0) = u_0 \end{cases}$$

- $u(t) \in \mathrm{H}^1 = \{f \in \mathrm{L}^2(\mathbb{R}^d,\mathbb{C}) : \nabla f \in \mathrm{L}^2(\mathbb{R}^d,\mathbb{C})\}$ where $\mathrm{L}^2(\mathbb{R}^d,\mathbb{C})$ is equipped w/ $(f,g)_{\mathrm{L}^2(\mathbb{R}^d,\mathbb{C})} = \mathfrak{Re}\int_{\mathbb{R}^d} f(x)\overline{g}(x)dx$.
- $(-i\Delta, \mathrm{H}^3)$ generates a unitary group $(U(t))_{t\in\mathbb{R}}$, $U(t)$ are isometries on the spaces H^s.
- Dispersive property: $\exp\left(i\overrightarrow{k}\cdot\overrightarrow{x} - i\omega t\right)$ solution iff $\omega = |k|^2$.
- $\forall p \geq 2,\ \forall t \neq 0,\ \forall u_0 \in \mathrm{L}^{p'}, \|U(t)u_0\|_{\mathrm{L}^p} \leq (4\pi|t|)^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|u_0\|_{\mathrm{L}^{p'}}$

The stochastic nonlinear Schrödinger equations

- Deterministic equation: generic model for the propagation of wave packets in weakly nonlinear and weakly dispersive media
- The linear equation as an evolution equation:

$$\begin{cases} i\frac{\mathrm{d}}{\mathrm{d}t}u = \Delta u, & t \in \mathbb{R} \\ u(0) = u_0 \end{cases}$$

- $u(t) \in \mathrm{H}^1 = \{f \in \mathrm{L}^2(\mathbb{R}^d, \mathbb{C}) : \nabla f \in \mathrm{L}^2(\mathbb{R}^d, \mathbb{C})\}$ where $\mathrm{L}^2(\mathbb{R}^d, \mathbb{C})$ is equipped w/ $(f, g)_{\mathrm{L}^2(\mathbb{R}^d,\mathbb{C})} = \mathfrak{Re} \int_{\mathbb{R}^d} f(x)\overline{g}(x)dx$.
- $(-i\Delta, \mathrm{H}^3)$ generates a unitary group $(U(t))_{t\in\mathbb{R}}$, $U(t)$ are isometries on the spaces H^s.
- Dispersive property: $\exp\left(i\vec{k}\cdot\vec{x} - i\omega t\right)$ solution iff $\omega = |k|^2$.
- $\forall p \geq 2,\ \forall t \neq 0,\ \forall u_0 \in \mathrm{L}^{p'}, \|U(t)u_0\|_{\mathrm{L}^p} \leq (4\pi|t|)^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|u_0\|_{\mathrm{L}^{p'}}$

The stochastic nonlinear Schrödinger equations

- Deterministic equation: generic model for the propagation of wave packets in weakly nonlinear and weakly dispersive media
- The linear equation as an evolution equation:

$$\begin{cases} i\frac{\mathrm{d}}{\mathrm{d}t}u = \Delta u, & t \in \mathbb{R} \\ u(0) = u_0 \end{cases}$$

- $u(t) \in \mathrm{H}^1 = \{f \in \mathrm{L}^2(\mathbb{R}^d, \mathbb{C}) : \nabla f \in \mathrm{L}^2(\mathbb{R}^d, \mathbb{C})\}$ where $\mathrm{L}^2(\mathbb{R}^d, \mathbb{C})$ is equipped w/ $(f, g)_{\mathrm{L}^2(\mathbb{R}^d,\mathbb{C})} = \mathfrak{Re} \int_{\mathbb{R}^d} f(x)\overline{g}(x)dx$.
- $(-i\Delta, \mathrm{H}^3)$ generates a unitary group $(U(t))_{t\in\mathbb{R}}$, $U(t)$ are isometries on the spaces H^s.
- Dispersive property: $\exp\left(i\overrightarrow{k}\cdot\overrightarrow{x} - i\omega t\right)$ solution iff $\omega = |k|^2$.
- $\forall p \geq 2,\ \forall t \neq 0,\ \forall u_0 \in \mathrm{L}^{p'}, \|U(t)u_0\|_{\mathrm{L}^p} \leq (4\pi|t|)^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|u_0\|_{\mathrm{L}^{p'}}$

The stochastic nonlinear Schrödinger equations

- Deterministic equation: generic model for the propagation of wave packets in weakly nonlinear and weakly dispersive media
- The linear equation as an evolution equation:

$$\begin{cases} i\frac{\mathrm{d}}{\mathrm{d}t}u = \Delta u, & t \in \mathbb{R} \\ u(0) = u_0 \end{cases}$$

- $u(t) \in \mathrm{H}^1 = \{f \in \mathrm{L}^2(\mathbb{R}^d, \mathbb{C}) : \nabla f \in \mathrm{L}^2(\mathbb{R}^d, \mathbb{C})\}$ where $\mathrm{L}^2(\mathbb{R}^d, \mathbb{C})$ is equipped w/ $(f, g)_{\mathrm{L}^2(\mathbb{R}^d,\mathbb{C})} = \mathfrak{Re} \int_{\mathbb{R}^d} f(x)\overline{g}(x)dx$.
- $(-i\Delta, \mathrm{H}^3)$ generates a unitary group $(U(t))_{t\in\mathbb{R}}$, $U(t)$ are isometries on the spaces H^s.
- Dispersive property: $\exp\left(i\overrightarrow{k} \cdot \overrightarrow{x} - i\omega t\right)$ solution iff $\omega = |k|^2$.
- $\forall p \geq 2,\ \forall t \neq 0,\ \forall u_0 \in \mathrm{L}^{p'}, \|U(t)u_0\|_{\mathrm{L}^p} \leq (4\pi|t|)^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|u_0\|_{\mathrm{L}^{p'}}$

The stochastic nonlinear Schrödinger equations

- Deterministic equation: generic model for the propagation of wave packets in weakly nonlinear and weakly dispersive media
- The linear equation as an evolution equation:

$$\begin{cases} i\frac{\mathrm{d}}{\mathrm{d}t}u = \Delta u, & t \in \mathbb{R} \\ u(0) = u_0 \end{cases}$$

- $u(t) \in \mathrm{H}^1 = \{f \in \mathrm{L}^2(\mathbb{R}^d, \mathbb{C}) : \nabla f \in \mathrm{L}^2(\mathbb{R}^d, \mathbb{C})\}$ where $\mathrm{L}^2(\mathbb{R}^d, \mathbb{C})$ is equipped w/ $(f, g)_{\mathrm{L}^2(\mathbb{R}^d,\mathbb{C})} = \mathfrak{Re} \int_{\mathbb{R}^d} f(x)\overline{g}(x)dx$.
- $(-i\Delta, \mathrm{H}^3)$ generates a unitary group $(U(t))_{t\in\mathbb{R}}$, $U(t)$ are isometries on the spaces H^s.
- Dispersive property: $\exp\left(i\overrightarrow{k} \cdot \overrightarrow{x} - i\omega t\right)$ solution iff $\omega = |k|^2$.
- $\forall p \geq 2,\ \forall t \neq 0,\ \forall u_0 \in \mathrm{L}^{p'}, \|U(t)u_0\|_{\mathrm{L}^p} \leq (4\pi|t|)^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|u_0\|_{\mathrm{L}^{p'}}$

The stochastic nonlinear Schrödinger equations

- We consider mild solutions in $\mathrm{H}^1 := \mathrm{H}^1(\mathbb{R}^d, \mathbb{C})$ of
  - $i\mathrm{d}u = \left(\Delta u + \lambda |u|^{2\sigma} u\right) \mathrm{d}t + \sqrt{\epsilon}\mathrm{d}W \quad (1),$
  - $i\mathrm{d}u = \left(\Delta u + \lambda |u|^{2\sigma} u\right) \mathrm{d}t + \sqrt{\epsilon} u \circ \mathrm{d}W \quad (2);$

Example (eq. (2))

$$u(t) = U(t)u_0 - i\lambda \int_0^t U(t-s)(|u(s)|^{2\sigma} u(s))ds - \frac{1}{2}\epsilon \int_0^t U(t-s)F_\Phi(x)u(s)ds - i\sqrt{\epsilon}\int_0^t U(t-s)u(s)dW(s)$$

where $F_\Phi(t) = \sum_{i=1}^\infty (\Phi e_i(t))^2$, $(e_i)_{i\geq 1}$ c.o.s. of L^2.

- Nonlinearity and noise treated as perturbations
- $W = \Phi W_c$, Φ is Hilbert-Schmidt from L^2 to H^1 (eq. (1)) or $\mathrm{H}^s(\mathbb{R}^d, \mathbb{R})$ where $s > \frac{d}{2} + 1$ (eq. (2)), W_c is a cylindrical Wiener process "on" L^2:

$$W_c = \sum_{i=1}^\infty \beta_i(t) e_i \ ;$$

- $u(0) = u_0 \in \mathrm{H}^1$, $\lambda = \pm 1$ and $\sigma > 0$.

The stochastic nonlinear Schrödinger equations

- We consider mild solutions in $\mathrm{H}^1 := \mathrm{H}^1(\mathbb{R}^d, \mathbb{C})$ of
  - $i\mathrm{d}u = \left(\Delta u + \lambda |u|^{2\sigma} u\right) \mathrm{d}t + \sqrt{\epsilon}\mathrm{d}W \qquad (1),$
  - $i\mathrm{d}u = \left(\Delta u + \lambda |u|^{2\sigma} u\right) \mathrm{d}t + \sqrt{\epsilon} u \circ \mathrm{d}W \quad (2);$

Example (eq. (2))

$$u(t) = U(t)u_0 - i\lambda \int_0^t U(t-s)(|u(s)|^{2\sigma} u(s))ds - \frac{1}{2}\epsilon \int_0^t U(t-s)F_\Phi(x)u(s)ds - i\sqrt{\epsilon}\int_0^t U(t-s)u(s)dW(s)$$

where $F_\Phi(t) = \sum_{i=1}^\infty (\Phi e_i(t))^2$, $(e_i)_{i\geq 1}$ c.o.s. of L^2.

- Nonlinearity and noise treated as perturbations
- $W = \Phi W_c$, Φ is Hilbert-Schmidt from L^2 to H^1 (eq. (1)) or $\mathrm{H}^s(\mathbb{R}^d, \mathbb{R})$ where $s > \frac{d}{2} + 1$ (eq. (2)), W_c is a cylindrical Wiener process "on" L^2:

$$W_c = \sum_{i=1}^{\infty} \beta_i(t) e_i \ ;$$

- $u(0) = u_0 \in \mathrm{H}^1$, $\lambda = \pm 1$ and $\sigma > 0$.

The stochastic nonlinear Schrödinger equations

- We consider mild solutions in $\mathrm{H}^1 := \mathrm{H}^1(\mathbb{R}^d, \mathbb{C})$ of
  - $i\mathrm{d}u = \left(\Delta u + \lambda |u|^{2\sigma} u\right) \mathrm{d}t + \sqrt{\epsilon}\mathrm{d}W \quad (1),$
  - $i\mathrm{d}u = \left(\Delta u + \lambda |u|^{2\sigma} u\right) \mathrm{d}t + \sqrt{\epsilon} u \circ \mathrm{d}W \quad (2);$

Example (eq. (2))

$$u(t) = U(t)u_0 - i\lambda \int_0^t U(t-s)(|u(s)|^{2\sigma} u(s))ds - \frac{1}{2}\epsilon \int_0^t U(t-s)F_\Phi(x)u(s)ds - i\sqrt{\epsilon}\int_0^t U(t-s)u(s)dW(s)$$

where $F_\Phi(t) = \sum_{i=1}^{\infty} (\Phi e_i(t))^2$, $(e_i)_{i\geq 1}$ c.o.s. of L^2.

- Nonlinearity and noise treated as perturbations
- $W = \Phi W_c$, Φ is Hilbert-Schmidt from L^2 to H^1 (eq. (1)) or $\mathrm{H}^s(\mathbb{R}^d, \mathbb{R})$ where $s > \frac{d}{2} + 1$ (eq. (2)), W_c is a cylindrical Wiener process "on" L^2:

$$W_c = \sum_{i=1}^{\infty} \beta_i(t) e_i \ ;$$

- $u(0) = u_0 \in \mathrm{H}^1$, $\lambda = \pm 1$ and $\sigma > 0$.

The stochastic nonlinear Schrödinger equations

- We consider mild solutions in $\mathrm{H}^1 := \mathrm{H}^1(\mathbb{R}^d, \mathbb{C})$ of
  - $i\mathrm{d}u = \left(\Delta u + \lambda |u|^{2\sigma} u\right) \mathrm{d}t + \sqrt{\epsilon}\mathrm{d}W \quad (1)$,
  - $i\mathrm{d}u = \left(\Delta u + \lambda |u|^{2\sigma} u\right) \mathrm{d}t + \sqrt{\epsilon} u \circ \mathrm{d}W \quad (2)$;

Example (eq. (2))

$$u(t) = U(t)u_0 - i\lambda \int_0^t U(t-s)(|u(s)|^{2\sigma} u(s))ds$$
$$-\frac{1}{2}\epsilon \int_0^t U(t-s)F_\Phi(x)u(s)ds - i\sqrt{\epsilon}\int_0^t U(t-s)u(s)dW(s)$$

where $F_\Phi(t) = \sum_{i=1}^\infty (\Phi e_i(t))^2$, $(e_i)_{i\geq 1}$ c.o.s. of L^2.

- Nonlinearity and noise treated as perturbations
- $W = \Phi W_c$, Φ is Hilbert-Schmidt from L^2 to H^1 (eq. (1)) or $\mathrm{H}^s(\mathbb{R}^d, \mathbb{R})$ where $s > \frac{d}{2} + 1$ (eq. (2)), W_c is a cylindrical Wiener process "on" L^2:

$$W_c = \sum_{i=1}^\infty \beta_i(t) e_i \; ;$$

- $u(0) = u_0 \in \mathrm{H}^1$, $\lambda = \pm 1$ and $\sigma > 0$.

THE STOCHASTIC NONLINEAR SCHRÖDINGER EQUATIONS

- We consider mild solutions in $\mathrm{H}^1 := \mathrm{H}^1(\mathbb{R}^d, \mathbb{C})$ of
  - $i\mathrm{d}u = \left(\Delta u + \lambda |u|^{2\sigma} u\right) \mathrm{d}t + \sqrt{\epsilon}\mathrm{d}W \quad (1),$
  - $i\mathrm{d}u = \left(\Delta u + \lambda |u|^{2\sigma} u\right) \mathrm{d}t + \sqrt{\epsilon} u \circ \mathrm{d}W \quad (2);$

EXAMPLE (EQ. (2))

$$u(t) = U(t)u_0 - i\lambda \int_0^t U(t-s)(|u(s)|^{2\sigma} u(s))ds - \frac{1}{2}\epsilon \int_0^t U(t-s)F_\Phi(x)u(s)ds - i\sqrt{\epsilon}\int_0^t U(t-s)u(s)dW(s)$$

where $F_\Phi(t) = \sum_{i=1}^\infty (\Phi e_i(t))^2$, $(e_i)_{i\geq 1}$ c.o.s. of L^2.

- Nonlinearity and noise treated as perturbations
- $W = \Phi W_c$, Φ is Hilbert-Schmidt from L^2 to H^1 (eq. (1)) or $\mathrm{H}^s(\mathbb{R}^d, \mathbb{R})$ where $s > \frac{d}{2} + 1$ (eq. (2)), W_c is a cylindrical Wiener process "on" L^2:

$$W_c = \sum_{i=1}^{\infty} \beta_i(t) e_i \ ;$$

- $u(0) = u_0 \in \mathrm{H}^1$, $\lambda = \pm 1$ and $\sigma > 0$.

The stochastic nonlinear Schrödinger equations

- We consider mild solutions in $\mathrm{H}^1 := \mathrm{H}^1(\mathbb{R}^d, \mathbb{C})$ of
  - $i\mathrm{d}u = \left(\Delta u + \lambda |u|^{2\sigma} u\right) \mathrm{d}t + \sqrt{\epsilon}\mathrm{d}W$ (1),
  - $i\mathrm{d}u = \left(\Delta u + \lambda |u|^{2\sigma} u\right) \mathrm{d}t + \sqrt{\epsilon} u \circ \mathrm{d}W$ (2);

Example (eq. (2))

$$u(t) = U(t)u_0 - i\lambda \int_0^t U(t-s)(|u(s)|^{2\sigma} u(s))ds - \frac{1}{2}\epsilon \int_0^t U(t-s)F_\Phi(x)u(s)ds - i\sqrt{\epsilon}\int_0^t U(t-s)u(s)dW(s)$$

where $F_\Phi(t) = \sum_{i=1}^\infty (\Phi e_i(t))^2$, $(e_i)_{i\geq 1}$ c.o.s. of L^2.

- Nonlinearity and noise treated as perturbations
- $W = \Phi W_c$, Φ is Hilbert-Schmidt from L^2 to H^1 (eq. (1)) or $\mathrm{H}^s(\mathbb{R}^d, \mathbb{R})$ where $s > \frac{d}{2} + 1$ (eq. (2)), W_c is a cylindrical Wiener process "on" L^2:

$$W_c = \sum_{i=1}^\infty \beta_i(t) e_i \ ;$$

- $u(0) = u_0 \in \mathrm{H}^1$, $\lambda = \pm 1$ and $\sigma > 0$.

The stochastic nonlinear Schrödinger equations

- We consider mild solutions in $\mathrm{H}^1 := \mathrm{H}^1(\mathbb{R}^d, \mathbb{C})$ of
  - $i\mathrm{d}u = \left(\Delta u + \lambda |u|^{2\sigma} u\right) \mathrm{d}t + \sqrt{\epsilon}\mathrm{d}W \quad (1)$,
  - $i\mathrm{d}u = \left(\Delta u + \lambda |u|^{2\sigma} u\right) \mathrm{d}t + \sqrt{\epsilon} u \circ \mathrm{d}W \quad (2)$;

Example (eq. (2))

$$u(t) = U(t)u_0 - i\lambda \int_0^t U(t-s)(|u(s)|^{2\sigma} u(s))ds - \frac{1}{2}\epsilon \int_0^t U(t-s)F_\Phi(x)u(s)ds - i\sqrt{\epsilon} \int_0^t U(t-s)u(s)dW(s)$$

where $F_\Phi(t) = \sum_{i=1}^\infty (\Phi e_i(t))^2$, $(e_i)_{i\geq 1}$ c.o.s. of L^2.

- Nonlinearity and noise treated as perturbations
- $W = \Phi W_c$, Φ is Hilbert-Schmidt from L^2 to H^1 (eq. (1)) or $\mathrm{H}^s(\mathbb{R}^d, \mathbb{R})$ where $s > \frac{d}{2} + 1$ (eq. (2)), W_c is a cylindrical Wiener process "on" L^2:

$$W_c = \sum_{i=1}^\infty \beta_i(t) e_i \ ;$$

- $u(0) = u_0 \in \mathrm{H}^1$, $\lambda = \pm 1$ and $\sigma > 0$.

The stochastic nonlinear Schrödinger equations

Theorem (de Bouard & Debussche 03')

- *When the noise is additive the Cauchy problem is*
  ▷ *locally well posed for* $0 \leq \sigma < \frac{2}{d-2}$ $(\sigma \geq 0$ *if* $d = 1, 2)$
  ▷ *globally well posed if* $0 \leq \sigma < \frac{2}{d-2}$ *and* $\lambda = -1$ *or* $0 \leq \sigma < \frac{2}{d}$ *and* $\lambda = 1$.
- *When the noise is multiplicative, the Cauchy problem is*
  ▷ *locally well posed under weaker assum. on* Φ *than ours and* σ *s.t.*
  $$\begin{cases} 0 < \sigma \text{ if } d = 1, 2, \ 0 < \sigma < 2 \text{ if } d = 3 \\ \frac{1}{2} \leq \sigma < \frac{2}{d-2} \text{ or } \sigma < \frac{1}{d-1} \text{ if } d \geq 4 \end{cases},$$
  ▷ $\forall t < T^*$, $\mathbf{N}(u(t)) = \int_{\mathbb{R}^d} |u(t,x)|^2 dx = \|u(t)\|_{L^2}^2 = cst$
  ▷ *same result on global existence than for an additive noise.*

In a series of paper they study the influence of a noise on the blow-up phenomenon.

Under some restrictions, local well-posedness is obtained for some SNLS equations driven by a fractional (in time) additive noises in *EJP* 07' for any Hurst parameter H and with Hölder continuity in time.

The stochastic nonlinear Schrödinger equations

Theorem (de Bouard & Debussche 03')

- *When the noise is additive the Cauchy problem is*
  ▷ *locally well posed for* $0 \le \sigma < \frac{2}{d-2}$ *(*$\sigma \ge 0$ *if* $d = 1, 2$*)*
  ▷ *globally well posed if* $0 \le \sigma < \frac{2}{d-2}$ *and* $\lambda = -1$ *or* $0 \le \sigma < \frac{2}{d}$ *and* $\lambda = 1$.
- *When the noise is multiplicative, the Cauchy problem is*
  ▷ *locally well posed under weaker assum. on* Φ *than ours and* σ *s.t.*
  $$\begin{cases} 0 < \sigma \text{ if } d = 1, 2, \ 0 < \sigma < 2 \text{ if } d = 3 \\ \frac{1}{2} \le \sigma < \frac{2}{d-2} \text{ or } \sigma < \frac{1}{d-1} \text{ if } d \ge 4 \end{cases},$$
  ▷ $\forall t < T^*$, $\mathbf{N}(u(t)) = \int_{\mathbb{R}^d} |u(t,x)|^2 dx = \|u(t)\|_{L^2}^2 = cst$
  ▷ *same result on global existence than for an additive noise.*

In a series of paper they study the influence of a noise on the blow-up phenomenon.

Under some restrictions, local well-posedness is obtained for some SNLS equations driven by a fractional (in time) additive noises in *EJP* 07' for any Hurst parameter H and with Hölder continuity in time.

The stochastic nonlinear Schrödinger equations

Theorem (de Bouard & Debussche 03')

- *When the noise is additive the Cauchy problem is*
  ▷ *locally well posed for* $0 \leq \sigma < \frac{2}{d-2}$ *(*$\sigma \geq 0$ *if* $d = 1, 2$*)*
  ▷ *globally well posed if* $0 \leq \sigma < \frac{2}{d-2}$ *and* $\lambda = -1$ *or* $0 \leq \sigma < \frac{2}{d}$ *and* $\lambda = 1$.
- *When the noise is multiplicative, the Cauchy problem is*
  ▷ *locally well posed under weaker assum. on* Φ *than ours and* σ *s.t.*
  $$\begin{cases} 0 < \sigma \text{ if } d = 1, 2, \ 0 < \sigma < 2 \text{ if } d = 3 \\ \frac{1}{2} \leq \sigma < \frac{2}{d-2} \text{ or } \sigma < \frac{1}{d-1} \text{ if } d \geq 4 \end{cases},$$
  ▷ $\forall t < T^*$, $\mathbf{N}(u(t)) = \int_{\mathbb{R}^d} |u(t,x)|^2 dx = \|u(t)\|^2_{L^2} = cst$
  ▷ *same result on global existence than for an additive noise.*

In a series of paper they study the influence of a noise on the blow-up phenomenon.

Under some restrictions, local well-posedness is obtained for some SNLS equations driven by a fractional (in time) additive noises in *EJP* 07' for any Hurst parameter H and with Hölder continuity in time.

The stochastic nonlinear Schrödinger equations

Theorem (de Bouard & Debussche 03')

- *When the noise is additive the Cauchy problem is*
  ▷ *locally well posed for* $0 \leq \sigma < \frac{2}{d-2}$ *(*$\sigma \geq 0$ *if* $d = 1, 2$*)*
  ▷ *globally well posed if* $0 \leq \sigma < \frac{2}{d-2}$ *and* $\lambda = -1$ *or* $0 \leq \sigma < \frac{2}{d}$ *and* $\lambda = 1$.
- *When the noise is multiplicative, the Cauchy problem is*
  ▷ *locally well posed under weaker assum. on* Φ *than ours and* σ *s.t.*
  $$\begin{cases} 0 < \sigma \text{ if } d = 1, 2, \ 0 < \sigma < 2 \text{ if } d = 3 \\ \frac{1}{2} \leq \sigma < \frac{2}{d-2} \text{ or } \sigma < \frac{1}{d-1} \text{ if } d \geq 4 \end{cases},$$
  ▷ $\forall t < T^*$, $\mathbf{N}(u(t)) = \int_{\mathbb{R}^d} |u(t,x)|^2 dx = \|u(t)\|^2_{L^2} = cst$
  ▷ *same result on global existence than for an additive noise.*

In a series of paper they study the influence of a noise on the blow-up phenomenon.

Under some restrictions, local well-posedness is obtained for some SNLS equations driven by a fractional (in time) additive noises in *EJP* 07' for any Hurst parameter H and with Hölder continuity in time.

The stochastic nonlinear Schrödinger equations

Theorem (de Bouard & Debussche 03')

- *When the noise is additive the Cauchy problem is*
  ▷ *locally well posed for* $0 \leq \sigma < \frac{2}{d-2}$ *(*$\sigma \geq 0$ *if* $d = 1, 2$*)*
  ▷ *globally well posed if* $0 \leq \sigma < \frac{2}{d-2}$ *and* $\lambda = -1$ *or* $0 \leq \sigma < \frac{2}{d}$ *and* $\lambda = 1$.
- *When the noise is multiplicative, the Cauchy problem is*
  ▷ *locally well posed under weaker assum. on* Φ *than ours and* σ *s.t.*
  $$\begin{cases} 0 < \sigma \text{ if } d = 1, 2, \ 0 < \sigma < 2 \text{ if } d = 3 \\ \frac{1}{2} \leq \sigma < \frac{2}{d-2} \text{ or } \sigma < \frac{1}{d-1} \text{ if } d \geq 4 \end{cases},$$
  ▷ $\forall t < T^*, \ \mathbf{N}(u(t)) = \int_{\mathbb{R}^d} |u(t,x)|^2 dx = \|u(t)\|_{L^2}^2 = cst$
  ▷ *same result on global existence than for an additive noise.*

In a series of paper they study the influence of a noise on the blow-up phenomenon.

Under some restrictions, local well-posedness is obtained for some SNLS equations driven by a fractional (in time) additive noises in *EJP* 07' for any Hurst parameter H and with Hölder continuity in time.

Large deviations

For A measurable set of paths that does not contain the deterministic solution, LD quantify the convergence of $\mathbb{P}\left(u^{\epsilon,u_0} \in A\right)$ to 0 with ϵ.

Definition

LDP: for every Borel set A, we have the sequence of inequalities

$$-\inf_{w\in\mathrm{Int}(A)} I_{u_0}(w) \leq \underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(u^{\epsilon,u_0} \in A\right) \leq \overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(u^{\epsilon,u_0} \in A\right) \leq -\inf_{w\in\overline{A}} I_{u_0}(w),$$

where I_{u_0} (rate function) is l.s.c. (LDP of speed ϵ)

When $\forall r > 0$ $I_{u_0}^{-1}([0,r])$ is compact, I_{u_0} is good.

Large deviations

For A measurable set of paths that does not contain the deterministic solution, LD quantify the convergence of $\mathbb{P}\left(u^{\epsilon,u_0} \in A\right)$ to 0 with ϵ.

Definition

LDP: for every Borel set A, we have the sequence of inequalities

$$-\inf_{w\in \mathrm{Int}(A)} I_{u_0}(w) \leq \underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(u^{\epsilon,u_0} \in A\right) \leq \overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(u^{\epsilon,u_0} \in A\right) \leq -\inf_{w\in \overline{A}} I_{u_0}(w),$$

where I_{u_0} (rate function) is l.s.c. (LDP of speed ϵ)

When $\forall r > 0$ $I_{u_0}^{-1}([0,r])$ is compact, I_{u_0} is good.

Large deviations

When σ is large (except when $\lambda = -1$ and $d = 1, 2$) solutions blow-up in finite time $\Rightarrow$ we consider spaces of exploding paths.

1. define $C([0,+\infty); H^1 \cup \{\Delta\})$ and $\mathcal{T}(\varphi) = \inf\{t \in [0,+\infty) : \varphi(t) = \Delta\}$,
2. allows to define as a set

$$\mathcal{E}(H^1) = \{\varphi \in C([0,+\infty); H^1 \cup \{\Delta\}) \text{ s.t. } \varphi(t_0) = \Delta \Rightarrow \forall t \geq t_0, f(t) = \Delta\}$$

3. we equip this set with the topology defined by the neighborhood basis

$$\left\{\varphi \in \mathcal{E}(H^1) : \mathcal{T}(\varphi) \geq \mathcal{T}(\varphi_1), \ \|\varphi_1 - \varphi\|_{C([0,T];H^1)} \leq r, T < \mathcal{T}(\varphi_1)\right\}.$$

It is a Haudorff topological space.

▷ The stronger the topology the sharper the LDP: we state the LDP in a subspace space $\mathcal{E}_\infty$ where solutions indeed live embedded with a topology that takes into account all the integrability properties of the Schrödinger group.

Large deviations

When σ is large (except when $\lambda = -1$ and $d = 1, 2$) solutions blow-up in finite time $\Rightarrow$ we consider spaces of exploding paths.

1. define $\mathrm{C}([0,+\infty);\mathrm{H}^1 \cup \{\Delta\})$ and $\mathcal{T}(\varphi) = \inf\{t \in [0,+\infty) : \varphi(t) = \Delta\}$,
2. allows to define as a set

$$\mathcal{E}(\mathrm{H}^1) = \{\varphi \in \mathrm{C}([0,+\infty);\mathrm{H}^1 \cup \{\Delta\}) \text{ s.t. } \varphi(t_0) = \Delta \Rightarrow \forall t \geq t_0, f(t) = \Delta\}$$

3. we equip this set with the topology defined by the neighborhood basis

$$\left\{\varphi \in \mathcal{E}(\mathrm{H}^1) : \mathcal{T}(\varphi) \geq \mathcal{T}(\varphi_1),\ \|\varphi_1 - \varphi\|_{\mathrm{C}([0,T];\mathrm{H}^1)} \leq r, T < \mathcal{T}(\varphi_1)\right\}.$$

It is a Haudorff topological space.

▷ The stronger the topology the sharper the LDP: we state the LDP in a subspace space $\mathcal{E}_\infty$ where solutions indeed live embedded with a topology that takes into account all the integrability properties of the Schrödinger group.

Large deviations

When σ is large (except when $\lambda = -1$ and $d = 1, 2$) solutions blow-up in finite time $\Rightarrow$ we consider spaces of exploding paths.

1. define $\mathrm{C}([0, +\infty); \mathrm{H}^1 \cup \{\Delta\})$ and $\mathcal{T}(\varphi) = \inf\{t \in [0, +\infty) : \varphi(t) = \Delta\}$,
2. allows to define as a set

$$\mathcal{E}(\mathrm{H}^1) = \{\varphi \in \mathrm{C}([0, +\infty); \mathrm{H}^1 \cup \{\Delta\}) \text{ s.t. } \varphi(t_0) = \Delta \Rightarrow \forall t \geq t_0, f(t) = \Delta\}$$

3. we equip this set with the topology defined by the neighborhood basis

$$\left\{\varphi \in \mathcal{E}(\mathrm{H}^1) : \mathcal{T}(\varphi) \geq \mathcal{T}(\varphi_1),\ \|\varphi_1 - \varphi\|_{\mathrm{C}\left([0,T];\mathrm{H}^1\right)} \leq r, T < \mathcal{T}(\varphi_1)\right\}.$$

It is a Haudorff topological space.

▷ The stronger the topology the sharper the LDP: we state the LDP in a subspace space $\mathcal{E}_\infty$ where solutions indeed live embedded with a topology that takes into account all the integrability properties of the Schrödinger group.

Large deviations

When σ is large (except when $\lambda = -1$ and $d = 1, 2$) solutions blow-up in finite time $\Rightarrow$ we consider spaces of exploding paths.

1. define $\mathrm{C}([0,+\infty);\mathrm{H}^1 \cup \{\Delta\})$ and $\mathcal{T}(\varphi) = \inf\{t \in [0,+\infty) : \varphi(t) = \Delta\}$,
2. allows to define as a set
$$\mathcal{E}(\mathrm{H}^1) = \{\varphi \in \mathrm{C}([0,+\infty);\mathrm{H}^1 \cup \{\Delta\}) \text{ s.t. } \varphi(t_0) = \Delta \Rightarrow \forall t \geq t_0, f(t) = \Delta\}$$
3. we equip this set with the topology defined by the neighborhood basis
$$\left\{\varphi \in \mathcal{E}(\mathrm{H}^1) : \mathcal{T}(\varphi) \geq \mathcal{T}(\varphi_1),\ \|\varphi_1 - \varphi\|_{\mathrm{C}\left([0,T];\mathrm{H}^1\right)} \leq r, T < \mathcal{T}(\varphi_1)\right\}.$$
It is a Haudorff topological space.

▷ The stronger the topology the sharper the LDP: we state the LDP in a subspace space $\mathcal{E}_\infty$ where solutions indeed live embedded with a topology that takes into account all the integrability properties of the Schrödinger group.

Large deviations

Theorem (*SPA* and *ESAIM: PS* 05')

μ^{ϵ,u_0} *satisfy on* $\mathcal{E}_\infty$, *with the restriction for multiplicative noises*
$\left\{ \begin{array}{ll} \frac{1}{2} \leq \sigma & si\ d = 1,2 \\ \frac{1}{2} \leq \sigma < \frac{2}{d-2} & si\ d \geq 3 \end{array} \right|$, *a LDP of speed* ϵ *and good rate function*

$$I_{u_0}(u) = \frac{1}{2} \inf_{h \in \mathrm{L}^2(0,\infty;\mathrm{L}^2):\ \mathbf{S}(u_0,h)=u} \left\{ \|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)} \right\},$$

where $\inf \emptyset = \infty$ *and* $\mathbf{S}(u_0,h)$ *is the unique mild solution of*

$$\left\{ \begin{array}{l} i\frac{\partial u}{\partial t} = \Delta u + \lambda |u|^{2\sigma} u + \Theta(u,h), \\ u(0) = u_0 \ \in \mathrm{H}^1,\ h \in \mathrm{L}^2\left(0,\infty,\mathrm{L}^2\right). \end{array} \right.$$

$\Theta(u,h) = \Phi h$ *(additive noise)*, $\Theta(u,h) = u\Phi h$ *(multiplicative noise)*

- Case of fractional in time noise studied in *EJP* 07'.

Large deviations

Theorem (*SPA* and *ESAIM: PS* 05')

μ^{ϵ,u_0} satisfy on $\mathcal{E}_\infty$, with the restriction for multiplicative noises

$$\left\{ \begin{array}{ll} \frac{1}{2} \leq \sigma & si\ d = 1, 2 \\ \frac{1}{2} \leq \sigma < \frac{2}{d-2} & si\ d \geq 3 \end{array} \right| ,$$

a LDP of speed ϵ and good rate function

$$I_{u_0}(u) = \frac{1}{2} \inf_{h \in \mathrm{L}^2(0,\infty;\mathrm{L}^2):\ \mathbf{S}(u_0,h)=u} \left\{ \|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)} \right\},$$

where $\inf \emptyset = \infty$ *and* $\mathbf{S}(u_0, h)$ *is the unique mild solution of*

$$\left\{ \begin{array}{l} i\frac{\partial u}{\partial t} = \Delta u + \lambda |u|^{2\sigma} u + \Theta(u, h), \\ u(0) = u_0 \ \in \mathrm{H}^1,\ h \in \mathrm{L}^2\left(0, \infty, \mathrm{L}^2\right). \end{array} \right.$$

$\Theta(u, h) = \Phi h$ *(additive noise)*, $\Theta(u, h) = u\Phi h$ *(multiplicative noise)*

- Case of fractional in time noise studied in *EJP* 07'.

Application to error in transmission

- We consider the case where $d = \sigma = \lambda = 1$;
- $t \in [0, L]$ is the coordinate along the line, L is of the order of 1000 km ;
- x is some retarded time ;
- Solitons are particular solutions

$$\Psi_A(t,x) = \sqrt{2}A\mathrm{sech}(A(x - x_0)) \exp\left(-iA^2 t + i\theta_0\right)$$

- To compensate for loss in the fiber an amplification device is introduced: Regularly spaced Erbium Doped Amplifiers, or distributed amplification $\Rightarrow$ complex additive noise
  Raman or 4 wave mixing amplification $\Rightarrow$ real multiplicative noises (we neglect the Raman non linear response...).
- Heisenberg principle $\Rightarrow$ uncertainty on the amplified signal $\Rightarrow$ noise.
- Noise is assumed to be small compared to L (fixed) $\epsilon L << 1$;

Application to error in transmission

- We consider the case where $d = \sigma = \lambda = 1$;
- $t \in [0, L]$ is the coordinate along the line, L is of the order of 1000 km ;
- x is some retarded time ;
- Solitons are particular solutions

$$\Psi_A(t,x) = \sqrt{2}A\mathrm{sech}(A(x - x_0)) \exp\left(-iA^2 t + i\theta_0\right)$$

- To compensate for loss in the fiber an amplification device is introduced: Regularly spaced Erbium Doped Amplifiers, or distributed amplification $\Rightarrow$ complex additive noise
  Raman or 4 wave mixing amplification $\Rightarrow$ real multiplicative noises (we neglect the Raman non linear response...).
- Heisenberg principle $\Rightarrow$ uncertainty on the amplified signal $\Rightarrow$ noise.
- Noise is assumed to be small compared to L (fixed) $\epsilon L << 1$;

Application to error in transmission

- We consider the case where $d = \sigma = \lambda = 1$;
- $t \in [0, L]$ is the coordinate along the line, L is of the order of 1000 km ;
- x is some retarded time ;
- Solitons are particular solutions

$$\Psi_A(t,x) = \sqrt{2}A\mathrm{sech}(A(x - x_0)) \exp\left(-iA^2 t + i\theta_0\right)$$

- To compensate for loss in the fiber an amplification device is introduced:
  Regularly spaced Erbium Doped Amplifiers, or distributed amplification
  $\Rightarrow$ complex additive noise
  Raman or 4 wave mixing amplification $\Rightarrow$ real multiplicative noises (we neglect the Raman non linear response...).
- Heisenberg principle $\Rightarrow$ uncertainty on the amplified signal $\Rightarrow$ noise.
- Noise is assumed to be small compared to L (fixed) $\epsilon L << 1$;

Application to error in transmission

- We consider the case where $d = \sigma = \lambda = 1$;
- $t \in [0, L]$ is the coordinate along the line, L is of the order of 1000 km ;
- x is some retarded time ;
- Solitons are particular solutions

$$\Psi_A(t,x) = \sqrt{2} A \mathrm{sech}(A(x - x_0)) \exp\left(-iA^2 t + i\theta_0\right)$$

- To compensate for loss in the fiber an amplification device is introduced: Regularly spaced Erbium Doped Amplifiers, or distributed amplification $\Rightarrow$ complex additive noise
  Raman or 4 wave mixing amplification $\Rightarrow$ real multiplicative noises (we neglect the Raman non linear response...).
- Heisenberg principle $\Rightarrow$ uncertainty on the amplified signal $\Rightarrow$ noise.
- Noise is assumed to be small compared to L (fixed) $\epsilon L << 1$;

Application to error in transmission

- We consider the case where $d = \sigma = \lambda = 1$;
- $t \in [0, L]$ is the coordinate along the line, L is of the order of 1000 km ;
- x is some retarded time ;
- Solitons are particular solutions

$$\Psi_A(t,x) = \sqrt{2}A\mathrm{sech}(A(x-x_0))\exp\left(-iA^2t + i\theta_0\right)$$

- To compensate for loss in the fiber an amplification device is introduced: Regularly spaced Erbium Doped Amplifiers, or distributed amplification $\Rightarrow$ complex additive noise
  Raman or 4 wave mixing amplification $\Rightarrow$ real multiplicative noises (we neglect the Raman non linear response...).
- Heisenberg principle $\Rightarrow$ uncertainty on the amplified signal $\Rightarrow$ noise.
- Noise is assumed to be small compared to L (fixed) $\epsilon L << 1$;

Application to error in transmission

- We consider the case where $d = \sigma = \lambda = 1$;
- $t \in [0, L]$ is the coordinate along the line, L is of the order of 1000 km ;
- x is some retarded time ;
- Solitons are particular solutions

$$\Psi_A(t,x) = \sqrt{2}A\mathrm{sech}(A(x - x_0)) \exp\left(-iA^2 t + i\theta_0\right)$$

- To compensate for loss in the fiber an amplification device is introduced: Regularly spaced Erbium Doped Amplifiers, or distributed amplification $\Rightarrow$ complex additive noise
  Raman or 4 wave mixing amplification $\Rightarrow$ real multiplicative noises (we neglect the Raman non linear response...).
- Heisenberg principle $\Rightarrow$ uncertainty on the amplified signal $\Rightarrow$ noise.
- Noise is assumed to be small compared to L (fixed) $\epsilon L << 1$;

Application to error in transmission

- We consider the case where $d = \sigma = \lambda = 1$;
- $t \in [0, L]$ is the coordinate along the line, L is of the order of 1000 km ;
- x is some retarded time ;
- Solitons are particular solutions

$$\Psi_A(t,x) = \sqrt{2}A\mathrm{sech}(A(x-x_0)) \exp\left(-iA^2 t + i\theta_0\right)$$

- To compensate for loss in the fiber an amplification device is introduced: Regularly spaced Erbium Doped Amplifiers, or distributed amplification $\Rightarrow$ complex additive noise
  Raman or 4 wave mixing amplification $\Rightarrow$ real multiplicative noises (we neglect the Raman non linear response...).
- Heisenberg principle $\Rightarrow$ uncertainty on the amplified signal $\Rightarrow$ noise.
- Noise is assumed to be small compared to L (fixed) $\epsilon L << 1$;

Application to error in transmission

- Soliton profile allow to code bits
  $1 := \Psi_A^0(x) = \sqrt{2}A\mathrm{sech}(Ax)$
  $0 := 0$
- At the end of the line we measure:

$$\frac{1}{T}\int_{-T/2}^{T/2}|u(L,x)|^2dx$$

  and compare the number to a threshold.
- Due to noise error in transmission may occur
  A 1 is wrongly discarded and it is decided that a 0 has been emitted
  or the contrary may also happen
- Two processes are mainly responsible for these errors:
  1. A fluctuation of the arrival time
     $\mathbf{Y}(u(L,x)) = \int_{\mathbb{R}^d} x|u(L,x)|^2dx$
  2. A fluctuation of the mass (with additive noise only)
- Error rate $\approx 10^{-12} \Rightarrow$ possibilities IS MC methods or genealogical particle systems (c.f. Del Moral & Garnier 05' used for a similar problem) ;

Application to error in transmission

- Soliton profile allow to code bits
  $1 := \Psi_A^0(x) = \sqrt{2}A\mathrm{sech}(Ax)$
  $0 := 0$
- At the end of the line we measure:

$$\frac{1}{T}\int_{-T/2}^{T/2} |u(L,x)|^2 dx$$

and compare the number to a threshold.

- Due to noise error in transmission may occur
  A 1 is wrongly discarded and it is decided that a 0 has been emitted or the contrary may also happen
- Two processes are mainly responsible for these errors:
  1. A fluctuation of the arrival time $\mathbf{Y}(u(L,x)) = \int_{\mathbb{R}^d} x|u(L,x)|^2 dx$
  2. A fluctuation of the mass (with additive noise only)
- Error rate $\approx 10^{-12} \Rightarrow$ possibilities IS MC methods or genealogical particle systems (c.f. Del Moral & Garnier 05' used for a similar problem) ;

Application to error in transmission

- Soliton profile allow to code bits
  $1 := \Psi_A^0(x) = \sqrt{2}A\mathrm{sech}(Ax)$
  $0 := 0$
- At the end of the line we measure:

$$\frac{1}{T}\int_{-T/2}^{T/2} |u(L,x)|^2 dx$$

  and compare the number to a threshold.
- Due to noise error in transmission may occur
  A 1 is wrongly discarded and it is decided that a 0 has been emitted
  or the contrary may also happen
- Two processes are mainly responsible for these errors:
  1. A fluctuation of the arrival time
     $\mathbf{Y}(u(L,x)) = \int_{\mathbb{R}^d} x|u(L,x)|^2 dx$
  2. A fluctuation of the mass (with additive noise only)
- Error rate $\approx 10^{-12} \Rightarrow$ possibilities IS MC methods or genealogical particle systems (c.f. Del Moral & Garnier 05' used for a similar problem) ;

Application to error in transmission

- Soliton profile allow to code bits
  $1 := \Psi_A^0(x) = \sqrt{2}A\mathrm{sech}(Ax)$
  $0 := 0$
- At the end of the line we measure:

$$\frac{1}{T}\int_{-T/2}^{T/2} |u(L,x)|^2 dx$$

  and compare the number to a threshold.
- Due to noise error in transmission may occur
  A 1 is wrongly discarded and it is decided that a 0 has been emitted
  or the contrary may also happen
- Two processes are mainly responsible for these errors:
  1. A fluctuation of the arrival time
     $\mathbf{Y}(u(L,x)) = \int_{\mathbb{R}^d} x|u(L,x)|^2 dx$
  2. A fluctuation of the mass (with additive noise only)
- Error rate $\approx 10^{-12} \Rightarrow$ possibilities IS MC methods or genealogical particle systems (c.f. Del Moral & Garnier 05' used for a similar problem) ;

Application to error in transmission

- Soliton profile allow to code bits
  $1 := \Psi_A^0(x) = \sqrt{2}A\mathrm{sech}(Ax)$
  $0 := 0$
- At the end of the line we measure:
  $$\frac{1}{T}\int_{-T/2}^{T/2} |u(L,x)|^2 dx$$
  and compare the number to a threshold.
- Due to noise error in transmission may occur
  A 1 is wrongly discarded and it is decided that a 0 has been emitted
  or the contrary may also happen
- Two processes are mainly responsible for these errors:
  1. A fluctuation of the arrival time
     $\mathbf{Y}(u(L,x)) = \int_{\mathbb{R}^d} x|u(L,x)|^2 dx$
  2. A fluctuation of the mass (with additive noise only)
- Error rate $\approx 10^{-12} \Rightarrow$ possibilities IS MC methods or genealogical particle systems (c.f. Del Moral & Garnier 05' used for a similar problem) ;

Application to error in transmission

- Soliton profile allow to code bits
  $1 := \Psi_A^0(x) = \sqrt{2}A\mathrm{sech}(Ax)$
  $0 := 0$
- At the end of the line we measure:

$$\frac{1}{T}\int_{-T/2}^{T/2} |u(L,x)|^2 dx$$

  and compare the number to a threshold.
- Due to noise error in transmission may occur
  A 1 is wrongly discarded and it is decided that a 0 has been emitted or the contrary may also happen
- Two processes are mainly responsible for these errors:
  1. A fluctuation of the arrival time
     $\mathbf{Y}(u(L,x)) = \int_{\mathbb{R}^d} x|u(L,x)|^2 dx$
  2. A fluctuation of the mass (with additive noise only)
- Error rate $\approx 10^{-12} \Rightarrow$ possibilities IS MC methods or genealogical particle systems (c.f. Del Moral & Garnier 05' used for a similar problem) ;

Application to error in transmission

- Soliton profile allow to code bits
  $1 := \Psi_A^0(x) = \sqrt{2} A \mathrm{sech}(Ax)$
  $0 := 0$
- At the end of the line we measure:

$$\frac{1}{T}\int_{-T/2}^{T/2} |u(L,x)|^2 dx$$

  and compare the number to a threshold.
- Due to noise error in transmission may occur
  A 1 is wrongly discarded and it is decided that a 0 has been emitted
  or the contrary may also happen
- Two processes are mainly responsible for these errors:
  1. A fluctuation of the arrival time
     $\mathbf{Y}(u(L,x)) = \int_{\mathbb{R}^d} x|u(L,x)|^2 dx$
  2. A fluctuation of the mass (with additive noise only)
- Error rate $\approx 10^{-12} \Rightarrow$ possibilities IS MC methods or genealogical particle systems (c.f. Del Moral & Garnier 05' used for a similar problem) ;

Application to error in transmission

- Some results in the physics litterature:
  1. Variances (e.g. Gordon & Haus 86', Drummond & Corney 01') (that of the arrival time is $\propto L^3$ while that of the mass is $\propto L$) and deduce the limitation on the transmission rate via a Gaussian approximation ;
  2. Densities
     ▷ using the Martin-Siggia-Rose formalism (use ansatz) (Falkovich, Kolokov, Lebedev, Mezentsev & Turitsyn 04'),
     ▷ or the Fokker-Planck equations (use ansatz) (ex. Derevyanko, Turitsyn & Yakushev 03') ;
  3. CDFs numerically using IS and considering ansatz (Moore, Biondini, Kath 03')
- LD allows for theoretical predictions of the error rate ;
- Our Goal: Evaluate the tails of $\mathbf{N}\left(u^{\epsilon,u_0}(L)\right)$ and $\mathbf{Y}\left(u^{\epsilon,u_0}(L)\right)$ pushing forward the LDPs ;

Application to error in transmission

- Some results in the physics litterature:
  1. Variances (e.g. Gordon & Haus 86', Drummond & Corney 01') (that of the arrival time is $\propto L^3$ while that of the mass is $\propto L$) and deduce the limitation on the transmission rate via a Gaussian approximation ;
  2. Densities
     ▷ using the Martin-Siggia-Rose formalism (use ansatz) (Falkovich, Kolokov, Lebedev, Mezentsev & Turitsyn 04'),
     ▷ or the Fokker-Planck equations (use ansatz) (ex. Derevyanko, Turitsyn & Yakushev 03') ;
  3. CDFs numerically using IS and considering ansatz (Moore, Biondini, Kath 03')
- LD allows for theoretical predictions of the error rate ;
- Our Goal: Evaluate the tails of $\mathbf{N}\left(u^{\epsilon,u_0}(L)\right)$ and $\mathbf{Y}\left(u^{\epsilon,u_0}(L)\right)$ pushing forward the LDPs ;

Application to error in transmission

- Some results in the physics litterature:
  1. Variances (e.g. Gordon & Haus 86', Drummond & Corney 01') (that of the arrival time is $\propto L^3$ while that of the mass is $\propto L$) and deduce the limitation on the transmission rate via a Gaussian approximation ;
  2. Densities
     ▷ using the Martin-Siggia-Rose formalism (use ansatz) (Falkovich, Kolokov, Lebedev, Mezentsev & Turitsyn 04'),
     ▷ or the Fokker-Planck equations (use ansatz) (ex. Derevyanko, Turitsyn & Yakushev 03') ;
  3. CDFs numerically using IS and considering ansatz (Moore, Biondini, Kath 03')
- LD allows for theoretical predictions of the error rate ;
- Our Goal: Evaluate the tails of $\mathbf{N}\left(u^{\epsilon,u_0}(L)\right)$ and $\mathbf{Y}\left(u^{\epsilon,u_0}(L)\right)$ pushing forward the LDPs ;

Application to error in transmission

- Some results in the physics litterature:
  1. Variances (e.g. Gordon & Haus 86', Drummond & Corney 01') (that of the arrival time is $\propto L^3$ while that of the mass is $\propto L$) and deduce the limitation on the transmission rate via a Gaussian approximation ;
  2. Densities
     ▷ using the Martin-Siggia-Rose formalism (use ansatz) (Falkovich, Kolokov, Lebedev, Mezentsev & Turitsyn 04'),
     ▷ or the Fokker-Planck equations (use ansatz) (ex. Derevyanko, Turitsyn & Yakushev 03') ;
  3. CDFs numerically using IS and considering ansatz (Moore, Biondini, Kath 03')
- LD allows for theoretical predictions of the error rate ;
- Our Goal: Evaluate the tails of $\mathbf{N}\left(u^{\epsilon,u_0}(L)\right)$ and $\mathbf{Y}\left(u^{\epsilon,u_0}(L)\right)$ pushing forward the LDPs ;

Application to error in transmission

- Some results in the physics litterature:
  1. Variances (e.g. Gordon & Haus 86', Drummond & Corney 01') (that of the arrival time is $\propto L^3$ while that of the mass is $\propto L$) and deduce the limitation on the transmission rate via a Gaussian approximation ;
  2. Densities
     ▷ using the Martin-Siggia-Rose formalism (use ansatz) (Falkovich, Kolokov, Lebedev, Mezentsev & Turitsyn 04'),
     ▷ or the Fokker-Planck equations (use ansatz) (ex. Derevyanko, Turitsyn & Yakushev 03') ;
  3. CDFs numerically using IS and considering ansatz (Moore, Biondini, Kath 03')
- LD allows for theoretical predictions of the error rate ;
- Our Goal: Evaluate the tails of $\mathbf{N}\left(u^{\epsilon,u_0}(L)\right)$ and $\mathbf{Y}\left(u^{\epsilon,u_0}(L)\right)$ pushing forward the LDPs ;

Application to error in transmission

- Some results in the physics litterature:
  1. Variances (e.g. Gordon & Haus 86', Drummond & Corney 01') (that of the arrival time is $\propto L^3$ while that of the mass is $\propto L$) and deduce the limitation on the transmission rate via a Gaussian approximation ;
  2. Densities
     ▷ using the Martin-Siggia-Rose formalism (use ansatz) (Falkovich, Kolokov, Lebedev, Mezentsev & Turitsyn 04'),
     ▷ or the Fokker-Planck equations (use ansatz) (ex. Derevyanko, Turitsyn & Yakushev 03') ;
  3. CDFs numerically using IS and considering ansatz (Moore, Biondini, Kath 03')
- LD allows for theoretical predictions of the error rate ;
- Our Goal: Evaluate the tails of $\mathbf{N}\left(u^{\epsilon,u_0}(L)\right)$ and $\mathbf{Y}\left(u^{\epsilon,u_0}(L)\right)$ pushing forward the LDPs ;

Application to error in transmission

- Obtain upper and lower bounds of the log of the tails, e.g.

$$\log \mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,u_0}(L)\right) \geq R\right)$$

and try to obtain the same order in the parameters L,A,R ;

- The log of the tails is of the order of (e.g.)

$$-\frac{1}{2} \inf_{h \in \mathrm{L}^2(0,L;\mathrm{L}^2)} \left\{ \|h\|^2_{\mathrm{L}^2(0,L;\mathrm{L}^2)} : \mathbf{Y}\left(\mathbf{S}\left(u_0, h\right) \geq R\right) \right\}$$

- Obtain lower bounds by minimizing on a smaller set (parameterized h) $\Rightarrow$ Calculus of the Variations ;
- Obtain upper bounds using energy inequalities ;
- We do not want to use the approximation by an ansatz (this approximation only gives lower bounds). We want to compare with the results from physics ;
- We present results for the mass and arrival time and additive noise (see Debussche & EG *AAP* 07' where the case of multiplicative noise is also studied).

Application to error in transmission

- Obtain upper and lower bounds of the log of the tails, e.g.

$$\log \mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,u_0}(L)\right) \geq R\right)$$

  and try to obtain the same order in the parameters L,A,R ;

- The log of the tails is of the order of (e.g.)

$$-\frac{1}{2} \inf_{h\in \mathrm{L}^2(0,L;\mathrm{L}^2)} \left\{ \|h\|^2_{\mathrm{L}^2(0,L;\mathrm{L}^2)} : \mathbf{Y}\left(\mathbf{S}\left(u_0, h\right) \geq R\right) \right\}$$

- Obtain lower bounds by minimizing on a smaller set (parameterized h) ⇒ Calculus of the Variations ;
- Obtain upper bounds using energy inequalities ;
- We do not want to use the approximation by an ansatz (this approximation only gives lower bounds). We want to compare with the results from physics ;
- We present results for the mass and arrival time and additive noise (see Debussche & EG *AAP* 07' where the case of multiplicative noise is also studied).

Application to error in transmission

- Obtain upper and lower bounds of the log of the tails, e.g.

$$\log \mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,u_0}(L)\right) \geq R\right)$$

and try to obtain the same order in the parameters L,A,R ;

- The log of the tails is of the order of (e.g.)

$$-\frac{1}{2}\inf_{h\in \mathrm{L}^2(0,L;\mathrm{L}^2)}\left\{\|h\|^2_{\mathrm{L}^2(0,L;\mathrm{L}^2)} : \mathbf{Y}\left(\mathbf{S}\left(u_0,h\right) \geq R\right)\right\}$$

- Obtain lower bounds by minimizing on a smaller set (parameterized h) ⇒ Calculus of the Variations ;
- Obtain upper bounds using energy inequalities ;
- We do not want to use the approximation by an ansatz (this approximation only gives lower bounds). We want to compare with the results from physics ;
- We present results for the mass and arrival time and additive noise (see Debussche & EG *AAP* 07' where the case of multiplicative noise is also studied).

Application to error in transmission

- Obtain upper and lower bounds of the log of the tails, e.g.

$$\log \mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,u_0}(L)\right) \geq R\right)$$

and try to obtain the same order in the parameters L,A,R ;

- The log of the tails is of the order of (e.g.)

$$-\frac{1}{2}\inf_{h\in \mathrm{L}^2(0,L;\mathrm{L}^2)}\left\{\|h\|^2_{\mathrm{L}^2(0,L;\mathrm{L}^2)} : \mathbf{Y}\left(\mathbf{S}\left(u_0,h\right) \geq R\right)\right\}$$

- Obtain lower bounds by minimizing on a smaller set (parameterized h) ⇒ Calculus of the Variations ;
- Obtain upper bounds using energy inequalities ;
- We do not want to use the approximation by an ansatz (this approximation only gives lower bounds). We want to compare with the results from physics ;
- We present results for the mass and arrival time and additive noise (see Debussche & EG *AAP* 07' where the case of multiplicative noise is also studied).

Application to error in transmission

- Obtain upper and lower bounds of the log of the tails, e.g.

$$\log \mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,u_0}(L)\right) \geq R\right)$$

and try to obtain the same order in the parameters L,A,R ;

- The log of the tails is of the order of (e.g.)

$$-\frac{1}{2}\inf_{h\in \mathrm{L}^2(0,L;\mathrm{L}^2)}\left\{\|h\|^2_{\mathrm{L}^2(0,L;\mathrm{L}^2)} : \mathbf{Y}\left(\mathbf{S}\left(u_0,h\right) \geq R\right)\right\}$$

- Obtain lower bounds by minimizing on a smaller set (parameterized h) $\Rightarrow$ Calculus of the Variations ;
- Obtain upper bounds using energy inequalities ;
- We do not want to use the approximation by an ansatz (this approximation only gives lower bounds). We want to compare with the results from physics ;
- We present results for the mass and arrival time and additive noise (see Debussche & EG *AAP* 07' where the case of multiplicative noise is also studied).

Application to error in transmission

- Obtain upper and lower bounds of the log of the tails, e.g.

$$\log \mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,u_0}(L)\right) \geq R\right)$$

and try to obtain the same order in the parameters L,A,R ;

- The log of the tails is of the order of (e.g.)

$$-\frac{1}{2}\inf_{h\in \mathrm{L}^2(0,L;\mathrm{L}^2)}\left\{\|h\|^2_{\mathrm{L}^2(0,L;\mathrm{L}^2)} : \mathbf{Y}\left(\mathbf{S}\left(u_0,h\right) \geq R\right)\right\}$$

- Obtain lower bounds by minimizing on a smaller set (parameterized h) $\Rightarrow$ Calculus of the Variations ;
- Obtain upper bounds using energy inequalities ;
- We do not want to use the approximation by an ansatz (this approximation only gives lower bounds). We want to compare with the results from physics ;
- We present results for the mass and arrival time and additive noise (see Debussche & EG *AAP* 07' where the case of multiplicative noise is also studied).

Tails of the mass, lower bounds

For $D \subset \mathbb{R}_+^*$ set : $\mathcal{A}_D^1 = \left\{ A : [0, L] \to \mathbb{R},\ \exists \tilde{R} \in D : A(t) = \tilde{R}\left(\frac{t}{2L}\right)^2 \right\}$

$$\mathcal{A}_D^2 = \left\{ A : [0, L] \to \mathbb{R},\ \exists \tilde{R} \in D : A(t) = \left(8 - \tilde{R} - 4\sqrt{4-\tilde{R}}\right)\left(\frac{t}{2L}\right)^2 + \left(-4 + 2\sqrt{4-\tilde{R}}\right)\frac{t}{2L} + 1 \right\}.$$

$$\mathcal{C}_D^i = \left\{ h \in \mathrm{L}^2(0, T; \mathrm{L}^2),\ \exists A \in \mathcal{A}_D^i : h(t,x) = i\frac{A'(t)}{A(t)}\Psi_A(t,x) - i\sqrt{2}A'(t)\exp\left(-i\int_0^t A^2(s)ds\right)A(t)x\frac{\sinh}{\cosh^2}(A(t)x)\right\}, \quad i = 1, 2$$

Proposition

For $L, R > 0$ ($R \in (0,4)$ in (2)), D dense in $[R, R+1]$ and $(\Phi_n)_{n\in\mathbb{N}}$ H.S. with values in L^2 s.t. $\forall h \in \mathcal{C}_D^1$, $\Phi_n h \to h$ in $\mathrm{L}^1(0, L; \mathrm{L}^2)$, then

$$\underline{\lim}_{n\to\infty,\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathbf{N}\left(u^{\epsilon,0,n}(L)\right) \geq R\right) \geq -\frac{R(12+\pi^2)}{18L} \quad (1)$$

Replacing $\mathcal{C}_D^1$ by $\mathcal{C}_D^2$ we get

$$\underline{\lim}_{n\to\infty,\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathbf{N}\left(u^{\epsilon,\Psi_1^0,n}(L)\right) - 4 < -R\right) \geq -\frac{(2-\sqrt{4-R})^2(12+\pi^2)}{9L} (2)$$

n recalls that Φ is replaced by Φ_n.

Tails of the mass, lower bounds

For $D \subset \mathbb{R}_+^*$ set : $\mathcal{A}_D^1 = \left\{ A : [0, L] \to \mathbb{R},\ \exists \tilde{R} \in D : A(t) = \tilde{R}\left(\frac{t}{2L}\right)^2 \right\}$

$$\mathcal{A}_D^2 = \left\{ A : [0, L] \to \mathbb{R},\ \exists \tilde{R} \in D : A(t) = \left(8 - \tilde{R} - 4\sqrt{4-\tilde{R}}\right)\left(\frac{t}{2L}\right)^2 + \left(-4 + 2\sqrt{4-\tilde{R}}\right)\frac{t}{2L} + 1 \right\}.$$

$$\mathcal{C}_D^i = \left\{ h \in \mathrm{L}^2(0, T; \mathrm{L}^2),\ \exists A \in \mathcal{A}_D^i : h(t,x) = i\frac{A'(t)}{A(t)}\Psi_A(t,x) - i\sqrt{2}A'(t)\exp\left(-i\int_0^t A^2(s)ds\right) A(t)x\frac{\sinh}{\cosh^2}(A(t)x) \right\}, \quad i = 1, 2$$

Proposition

For $L, R > 0$ ($R \in (0,4)$ in (2)), D dense in $[R, R+1]$ and $(\Phi_n)_{n\in\mathbb{N}}$ H.S. with values in L^2 s.t. $\forall h \in \mathcal{C}_D^1$, $\Phi_n h \to h$ in $\mathrm{L}^1\left(0, L; \mathrm{L}^2\right)$, then

$$\underline{\lim}_{n\to\infty,\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathbf{N}\left(u^{\epsilon,0,n}(L)\right) \geq R\right) \geq -\frac{R(12+\pi^2)}{18L} \quad (1)$$

Replacing $\mathcal{C}_D^1$ by $\mathcal{C}_D^2$ we get

$$\underline{\lim}_{n\to\infty,\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathbf{N}\left(u^{\epsilon,\Psi_1^0,n}(L)\right) - 4 < -R\right) \geq -\frac{(2-\sqrt{4-R})^2(12+\pi^2)}{9L} (2)$$

n recalls that Φ is replaced by Φ_n.

Idea of the proof, case where initial datum is 0

- Look for solutions of the form

$$\mathbf{S}^0_{WN}(t,x) = \sqrt{2}A(t)\exp\left(-i\int_0^t A^2(s)ds\right)\mathrm{sech}(A(t)x)$$

- Consider first the case of a controlled equation where $\Phi = I$
- The optimal control problem becomes a problem of the calculus of variations where we have to find an optimal A

$$\inf_{h\in \mathrm{L}^2(0,L;\mathrm{L}^2):\ \mathbf{N}\left(\mathbf{S}^0_{WN}(h)(L)\right)\geq R} \frac{\|h\|^2_{\mathrm{L}^2(0,L;\mathrm{L}^2)}}{2} \leq \inf_{A\in\ \mathrm{C}^1([0,T]),b.c.} \int_0^L \frac{(12+\pi^2)}{18}\frac{(A'(t))^2}{A(t)}dt,$$

- Boundary conditions are derived from the constraints on the mass: $A_0(0)=0$ and $4A_0(L)=\tilde{R}>R$
- The singular Euler-Lagrange equation

$$2\frac{A''}{A} = \left(\frac{A'}{A}\right)^2$$

allows to make the guesses corresponding to $\mathcal{A}^1_D$.

- We use the fact that Φ_n approximate I and $\tilde{R}$ arbitrary close to R.

Idea of the proof, case where initial datum is 0

- Look for solutions of the form

$$\mathbf{S}^0_{WN}(t,x) = \sqrt{2}A(t)\exp\left(-i\int_0^t A^2(s)ds\right)\operatorname{sech}(A(t)x)$$

- Consider first the case of a controlled equation where $\Phi = I$
- The optimal control problem becomes a problem of the calculus of variations where we have to find an optimal A

$$\inf_{h\in L^2(0,L;L^2):\ \mathbf{N}\left(\mathbf{S}^0_{WN}(h)(L)\right)\geq R} \frac{\|h\|^2_{L^2(0,L;L^2)}}{2} \leq \inf_{A\in\ C^1([0,T]),b.c.} \int_0^L \frac{(12+\pi^2)}{18}\frac{(A'(t))^2}{A(t)}dt,$$

- Boundary conditions are derived from the constraints on the mass: $A_0(0)=0$ and $4A_0(L)=\tilde{R}>R$
- The singular Euler-Lagrange equation

$$2\frac{A''}{A} = \left(\frac{A'}{A}\right)^2$$

allows to make the guesses corresponding to $\mathcal{A}^1_D$.

- We use the fact that Φ_n approximate I and $\tilde{R}$ arbitrary close to R.

Idea of the proof, case where initial datum is 0

- Look for solutions of the form

$$\mathbf{S}^0_{WN}(t,x)=\sqrt{2}A(t)\exp\left(-i\int_0^t A^2(s)ds\right)\mathrm{sech}(A(t)x)$$

- Consider first the case of a controlled equation where $\Phi = I$
- The optimal control problem becomes a problem of the calculus of variations where we have to find an optimal A

$$\inf_{h\in \mathrm{L}^2(0,L;\mathrm{L}^2):\ \mathbf{N}\left(\mathbf{S}^0_{WN}(h)(L)\right)\geq R}\frac{\|h\|^2_{\mathrm{L}^2(0,L;\mathrm{L}^2)}}{2}\leq \inf_{A\in\ \mathrm{C}^1([0,T]),b.c.}\int_0^L\frac{(12+\pi^2)}{18}\frac{(A'(t))^2}{A(t)}dt,$$

- Boundary conditions are derived from the constraints on the mass: $A_0(0)=0$ and $4A_0(L)=\tilde{R}>R$
- The singular Euler-Lagrange equation

$$2\frac{A''}{A}=\left(\frac{A'}{A}\right)^2$$

allows to make the guesses corresponding to $\mathcal{A}^1_D$.

- We use the fact that Φ_n approximate I and $\tilde{R}$ arbitrary close to R.

Idea of the proof, case where initial datum is 0

- Look for solutions of the form

$$\mathbf{S}^0_{WN}(t,x) = \sqrt{2}A(t)\exp\left(-i\int_0^t A^2(s)ds\right)\operatorname{sech}(A(t)x)$$

- Consider first the case of a controlled equation where $\Phi = I$
- The optimal control problem becomes a problem of the calculus of variations where we have to find an optimal A

$$\inf_{h\in \mathrm{L}^2(0,L;\mathrm{L}^2):\ \mathbf{N}\left(\mathbf{S}^0_{WN}(h)(L)\right)\geq R} \frac{\|h\|^2_{\mathrm{L}^2(0,L;\mathrm{L}^2)}}{2} \leq \inf_{A\in\ \mathrm{C}^1([0,T]),b.c.} \int_0^L \frac{(12+\pi^2)}{18}\frac{(A'(t))^2}{A(t)}dt,$$

- Boundary conditions are derived from the constraints on the mass: $A_0(0) = 0$ and $4A_0(L) = \tilde{R} > R$
- The singular Euler-Lagrange equation

$$2\frac{A''}{A} = \left(\frac{A'}{A}\right)^2$$

allows to make the guesses corresponding to $\mathcal{A}^1_D$.

- We use the fact that Φ_n approximate I and $\tilde{R}$ arbitrary close to R.

Idea of the proof, case where initial datum is 0

- Look for solutions of the form

$$\mathbf{S}^0_{WN}(t,x)=\sqrt{2}A(t)\exp\left(-i\int_0^t A^2(s)ds\right)\operatorname{sech}(A(t)x)$$

- Consider first the case of a controlled equation where $\Phi = I$
- The optimal control problem becomes a problem of the calculus of variations where we have to find an optimal A

$$\inf_{h\in \mathrm{L}^2(0,L;\mathrm{L}^2):\ \mathbf{N}\left(\mathbf{S}^0_{WN}(h)(L)\right)\geq R}\frac{\|h\|^2_{\mathrm{L}^2(0,L;\mathrm{L}^2)}}{2}\leq \inf_{A\in\ \mathrm{C}^1([0,T]),b.c.}\int_0^L\frac{(12+\pi^2)}{18}\frac{(A'(t))^2}{A(t)}dt,$$

- Boundary conditions are derived from the constraints on the mass: $A_0(0)=0$ and $4A_0(L)=\tilde{R}>R$
- The singular Euler-Lagrange equation

$$2\frac{A''}{A}=\left(\frac{A'}{A}\right)^2$$

allows to make the guesses corresponding to $\mathcal{A}^1_D$.

- We use the fact that Φ_n approximate I and $\tilde{R}$ arbitrary close to R.

Idea of the proof, case where initial datum is 0

- Look for solutions of the form

$$\mathbf{S}^0_{WN}(t,x)=\sqrt{2}A(t)\exp\left(-i\int_0^t A^2(s)ds\right)\mathrm{sech}(A(t)x)$$

- Consider first the case of a controlled equation where $\Phi = I$
- The optimal control problem becomes a problem of the calculus of variations where we have to find an optimal A

$$\inf_{h\in \mathrm{L}^2(0,L;\mathrm{L}^2):\ \mathbf{N}\left(\mathbf{S}^0_{WN}(h)(L)\right)\geq R}\frac{\|h\|^2_{\mathrm{L}^2(0,L;\mathrm{L}^2)}}{2}\leq \inf_{A\in\ \mathrm{C}^1([0,T]),b.c.}\int_0^L\frac{(12+\pi^2)}{18}\frac{(A'(t))^2}{A(t)}dt,$$

- Boundary conditions are derived from the constraints on the mass: $A_0(0)=0$ and $4A_0(L)=\tilde{R}>R$
- The singular Euler-Lagrange equation

$$2\frac{A''}{A}=\left(\frac{A'}{A}\right)^2$$

allows to make the guesses corresponding to $\mathcal{A}^1_D$.

- We use the fact that Φ_n approximate I and $\tilde{R}$ arbitrary close to R.

Tails of the mass, upper bounds

Proposition

$\forall L, R > 0$ ($R \in (0,4)$ in (2)), D dense in $[R, R+1]$ and $(\Phi_n)_{n\in\mathbb{N}}$ H.S. with values in L^2 s.t. $\forall h \in \mathcal{C}_D^1$, $\Phi_n h \to h$ in $\mathrm{L}^1\left(0, L; \mathrm{L}^2\right)$ (assumption for the lower bounds) and uniformly bounded as operators on L^2 by $C > 1$ independent of L,

$$\overline{\lim}_{n\to\infty,\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathbf{N}\left(u^{\epsilon,0,n}(L)\right) \geq R\right) \leq -\frac{R}{8LC^2} \quad (1),$$

Replacing $\mathcal{C}_D^1$ by $\mathcal{C}_D^2$ we obtain

$$\overline{\lim}_{n\to\infty,\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathbf{N}\left(u^{\epsilon,\Psi_1^0,n}(L)\right) - 4 < -R\right) \leq -\frac{R^2}{8LC^2(4+R)} \quad (2).$$

Idea of the proof, case 0 as initial datum

- Manipulations of the controlled equation

$$i\frac{\partial u}{\partial t} - \Delta u - \lambda |u|^{2\sigma} u = \Phi_n h,$$

allow to obtain

$$\|\mathbf{S}^0(h)(t)\|^2_{\mathrm{L}^2} - \|u_0\|^2_{\mathrm{L}^2} = 2\mathfrak{Re}\left(-i\int_0^t \int_{\mathbb{R}} \left((\Phi_n h)(s,x)\overline{\mathbf{S}^{a,0}(h)(s,x)}\right) dxds\right).$$

and after some computations

$$R \leq \|\mathbf{S}^{a,0}(h)(L)\|^2_{\mathrm{L}^2} \leq 4T\|\Phi_n\|^2_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{L}^2)} \int_0^L \|h(s)\|^2_{\mathrm{L}^2} ds.$$

- It allows to obtain lower bounds on the L^2 norm of any control allowing to get in the "large deviation set"
- With the - sign we obtain an upper bound which depends on the operator norm of Φ_n
- These operator norms are assumed to be uniformly bounded.

Idea of the proof, case 0 as initial datum

- Manipulations of the controlled equation

$$i\frac{\partial u}{\partial t} - \Delta u - \lambda |u|^{2\sigma} u = \Phi_n h,$$

allow to obtain

$$\|\mathbf{S}^0(h)(t)\|_{\mathrm{L}^2}^2 - \|u_0\|_{\mathrm{L}^2}^2 = 2\mathfrak{Re}\left(-i\int_0^t \int_{\mathbb{R}} \left((\Phi_n h)(s,x)\overline{\mathbf{S}^{a,0}(h)(s,x)}\right) dxds\right).$$

and after some computations

$$R \leq \|\mathbf{S}^{a,0}(h)(L)\|_{\mathrm{L}^2}^2 \leq 4T\|\Phi_n\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{L}^2)}^2 \int_0^L \|h(s)\|_{\mathrm{L}^2}^2 ds.$$

- It allows to obtain lower bounds on the L^2 norm of any control allowing to get in the "large deviation set"
- With the - sign we obtain an upper bound which depends on the operator norm of Φ_n
- These operator norms are assumed to be uniformly bounded.

Idea of the proof, case 0 as initial datum

- Manipulations of the controlled equation

$$i\frac{\partial u}{\partial t} - \Delta u - \lambda |u|^{2\sigma} u = \Phi_n h,$$

allow to obtain

$$\|\mathbf{S}^0(h)(t)\|_{\mathrm{L}^2}^2 - \|u_0\|_{\mathrm{L}^2}^2 = 2\mathfrak{Re}\left(-i\int_0^t \int_{\mathbb{R}} \left((\Phi_n h)(s,x)\overline{\mathbf{S}^{a,0}(h)(s,x)}\right) dxds\right).$$

and after some computations

$$R \leq \|\mathbf{S}^{a,0}(h)(L)\|_{\mathrm{L}^2}^2 \leq 4T\|\Phi_n\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{L}^2)}^2 \int_0^L \|h(s)\|_{\mathrm{L}^2}^2 ds.$$

- It allows to obtain lower bounds on the L^2 norm of any control allowing to get in the "large deviation set"
- With the - sign we obtain an upper bound which depends on the operator norm of Φ_n
- These operator norms are assumed to be uniformly bounded.

Idea of the proof, case 0 as initial datum

- Manipulations of the controlled equation

$$i\frac{\partial u}{\partial t} - \Delta u - \lambda |u|^{2\sigma} u = \Phi_n h,$$

allow to obtain

$$\|\mathbf{S}^0(h)(t)\|_{\mathrm{L}^2}^2 - \|u_0\|_{\mathrm{L}^2}^2 = 2\mathfrak{Re}\left(-i\int_0^t \int_{\mathbb{R}} \left((\Phi_n h)(s,x)\overline{\mathbf{S}^{a,0}(h)(s,x)}\right) dxds\right).$$

and after some computations

$$R \leq \|\mathbf{S}^{a,0}(h)(L)\|_{\mathrm{L}^2}^2 \leq 4T\|\Phi_n\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{L}^2)}^2 \int_0^L \|h(s)\|_{\mathrm{L}^2}^2 ds.$$

- It allows to obtain lower bounds on the L^2 norm of any control allowing to get in the "large deviation set"
- With the - sign we obtain an upper bound which depends on the operator norm of Φ_n
- These operator norms are assumed to be uniformly bounded.

Tails of the arrival time (w/ A. Debussche)

- To be able to consider the arrival time we work in space of localized functions. We introduce

$$\Sigma^{\frac{1}{2}} = \left\{ f \in \mathrm{H}^1 : x \mapsto \sqrt{|x|} f(x) \in \mathrm{L}^2 \right\}, \Sigma = \left\{ f \in \mathrm{H}^1 : \ x \mapsto x f(x) \in \mathrm{L}^2 \right\},$$

$$\|f\|^2_{\Sigma^{\frac{1}{2}}} = \|f\|^2_{\mathrm{H}^1} + \left\| x \mapsto \sqrt{|x|} f(x) \right\|^2_{\mathrm{L}^2}, \|f\|^2_{\Sigma} = \|f\|^2_{\mathrm{H}^1} + \|x \mapsto x f(x)\|^2_{\mathrm{L}^2}.$$

- We prove sample paths LDPs for paths in $\Sigma^{\frac{1}{2}}$

Theorem

If Φ is H.S. in Σ (additive case) or in $\mathrm{H}^s(\mathbb{R}, \mathbb{R})$ with $s > 3/2$ (multiplicative case) and $u_0 \in \Sigma$, then solutions define r.v. in $\mathrm{C}([0, L]; \Sigma^{\frac{1}{2}})$ and their laws satisfy a LDP of speed ϵ and good rate function
$I^L_{u_0}(w) = \frac{1}{2} \inf_{h \in \mathrm{L}^2(0,L;\mathrm{L}^2):\ w = \mathbf{S}(u_0,h)} \|h\|^2_{\mathrm{L}^2(0,L;\mathrm{L}^2)}$.

- uniformly bounded operators with values in Σ incompatible with convergence of $\Phi_n h$ to h in $\mathrm{L}^1(0, L, \Sigma)$. In the limit we assume that in the limit we have a colored noise say defined through $\Phi = (I - \Delta + |x|^2 I)^{-1/2}$.

Tails of the arrival time (w/ A. Debussche)

- To be able to consider the arrival time we work in space of localized functions. We introduce
$$\Sigma^{\frac{1}{2}} = \left\{ f \in \mathrm{H}^1 : x \mapsto \sqrt{|x|} f(x) \in \mathrm{L}^2 \right\}, \Sigma = \left\{ f \in \mathrm{H}^1 : \ x \mapsto x f(x) \in \mathrm{L}^2 \right\},$$
$$\|f\|^2_{\Sigma^{\frac{1}{2}}} = \|f\|^2_{\mathrm{H}^1} + \left\| x \mapsto \sqrt{|x|} f(x) \right\|^2_{\mathrm{L}^2}, \|f\|^2_{\Sigma} = \|f\|^2_{\mathrm{H}^1} + \|x \mapsto x f(x)\|^2_{\mathrm{L}^2}.$$
- We prove sample paths LDPs for paths in $\Sigma^{\frac{1}{2}}$

Theorem

If Φ is H.S. in Σ (additive case) or in $\mathrm{H}^s(\mathbb{R}, \mathbb{R})$ with $s > 3/2$ (multiplicative case) and $u_0 \in \Sigma$, then solutions define r.v. in $\mathrm{C}([0, L]; \Sigma^{\frac{1}{2}})$ and their laws satisfy a LDP of speed ϵ and good rate function
$I^L_{u_0}(w) = \frac{1}{2} \inf_{h \in \mathrm{L}^2(0,L;\mathrm{L}^2):\ w = \mathbf{S}(u_0, h)} \|h\|^2_{\mathrm{L}^2(0,L;\mathrm{L}^2)}$.

- uniformly bounded operators with values in Σ incompatible with convergence of $\Phi_n h$ to h in $\mathrm{L}^1(0, L, \Sigma)$. In the limit we assume that in the limit we have a colored noise say defined through $\Phi = (I - \Delta + |x|^2 I)^{-1/2}$.

Tails of the arrival time (w/ A. Debussche)

- To be able to consider the arrival time we work in space of localized functions. We introduce
$$\Sigma^{\frac{1}{2}} = \left\{ f \in \mathrm{H}^1 : x \mapsto \sqrt{|x|} f(x) \in \mathrm{L}^2 \right\}, \Sigma = \left\{ f \in \mathrm{H}^1 : \ x \mapsto x f(x) \in \mathrm{L}^2 \right\},$$
$$\|f\|^2_{\Sigma^{\frac{1}{2}}} = \|f\|^2_{\mathrm{H}^1} + \left\| x \mapsto \sqrt{|x|} f(x) \right\|^2_{\mathrm{L}^2}, \|f\|^2_{\Sigma} = \|f\|^2_{\mathrm{H}^1} + \|x \mapsto x f(x)\|^2_{\mathrm{L}^2}.$$
- We prove sample paths LDPs for paths in $\Sigma^{\frac{1}{2}}$

Theorem

If Φ is H.S. in Σ (additive case) or in $\mathrm{H}^s(\mathbb{R}, \mathbb{R})$ with $s > 3/2$ (multiplicative case) and $u_0 \in \Sigma$, then solutions define r.v. in $\mathrm{C}([0, L]; \Sigma^{\frac{1}{2}})$ and their laws satisfy a LDP of speed ϵ and good rate function
$I^L_{u_0}(w) = \frac{1}{2} \inf_{h \in \mathrm{L}^2(0,L;\mathrm{L}^2): \ w = \mathbf{S}(u_0,h)} \|h\|^2_{\mathrm{L}^2(0,L;\mathrm{L}^2)}$.

- uniformly bounded operators with values in Σ incompatible with convergence of $\Phi_n h$ to h in $\mathrm{L}^1(0, L, \Sigma)$. In the limit we assume that in the limit we have a colored noise say defined through $\Phi = (I - \Delta + |x|^2 I)^{-1/2}$.

Tails of the arrival time (w/ A. Debussche)

- To be able to consider the arrival time we work in space of localized functions. We introduce
$$\Sigma^{\frac{1}{2}} = \left\{ f \in \mathrm{H}^1 : x \mapsto \sqrt{|x|} f(x) \in \mathrm{L}^2 \right\}, \Sigma = \left\{ f \in \mathrm{H}^1 : \ x \mapsto x f(x) \in \mathrm{L}^2 \right\},$$
$$\|f\|^2_{\Sigma^{\frac{1}{2}}} = \|f\|^2_{\mathrm{H}^1} + \left\| x \mapsto \sqrt{|x|} f(x) \right\|^2_{\mathrm{L}^2}, \|f\|^2_{\Sigma} = \|f\|^2_{\mathrm{H}^1} + \|x \mapsto x f(x)\|^2_{\mathrm{L}^2}.$$
- We prove sample paths LDPs for paths in $\Sigma^{\frac{1}{2}}$

Theorem

If Φ is H.S. in Σ (additive case) or in $\mathrm{H}^s(\mathbb{R},\mathbb{R})$ with $s > 3/2$ (multiplicative case) and $u_0 \in \Sigma$, then solutions define r.v. in $\mathrm{C}([0,L];\Sigma^{\frac{1}{2}})$ and their laws satisfy a LDP of speed ϵ and good rate function
$I^L_{u_0}(w) = \frac{1}{2} \inf_{h \in \mathrm{L}^2(0,L;\mathrm{L}^2):\ w = \mathbf{S}(u_0,h)} \|h\|^2_{\mathrm{L}^2(0,L;\mathrm{L}^2)}$.

- uniformly bounded operators with values in Σ incompatible with convergence of $\Phi_n h$ to h in $\mathrm{L}^1(0,L,\Sigma)$. In the limit we assume that in the limit we have a colored noise say defined through $\Phi = (I - \Delta + |x|^2 I)^{-1/2}$.

Tails of the arrival time, lower bounds

For $A, L > 0$ and $D \subset \mathbb{R}_+^*$, we define the set of controls

$$\mathcal{H}_{A,L}^D = \quad \left\{h \in \mathrm{L}^2(0,L;\mathrm{L}^2),\ h(t,x) = \lambda(t)\left(x - 2\int_0^t \int_0^s \lambda(\sigma)d\sigma ds\right)\Psi_{A,\lambda}(t,x), \right.$$
$$\left. \text{with } \lambda(t) = \frac{3\tilde{R}(L-t)}{8AL^3},\ \tilde{R} \in D\right\},$$

$$\Psi_{A,\lambda}(t,x) = \sqrt{2}A\mathrm{sech}\left(A\left(x - 2\int_0^t \int_0^s \lambda(\sigma)d\sigma ds\right)\right)\exp\left(2i\int_0^t \lambda(s)\int_0^s\int_0^\sigma \lambda(\tau)d\tau d\sigma ds\right)$$
$$\exp\left[-iA^2t + i\int_0^t\left(\int_0^s \lambda(\sigma)d\sigma\right)^2 ds - ix\int_0^t \lambda(s)ds + 2i\left(\int_0^t \lambda(s)ds\right)\left(\int_0^t\int_0^s \lambda(\sigma)d\sigma ds\right)\right].$$

Proposition

For $L, A, R > 0$ and D dense in $[R, R+1]$, $(\Phi_n)_{n\in\mathbb{N}}$ H.S. in Σ s.t. $\forall h \in \mathcal{H}_{L,A}^D$, $\Phi_n h \to \Phi h$ in $\mathrm{L}^1(0,L;\Sigma)$, then

$$\underline{\lim}_{n\to\infty,\epsilon\to 0}\,\epsilon\log\mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,\Psi_A^0,n}(L)\right) \geq R\right) \geq -\frac{CR^2}{128L^3A}.$$

Tails of the arrival time, lower bounds

For $A, L > 0$ and $D \subset \mathbb{R}_+^*$, we define the set of controls

$$\mathcal{H}_{A,L}^D = \Big\{h \in \mathrm{L}^2(0,L;\mathrm{L}^2),\ h(t,x) = \lambda(t)\left(x - 2\int_0^t \int_0^s \lambda(\sigma)d\sigma ds\right)\Psi_{A,\lambda}(t,x), \text{ with } \lambda(t) = \frac{3\tilde{R}(L-t)}{8AL^3},\ \tilde{R} \in D\Big\},$$

$$\Psi_{A,\lambda}(t,x) = \sqrt{2}A\,\mathrm{sech}\left(A\left(x - 2\int_0^t \int_0^s \lambda(\sigma)d\sigma ds\right)\right)\exp\left(2i\int_0^t \lambda(s)\int_0^s \int_0^\sigma \lambda(\tau)d\tau d\sigma ds\right)$$
$$\exp\left[-iA^2 t + i\int_0^t \left(\int_0^s \lambda(\sigma)d\sigma\right)^2 ds - ix\int_0^t \lambda(s)ds + 2i\left(\int_0^t \lambda(s)ds\right)\left(\int_0^t \int_0^s \lambda(\sigma)d\sigma ds\right)\right].$$

Proposition

For $L, A, R > 0$ and D dense in $[R, R+1]$, $(\Phi_n)_{n\in\mathbb{N}}$ H.S. in Σ s.t. $\forall h \in \mathcal{H}_{L,A}^D$, $\Phi_n h \to \Phi h$ in $\mathrm{L}^1(0,L;\Sigma)$, then

$$\underline{\lim}_{n\to\infty,\epsilon\to 0}\,\epsilon \log \mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,\Psi_A^0,n}(L)\right) \geq R\right) \geq -\frac{CR^2}{128L^3A}.$$

Idea of the proof of the lower bound I

- We seek controls such that

$$i\frac{\mathrm{d}u}{\mathrm{d}t} = \Delta u + |u|^2 u + \lambda(t) x u$$

- $v_1(t) = \exp\left(i\left(\int_0^t \lambda(s)ds\right)x\right)u(t)$ is solution of

$$i\frac{\partial v_1}{\partial t} = \frac{\partial^2 v_1}{\partial x^2} + |v_1|^2 v_1 - \left(\int_0^t \lambda(s)ds\right)^2 v_1 - 2i\left(\int_0^t \lambda(s)ds\right)\frac{\partial v_1}{\partial x}$$

and $v_2(t) = \exp\left(-i\int_0^t \left(\int_0^s \lambda(\tau)d\tau\right)^2 ds\right) v_1(t)$ (gauge transform) satisfies

$$i\left(\frac{\partial v_2}{\partial t} + 2\left(\int_0^t \lambda(s)ds\right)\frac{\partial v_2}{\partial x}\right) = \frac{\partial^2 v_2}{\partial x^2} + |v_2|^2 v_2$$

- Using the methods of characteristics

$$v_3(t,x) = v_2\left(t, x + 2\int_0^t \int_0^s \lambda(u)duds\right)$$

is a solution of the usual NLS equation with initial datum Ψ_A^0.

Idea of the proof of the lower bound I

- We seek controls such that

$$i\frac{\mathrm{d}u}{\mathrm{d}t} = \Delta u + |u|^2 u + \lambda(t) x u$$

- $v_1(t) = \exp\left(i\left(\int_0^t \lambda(s)ds\right)x\right)u(t)$ is solution of

$$i\frac{\partial v_1}{\partial t} = \frac{\partial^2 v_1}{\partial x^2} + |v_1|^2 v_1 - \left(\int_0^t \lambda(s)ds\right)^2 v_1 - 2i\left(\int_0^t \lambda(s)ds\right)\frac{\partial v_1}{\partial x}$$

and $v_2(t) = \exp\left(-i\int_0^t\left(\int_0^s \lambda(\tau)d\tau\right)^2 ds\right)v_1(t)$ (gauge transform) satisfies

$$i\left(\frac{\partial v_2}{\partial t} + 2\left(\int_0^t \lambda(s)ds\right)\frac{\partial v_2}{\partial x}\right) = \frac{\partial^2 v_2}{\partial x^2} + |v_2|^2 v_2$$

- Using the methods of characteristics

$$v_3(t,x) = v_2\left(t, x + 2\int_0^t\int_0^s \lambda(u)duds\right)$$

is a solution of the usual NLS equation with initial datum Ψ_A^0.

Idea of the proof of the lower bound I

- We seek controls such that

$$i\frac{\mathrm{d}u}{\mathrm{d}t} = \Delta u + |u|^2 u + \lambda(t)xu$$

- $v_1(t) = \exp\left(i\left(\int_0^t \lambda(s)ds\right)x\right)u(t)$ is solution of

$$i\frac{\partial v_1}{\partial t} = \frac{\partial^2 v_1}{\partial x^2} + |v_1|^2 v_1 - \left(\int_0^t \lambda(s)ds\right)^2 v_1 - 2i\left(\int_0^t \lambda(s)ds\right)\frac{\partial v_1}{\partial x}$$

and $v_2(t) = \exp\left(-i\int_0^t\left(\int_0^s \lambda(\tau)d\tau\right)^2 ds\right)v_1(t)$ (gauge transform) satisfies

$$i\left(\frac{\partial v_2}{\partial t} + 2\left(\int_0^t \lambda(s)ds\right)\frac{\partial v_2}{\partial x}\right) = \frac{\partial^2 v_2}{\partial x^2} + |v_2|^2 v_2$$

- Using the methods of characteristics

$$v_3(t,x) = v_2\left(t, x + 2\int_0^t\int_0^s \lambda(u)duds\right)$$

is a solution of the usual NLS equation with initial datum Ψ_A^0.

Idea of the proof of the lower bound II

- It is even more convenient to work with

$$i\frac{\mathrm{d}v}{\mathrm{d}t} = \Delta v + |v|^2 v + \lambda(t)\left(x - 2\int_0^t \int_0^s \lambda(\tau)d\tau ds\right) v$$

$\Rightarrow$ We use yet another Gauge transform.

- Then

$$\inf_{h\in \mathrm{L}^2(0,T;\mathrm{L}^2):\ \mathbf{Y}\left(\mathbf{S}^{\Psi^0}_{\substack{A\\WN}}(h)(T)\right)\geq \tilde{R}} \frac{\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}}{2} \leq \inf_{\lambda\in \mathrm{L}^2(0,T;\mathbb{R}),\int_0^T\int_0^t \lambda(s)dsdt\geq \frac{\tilde{R}}{8A}} \left(\frac{\pi^2}{6A}\right)\int_0^T \lambda^2(t)dt$$

- We finally pass to the limit with the sequence of operators and for large A obtain a term of the order of the square of the norm of the gradient and use that $\tilde{R}$ is arbitrary and such that $\tilde{R} > R$
- It is precisely the boundary condition in integral form that gives the factor $1/T^3$.

Idea of the proof of the lower bound II

- It is even more convenient to work with

$$i\frac{\mathrm{d}v}{\mathrm{d}t} = \Delta v + |v|^2 v + \lambda(t)\left(x - 2\int_0^t \int_0^s \lambda(\tau) d\tau ds\right) v$$

$\Rightarrow$ We use yet another Gauge transform.

- Then

$$\inf_{h \in \mathrm{L}^2(0,T;\mathrm{L}^2):\ \mathbf{Y}\left(\mathbf{S}_{WN}^{\Psi^0_A}(h)(T)\right) \geq \tilde{R}} \frac{\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}}{2} \leq \inf_{\lambda \in \mathrm{L}^2(0,T;\mathbb{R}),\ \int_0^T \int_0^t \lambda(s) ds dt \geq \frac{\tilde{R}}{8A}} \left(\frac{\pi^2}{6A}\right) \int_0^T \lambda^2(t) dt$$

- We finally pass to the limit with the sequence of operators and for large A obtain a term of the order of the square of the norm of the gradient and use that $\tilde{R}$ is arbitrary and such that $\tilde{R} > R$
- It is precisely the boundary condition in integral form that gives the factor $1/T^3$.

Idea of the proof of the lower bound II

- It is even more convenient to work with

$$i\frac{\mathrm{d}v}{\mathrm{d}t} = \Delta v + |v|^2 v + \lambda(t)\left(x - 2\int_0^t \int_0^s \lambda(\tau)d\tau ds\right) v$$

$\Rightarrow$ We use yet another Gauge transform.

- Then

$$\inf_{h\in \mathrm{L}^2(0,T;\mathrm{L}^2):\ \mathbf{Y}\left(\mathbf{S}^{\Psi^0}_{A\,WN}(h)(T)\right)\geq \tilde{R}} \frac{\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}}{2} \leq \inf_{\lambda\in \mathrm{L}^2(0,T;\mathbb{R}),\int_0^T\int_0^t \lambda(s)dsdt\geq \frac{\tilde{R}}{8A}} \left(\frac{\pi^2}{6A}\right)\int_0^T \lambda^2(t)dt$$

- We finally pass to the limit with the sequence of operators and for large A obtain a term of the order of the square of the norm of the gradient and use that $\tilde{R}$ is arbitrary and such that $\tilde{R} > R$
- It is precisely the boundary condition in integral form that gives the factor $1/T^3$.

Idea of the proof of the lower bound II

- It is even more convenient to work with

$$i\frac{\mathrm{d}v}{\mathrm{d}t} = \Delta v + |v|^2 v + \lambda(t)\left(x - 2\int_0^t \int_0^s \lambda(\tau)d\tau ds\right) v$$

$\Rightarrow$ We use yet another Gauge transform.

- Then

$$\inf_{h\in \mathrm{L}^2(0,T;\mathrm{L}^2):\ \mathbf{Y}\left(\mathbf{S}^{\Psi^0}_{WN}{}^{A}(h)(T)\right)\geq \tilde{R}} \frac{\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}}{2} \leq \inf_{\lambda\in \mathrm{L}^2(0,T;\mathbb{R}),\int_0^T\int_0^t \lambda(s)dsdt\geq \frac{\tilde{R}}{8A}} \left(\frac{\pi^2}{6A}\right)\int_0^T \lambda^2(t)dt$$

- We finally pass to the limit with the sequence of operators and for large A obtain a term of the order of the square of the norm of the gradient and use that $\tilde{R}$ is arbitrary and such that $\tilde{R} > R$
- It is precisely the boundary condition in integral form that gives the factor $1/T^3$.

Tails of the arrival time, upper bounds

Proposition

$\forall L, A, R > 0$ and D dense in $[R, R+1]$, $(\Phi_n)_{n\in\mathbb{N}}$ H.S. in Σ s.t. $\forall h \in \mathcal{H}^D_{L,A}$, $\Phi_n h \to h$ in $\mathrm{L}^1(0, L; \Sigma)$ and uniformly bounded (norms bounded by 1) then

$$\overline{\lim}_{n\to\infty,\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,\Psi^0_A}(L)\right) \geq R\right) \leq -\frac{R^2}{128L^3(1+\frac{2}{L})^2\left(A+\frac{R}{8L+4}\right)}.$$

Thus for R of the order of 1 and L large (in practice $>> 1000$) the upper bound is of the order $-\frac{R^2}{128L^3A}$.
Such operators exist:

$$\Phi_n = \left(I - \Delta + |x|^2 I + \frac{1}{n}(-\Delta + |x|^2 I)^k\right)^{-1/2}.$$

Tails of the arrival time, upper bounds

Proposition

$\forall L, A, R > 0$ and D dense in $[R, R+1]$, $(\Phi_n)_{n\in\mathbb{N}}$ H.S. in Σ s.t. $\forall h \in \mathcal{H}_{L,A}^{D}$, $\Phi_n h \to h$ in $\mathrm{L}^1(0, L; \Sigma)$ and uniformly bounded (norms bounded by 1) then

$$\overline{\lim}_{n\to\infty,\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,\Psi_A^0}(L)\right) \geq R\right) \leq -\frac{R^2}{128L^3(1+\frac{2}{L})^2\left(A+\frac{R}{8L+4}\right)}.$$

Thus for R of the order of 1 and L large (in practice $>> 1000$) the upper bound is of the order $-\frac{R^2}{128L^3A}$.

Such operators exist:

$$\Phi_n = \left(I - \Delta + |x|^2 I + \frac{1}{n}(-\Delta + |x|^2 I)^k\right)^{-1/2}.$$

Tails of the arrival time, upper bounds

Proposition

$\forall L, A, R > 0$ and D dense in $[R, R+1]$, $(\Phi_n)_{n\in\mathbb{N}}$ H.S. in Σ s.t. $\forall h \in \mathcal{H}^D_{L,A}$, $\Phi_n h \to h$ in $\mathrm{L}^1(0, L; \Sigma)$ and uniformly bounded (norms bounded by 1) then

$$\overline{\lim}_{n\to\infty,\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,\Psi^0_A}(L)\right) \geq R\right) \leq -\frac{R^2}{128L^3(1+\frac{2}{L})^2\left(A+\frac{R}{8L+4}\right)}.$$

Thus for R of the order of 1 and L large (in practice $>> 1000$) the upper bound is of the order $-\frac{R^2}{128L^3A}$.
Such operators exist:

$$\Phi_n = \left(I - \Delta + |x|^2 I + \frac{1}{n}(-\Delta + |x|^2 I)^k\right)^{-1/2}.$$

Idea of the proof of the upper bound

For the upper bound we manipulate the controlled equation in order to obtain bounds of the L^2 norm of any control that allows to get to the "large deviation set".

Indeed we prove

$$\begin{aligned}\mathbf{Y}(\mathbf{S}^{a,\Psi_A^0}(h)(t)) = \ & 4\mathfrak{Re}\left(\int_0^t\int_0^s\int_{\mathbb{R}}\overline{\mathbf{S}^{a,\Psi_A^0}(h)(\sigma,x)}\left(\partial_x\Phi h\right)(\sigma,x)dxd\sigma ds\right)\\ & -2\mathfrak{Re}\left(i\int_0^t\int_{\mathbb{R}}x\overline{\mathbf{S}^{a,\Psi_A^0}(h)(s,x)}\left(\Phi h\right)(s,x)dxds\right).\end{aligned}$$

Then using $R \leq \mathbf{Y}\left(\mathbf{S}^{a,\Psi_A^0}(h)(L)\right)$ we deduce a lower bound for $\|h\|^2_{\mathrm{L}^2(0,L;\mathrm{L}^2)}$.

Idea of the proof of the upper bound

For the upper bound we manipulate the controlled equation in order to obtain bounds of the L^2 norm of any control that allows to get to the "large deviation set".
Indeed we prove

$$\mathbf{Y}(\mathbf{S}^{a,\Psi_A^0}(h)(t)) = 4\Re\mathfrak{e}\left(\int_0^t\int_0^s\int_{\mathbb{R}}\overline{\mathbf{S}^{a,\Psi_A^0}(h)(\sigma,x)}\left(\partial_x\Phi h\right)(\sigma,x)dxd\sigma ds\right) \\ -2\Re\mathfrak{e}\left(i\int_0^t\int_{\mathbb{R}}x\overline{\mathbf{S}^{a,\Psi_A^0}(h)(s,x)}\left(\Phi h\right)(s,x)dxds\right).$$

Then using $R \leq \mathbf{Y}\left(\mathbf{S}^{a,\Psi_A^0}(h)(L)\right)$ we deduce a lower bound for $\|h\|^2_{\mathrm{L}^2(0,L;\mathrm{L}^2)}$.

Idea of the proof of the upper bound

For the upper bound we manipulate the controlled equation in order to obtain bounds of the L^2 norm of any control that allows to get to the "large deviation set".
Indeed we prove

$$\begin{aligned}\mathbf{Y}(\mathbf{S}^{a,\Psi_A^0}(h)(t)) = \quad & 4\mathfrak{Re}\left(\int_0^t\int_0^s\int_{\mathbb{R}}\overline{\mathbf{S}^{a,\Psi_A^0}(h)(\sigma,x)}\left(\partial_x\Phi h\right)(\sigma,x)dxd\sigma ds\right)\\ & -2\mathfrak{Re}\left(i\int_0^t\int_{\mathbb{R}}x\overline{\mathbf{S}^{a,\Psi_A^0}(h)(s,x)}\left(\Phi h\right)(s,x)dxds\right).\end{aligned}$$

Then using $R \leq \mathbf{Y}\left(\mathbf{S}^{a,\Psi_A^0}(h)(L)\right)$ we deduce a lower bound for $\|h\|^2_{\mathrm{L}^2(0,L;\mathrm{L}^2)}$.

Comparison with the results in the physics literature

- The tails of the mass with 0 as initial datum are not Gaussian, undistinguishable from exponential tails on a log-scale ;
- The tails of the arrival time are undistinguishable from Gaussian tails on the log-scale ;
- On the log-scale the tails of the mass are of the order of $\exp\left(-\frac{c}{\epsilon L}\right)$, that of the arrival time of $\exp\left(-\frac{c}{\epsilon L^3}\right)$
  $\Rightarrow$ The tails of the arrival time are larger than that of the mass,
  $\Rightarrow$ The fluctuation of the arrival time is the main process impairing soliton optical communications
- Gordon-Hauss effect: the variance of the arrival time is of the order of L^3, if the law were indeed Gaussian we obtain the same result.
- In some articles the influence of the amplitude of the initial datum is studied and the variance of the arrival time is of the order of AL^3.

Comparison with the results in the physics literature

- The tails of the mass with 0 as initial datum are not Gaussian, undistinguishable from exponential tails on a log-scale ;
- The tails of the arrival time are undistinguishable from Gaussian tails on the log-scale ;
- On the log-scale the tails of the mass are of the order of $\exp\left(-\frac{c}{\epsilon L}\right)$, that of the arrival time of $\exp\left(-\frac{c}{\epsilon L^3}\right)$
  $\Rightarrow$ The tails of the arrival time are larger than that of the mass,
  $\Rightarrow$ The fluctuation of the arrival time is the main process impairing soliton optical communications
- Gordon-Hauss effect: the variance of the arrival time is of the order of L^3, if the law were indeed Gaussian we obtain the same result.
- In some articles the influence of the amplitude of the initial datum is studied and the variance of the arrival time is of the order of AL^3.

Comparison with the results in the physics literature

- The tails of the mass with 0 as initial datum are not Gaussian, undistinguishable from exponential tails on a log-scale ;
- The tails of the arrival time are undistinguishable from Gaussian tails on the log-scale ;
- On the log-scale the tails of the mass are of the order of $\exp\left(-\frac{c}{\epsilon L}\right)$, that of the arrival time of $\exp\left(-\frac{c}{\epsilon L^3}\right)$
  $\Rightarrow$ The tails of the arrival time are larger than that of the mass,
  $\Rightarrow$ The fluctuation of the arrival time is the main process impairing soliton optical communications
- Gordon-Hauss effect: the variance of the arrival time is of the order of L^3, if the law were indeed Gaussian we obtain the same result.
- In some articles the influence of the amplitude of the initial datum is studied and the variance of the arrival time is of the order of AL^3.

Comparison with the results in the physics literature

- The tails of the mass with 0 as initial datum are not Gaussian, undistinguishable from exponential tails on a log-scale ;
- The tails of the arrival time are undistinguishable from Gaussian tails on the log-scale ;
- On the log-scale the tails of the mass are of the order of $\exp\left(-\frac{c}{\epsilon L}\right)$, that of the arrival time of $\exp\left(-\frac{c}{\epsilon L^3}\right)$
  $\Rightarrow$ The tails of the arrival time are larger than that of the mass,
  $\Rightarrow$ The fluctuation of the arrival time is the main process impairing soliton optical communications
- Gordon-Hauss effect: the variance of the arrival time is of the order of L^3, if the law were indeed Gaussian we obtain the same result.
- In some articles the influence of the amplitude of the initial datum is studied and the variance of the arrival time is of the order of AL^3.

Comparison with the results in the physics literature

- The tails of the mass with 0 as initial datum are not Gaussian, undistinguishable from exponential tails on a log-scale ;
- The tails of the arrival time are undistinguishable from Gaussian tails on the log-scale ;
- On the log-scale the tails of the mass are of the order of $\exp\left(-\frac{c}{\epsilon L}\right)$, that of the arrival time of $\exp\left(-\frac{c}{\epsilon L^3}\right)$
  $\Rightarrow$ The tails of the arrival time are larger than that of the mass,
  $\Rightarrow$ The fluctuation of the arrival time is the main process impairing soliton optical communications
- Gordon-Hauss effect: the variance of the arrival time is of the order of L^3, if the law were indeed Gaussian we obtain the same result.
- In some articles the influence of the amplitude of the initial datum is studied and the variance of the arrival time is of the order of AL^3.

Conclusion

- Introducing inline control elements can alow to reduce exponentially the fluctuations of the mass and especially that of the arrival time. Optimizing the system with constraints on the cost would require an optimization on two sets of controls. Some particular in line control elements have been considered by physicists.
- Sharp LD could allow to obtain the pre-exponential factors of the tails.

Conclusion

- Introducing inline control elements can alow to reduce exponentially the fluctuations of the mass and especially that of the arrival time. Optimizing the system with constraints on the cost would require an optimization on two sets of controls. Some particular in line control elements have been considered by physicists.
- Sharp LD could allow to obtain the pre-exponential factors of the tails.

A stochastic KdV equation

- We consider
$$du + \left(\partial_x^3 u + \partial_x(u^2)\right) dt = \sqrt{\epsilon} dW$$
where Φ is H.S. from L^2 to H^1, $u(0) = u_0 \in \mathrm{H}^1$.
- Model for the evolution of weakly nonlinear shallow water waves with random pressure
- Existence of mild solutions in $\mathrm{C}([0,T];\mathrm{H}^1)$ and uniqueness in $X_T \subset \mathrm{C}\left([0,T];\mathrm{H}^1\right)$ proved by de Bouard & Debussche 98', they also studied rougher noises and less regular solutions in other papers
- The deterministic equation has soliton solutions of the form $\varphi_c(x - ct + x_0)$ where c is the velocity, $x_0 \in \mathbb{R}$ the initial phase and
$$\varphi_c(x,t) = \frac{3c}{2}\mathrm{sech}^2\left(\sqrt{c}x/2\right)$$
- Solitons are stable (deterministic case) even for more general nonlinearities: notion of orbital stability and some results regarding asymptotic stability Pego & Weinstein 94', Martel & Merle 01' (either weak convergence or CV in weighted Sobolev spaces)

A stochastic KdV equation

- We consider

$$du + \left(\partial_x^3 u + \partial_x(u^2)\right) dt = \sqrt{\epsilon} dW$$

where Φ is H.S. from L^2 to H^1, $u(0) = u_0 \in \mathrm{H}^1$.

- Model for the evolution of weakly nonlinear shallow water waves with random pressure
- Existence of mild solutions in $\mathrm{C}([0,T];\mathrm{H}^1)$ and uniqueness in $X_T \subset \mathrm{C}\left([0,T];\mathrm{H}^1\right)$ proved by de Bouard & Debussche 98', they also studied rougher noises and less regular solutions in other papers
- The deterministic equation has soliton solutions of the form $\varphi_c(x - ct + x_0)$ where c is the velocity, $x_0 \in \mathbb{R}$ the initial phase and

$$\varphi_c(x,t) = \frac{3c}{2} \mathrm{sech}^2\left(\sqrt{c}x/2\right)$$

- Solitons are stable (deterministic case) even for more general nonlinearities: notion of orbital stability and some results regarding asymptotic stability Pego & Weinstein 94', Martel & Merle 01' (either weak convergence or CV in weighted Sobolev spaces)

A stochastic KdV equation

- We consider
$$du + \left(\partial_x^3 u + \partial_x(u^2)\right) dt = \sqrt{\epsilon} dW$$
where Φ is H.S. from L^2 to H^1, $u(0) = u_0 \in \mathrm{H}^1$.
- Model for the evolution of weakly nonlinear shallow water waves with random pressure
- Existence of mild solutions in $\mathrm{C}([0, T]; \mathrm{H}^1)$ and uniqueness in $X_T \subset \mathrm{C}\left([0, T]; \mathrm{H}^1\right)$ proved by de Bouard & Debussche 98', they also studied rougher noises and less regular solutions in other papers
- The deterministic equation has soliton solutions of the form $\varphi_c(x - ct + x_0)$ where c is the velocity, $x_0 \in \mathbb{R}$ the initial phase and
$$\varphi_c(x, t) = \frac{3c}{2}\mathrm{sech}^2\left(\sqrt{c}x/2\right)$$
- Solitons are stable (deterministic case) even for more general nonlinearities: notion of orbital stability and some results regarding asymptotic stability Pego & Weinstein 94', Martel & Merle 01' (either weak convergence or CV in weighted Sobolev spaces)

A stochastic KdV equation

- We consider
$$du + \left(\partial_x^3 u + \partial_x(u^2)\right) dt = \sqrt{\epsilon} dW$$
where Φ is H.S. from L^2 to H^1, $u(0) = u_0 \in \mathrm{H}^1$.
- Model for the evolution of weakly nonlinear shallow water waves with random pressure
- Existence of mild solutions in $\mathrm{C}([0, T]; \mathrm{H}^1)$ and uniqueness in $X_T \subset \mathrm{C}\left([0, T]; \mathrm{H}^1\right)$ proved by de Bouard & Debussche 98', they also studied rougher noises and less regular solutions in other papers
- The deterministic equation has soliton solutions of the form $\varphi_c(x - ct + x_0)$ where c is the velocity, $x_0 \in \mathbb{R}$ the initial phase and
$$\varphi_c(x, t) = \frac{3c}{2}\mathrm{sech}^2\left(\sqrt{c}x/2\right)$$
- Solitons are stable (deterministic case) even for more general nonlinearities: notion of orbital stability and some results regarding asymptotic stability Pego & Weinstein 94', Martel & Merle 01' (either weak convergence or CV in weighted Sobolev spaces)

A stochastic KdV equation

- We consider
$$du + \left(\partial_x^3 u + \partial_x(u^2)\right) dt = \sqrt{\epsilon} dW$$
where Φ is H.S. from L^2 to H^1, $u(0) = u_0 \in \mathrm{H}^1$.
- Model for the evolution of weakly nonlinear shallow water waves with random pressure
- Existence of mild solutions in $\mathrm{C}([0,T];\mathrm{H}^1)$ and uniqueness in $X_T \subset \mathrm{C}\left([0,T];\mathrm{H}^1\right)$ proved by de Bouard & Debussche 98', they also studied rougher noises and less regular solutions in other papers
- The deterministic equation has soliton solutions of the form $\varphi_c(x - ct + x_0)$ where c is the velocity, $x_0 \in \mathbb{R}$ the initial phase and
$$\varphi_c(x,t) = \frac{3c}{2} \mathrm{sech}^2\left(\sqrt{c}x/2\right)$$
- Solitons are stable (deterministic case) even for more general nonlinearities: notion of orbital stability and some results regarding asymptotic stability Pego & Weinstein 94', Martel & Merle 01' (either weak convergence or CV in weighted Sobolev spaces)

Time scale for the solution to stay in the vincinity of the soliton

- The phenomenon of persistence of the soliton has been observed numerically in Debussche & Printems 99'
- Goal: Verify the claim that starting from a soliton profile φ_c the solution of the stochastic equation remains close to the deterministic soliton for times of the order at most $\epsilon^{-1/3}$ but stays close to a randomly modulated soliton for times of the order of ϵ^{-1}
  Joint work w/ A. de Bouard
- We can prove easily a LDP for for the paths of the solutions of the SPDE in $\mathrm{C}([0,T];\mathrm{H}^1)$, the rate function is again the minum energy of a control that allows to reach the large deviation event.
- The approximation by a randomly modulated soliton is the approximation we did not use in the section on error in soliton transmission.
- We denote by u^{ϵ,u_0} the solution with initial datum φ_{c_0}.

Time scale for the solution to stay in the vincinity of the soliton

- The phenomenon of persistence of the soliton has been observed numerically in Debussche & Printems 99'
- Goal: Verify the claim that starting from a soliton profile φ_c the solution of the stochastic equation remains close to the deterministic soliton for times of the order at most $\epsilon^{-1/3}$ but stays close to a randomly modulated soliton for times of the order of ϵ^{-1}
  Joint work w/ A. de Bouard
- We can prove easily a LDP for for the paths of the solutions of the SPDE in $C([0,T];H^1)$, the rate function is again the minum energy of a control that allows to reach the large deviation event.
- The approximation by a randomly modulated soliton is the approximation we did not use in the section on error in soliton transmission.
- We denote by u^{ϵ,u_0} the solution with initial datum φ_{c_0}.

Time scale for the solution to stay in the vincinity of the soliton

- The phenomenon of persistence of the soliton has been observed numerically in Debussche & Printems 99'
- Goal: Verify the claim that starting from a soliton profile φ_c the solution of the stochastic equation remains close to the deterministic soliton for times of the order at most $\epsilon^{-1/3}$ but stays close to a randomly modulated soliton for times of the order of ϵ^{-1}
  Joint work w/ A. de Bouard
- We can prove easily a LDP for for the paths of the solutions of the SPDE in $\mathrm{C}([0, T]; \mathrm{H}^1)$, the rate function is again the minum energy of a control that allows to reach the large deviation event.
- The approximation by a randomly modulated soliton is the approximation we did not use in the section on error in soliton transmission.
- We denote by u^{ϵ, u_0} the solution with initial datum φ_{c_0}.

Time scale for the solution to stay in the vincinity of the soliton

- The phenomenon of persistence of the soliton has been observed numerically in Debussche & Printems 99'
- Goal: Verify the claim that starting from a soliton profile φ_c the solution of the stochastic equation remains close to the deterministic soliton for times of the order at most $\epsilon^{-1/3}$ but stays close to a randomly modulated soliton for times of the order of ϵ^{-1}
  Joint work w/ A. de Bouard
- We can prove easily a LDP for for the paths of the solutions of the SPDE in $\mathrm{C}([0, T]; \mathrm{H}^1)$, the rate function is again the minum energy of a control that allows to reach the large deviation event.
- The approximation by a randomly modulated soliton is the approximation we did not use in the section on error in soliton transmission.
- We denote by u^{ϵ, u_0} the solution with initial datum φ_{c_0}.

Time scale for the solution to stay in the vincinity of the soliton

- The phenomenon of persistence of the soliton has been observed numerically in Debussche & Printems 99'
- Goal: Verify the claim that starting from a soliton profile φ_c the solution of the stochastic equation remains close to the deterministic soliton for times of the order at most $\epsilon^{-1/3}$ but stays close to a randomly modulated soliton for times of the order of ϵ^{-1}
  Joint work w/ A. de Bouard
- We can prove easily a LDP for for the paths of the solutions of the SPDE in $\mathrm{C}([0, T]; \mathrm{H}^1)$, the rate function is again the minum energy of a control that allows to reach the large deviation event.
- The approximation by a randomly modulated soliton is the approximation we did not use in the section on error in soliton transmission.
- We denote by u^{ϵ,u_0} the solution with initial datum φ_{c_0}.

Time scale for the solution to stay in the vincinity of the soliton

Exit time of a neighborhood of the soliton defined by

$$\tilde{\tau}_\alpha^\epsilon = \inf\left\{t \in [0,\infty) : \ \|u^{\epsilon,u_0}(t,\cdot + c_0 t) - \varphi_{c_0}\|_{\mathrm{H}^1} > \alpha\right\}$$

Proposition

For $T > 0$ and $\Phi = \left((1+x^2)I - \partial_x^2\right)^{-1/2}$, then, $\forall 0 < \alpha < \alpha_0 : \alpha_0$ is small enough, $\exists c(\alpha, c_0)$ s.t.

$$\underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\tilde{\tau}_\alpha^\epsilon \le T\right) \ge -\frac{c(\alpha, c_0)}{T^3}.$$

Idea of the proof: look for solutions of the controlled KdV equation of the form

$$\varphi_{c(t)}\left(x - \int_0^t c(s)\right)$$

which implies

$$\Phi h(t,x) = c'(t)\ \partial_c\varphi_c|_{c=c(t)}\left(x - \int_0^t c(s)ds\right)$$

the r.h.s is in the image of Φ for c close to c_0.

Time scale for the solution to stay in the vincinity of the soliton

Exit time of a neighborhood of the soliton defined by

$$\tilde{\tau}_\alpha^\epsilon = \inf\left\{t \in [0,\infty) : \ \|u^{\epsilon,u_0}(t,\cdot + c_0 t) - \varphi_{c_0}\|_{\mathrm{H}^1} > \alpha\right\}$$

Proposition

For $T > 0$ and $\Phi = \left((1+x^2)I - \partial_x^2\right)^{-1/2}$, then, $\forall 0 < \alpha < \alpha_0 : \alpha_0$ is small enough, $\exists c(\alpha, c_0)$ s.t.

$$\underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\tilde{\tau}_\alpha^\epsilon \leq T\right) \geq -\frac{c(\alpha, c_0)}{T^3}.$$

Idea of the proof: look for solutions of the controlled KdV equation of the form

$$\varphi_{c(t)}\left(x - \int_0^t c(s)\right)$$

which implies

$$\Phi h(t,x) = c'(t)\ \partial_c\varphi_c|_{c=c(t)}\left(x - \int_0^t c(s)ds\right)$$

the r.h.s is in the image of Φ for c close to c_0.

Time scale for the solution to stay in the vincinity of the soliton

Exit time of a neighborhood of the soliton defined by

$$\tilde{\tau}_\alpha^\epsilon = \inf\left\{t \in [0,\infty): \ \|u^{\epsilon,u_0}(t,\cdot + c_0 t) - \varphi_{c_0}\|_{\mathrm{H}^1} > \alpha\right\}$$

Proposition

For $T > 0$ *and* $\Phi = \left((1+x^2)I - \partial_x^2\right)^{-1/2}$*, then,* $\forall 0 < \alpha < \alpha_0 : \alpha_0$ *is small enough,* $\exists c(\alpha, c_0)$ *s.t.*

$$\underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\tilde{\tau}_\alpha^\epsilon \le T\right) \ge -\frac{c(\alpha, c_0)}{T^3}.$$

Idea of the proof: look for solutions of the controlled KdV equation of the form

$$\varphi_{c(t)}\left(x - \int_0^t c(s)\right)$$

which implies

$$\Phi h(t,x) = c'(t)\ \partial_c\varphi_c|_{c=c(t)}\left(x - \int_0^t c(s)ds\right)$$

the r.h.s is in the image of Φ for c close to c_0.

Proof II

The objective function is less than

$$c(\alpha, c_0)\int_0^T \left(c'(t)\right)^2 dt$$

and we check that the boundary condition implied by the rare event implies

$$\int_0^T (c_0 - c(s))ds > \frac{2}{e\sqrt{c_0 - 2c(\infty)\alpha/3}}$$

($c(\infty)$ is the constant in the constant in the Sobolev injection $\mathrm{H}^1 \subset \mathrm{L}^\infty$)
Change of variable $t = Tu$ and change of unknown function $v(u) = c(Tu)$ implies the scaling as $1/T^3$.

Random modulations of solitons

Theorem

$\exists \alpha_0 > 0 : \forall \alpha \in (0, \alpha_0]$, $\exists \tau_\alpha^\epsilon > 0$ a.s. stopping time, $\exists c^\epsilon(t), x^\epsilon(t)$ semi-martingales defined a.s. for $t \le \tau_\alpha^\epsilon$ with values in $(0, \infty)$ and $\mathbb{R}$ s.t. if we set

$$\sqrt{\epsilon}\eta^\epsilon(t) = u^{\epsilon, u_0}\left(t, \cdot + x^\epsilon(t)\right) - \varphi_{c^\epsilon(t)}$$

then

$$\int_{\mathbb{R}} \eta^\epsilon(t,x)\varphi_{c_0}(x)dx = \int_{\mathbb{R}} \eta^\epsilon(t,x)\partial_x\varphi_{c_0}(x)dx = 0, \quad \forall t \le \tau_\alpha^\epsilon \quad a.s.$$

and for all $t \le \tau_\alpha^\epsilon$,

$$\left\|\sqrt{\epsilon}\eta^\epsilon(t)\right\|_{H^1} \le \alpha, \quad |c^\epsilon(t) - c_0| \le \alpha.$$

Moreover, $\exists C > 0$ s.t. $\forall T > 0$, $\forall \alpha \le \alpha_0$, $\exists \epsilon_0 > 0$ s.t. $\forall \epsilon < \epsilon_0$,

$$\mathbb{P}\left(\tau_\alpha^\epsilon \le T\right) \le \frac{C\epsilon T \|\Phi\|_{\mathcal{L}_2^{0,1}}}{\alpha^4}.$$

Exit off neighborhood of randomly modulated soliton, upper bound

Proposition

$\forall T > 0$, $\forall \alpha_0 > 0$ *small enough*, $\exists C > 0 : \forall \alpha < \alpha_0$, $\exists \epsilon_0 > 0$ *small enough s.t.* $\forall \epsilon < \epsilon_0$,

$$\mathbb{P}\left(\tau_\alpha^\epsilon \le T\right) \le \exp\left(-\frac{\alpha^2}{C\epsilon T \|\Phi\|^2_{\mathcal{L}_2^{0,1}}}\right).$$

Idea of the proof: Work with the Lyapounov functional

$$Q_{c_0}(u) = \mathbf{H}(u) + c_0 \mathbf{N}(u)$$

where

$$\mathbf{H}(u) = \frac{1}{2}\int_{\mathbb{R}} (\partial_x u)^2 dx - \frac{1}{3}\int_{\mathbb{R}} u^3 dx, \quad \mathbf{N}(u) = \frac{1}{2}\int_{\mathbb{R}} u^2 dx$$

which is such that $Q''_{c_0}(\varphi_{c_0}) = -\partial_x + (c_0 - 2\varphi_{c_0})I$ has no unstable eigenvalue and a general null space spanned by φ_{c_0} and $\partial_x \varphi_{c_0}$.

Use the Itô formula for $\mathbf{H}\left(u^{\epsilon,u_0}\right)$ and $\mathbf{N}\left(u^{\epsilon,u_0}\right)$, smoothing and exponential tail estimates.

Exit off neighborhood of randomly modulated soliton, upper bound

Proposition

$\forall T > 0$, $\forall \alpha_0 > 0$ *small enough*, $\exists C > 0 : \forall \alpha < \alpha_0$, $\exists \epsilon_0 > 0$ *small enough s.t.* $\forall \epsilon < \epsilon_0$,

$$\mathbb{P}\left(\tau_\alpha^\epsilon \leq T\right) \leq \exp\left(-\frac{\alpha^2}{C\epsilon T \|\Phi\|^2_{\mathcal{L}_2^{0,1}}}\right).$$

Idea of the proof: Work with the Lyapounov functional

$$Q_{c_0}(u) = \mathbf{H}(u) + c_0 \mathbf{N}(u)$$

where

$$\mathbf{H}(u) = \frac{1}{2}\int_{\mathbb{R}} (\partial_x u)^2 dx - \frac{1}{3}\int_{\mathbb{R}} u^3 dx, \quad \mathbf{N}(u) = \frac{1}{2}\int_{\mathbb{R}} u^2 dx$$

which is such that $Q''_{c_0}(\varphi_{c_0}) = -\partial_x + (c_0 - 2\varphi_{c_0})I$ has no unstable eigenvalue and a general null space spanned by φ_{c_0} and $\partial_x \varphi_{c_0}$.

Use the Itô formula for $\mathbf{H}\left(u^{\epsilon,u_0}\right)$ and $\mathbf{N}\left(u^{\epsilon,u_0}\right)$, smoothing and exponential tail estimates.

Exit off neighborhood of randomly modulated soliton, lower bound

Proposition

For $T > 0$, $\Phi = \left((1+x^2)I - \partial_x^2\right)^{-1/2}$ *and* $0 < \alpha < \alpha_0$ *small enough,* $\exists c(\Phi, \alpha)$ *s.t.*

$$\underline{\lim}_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(\tau_\alpha^\epsilon \leq T\right) \geq -\frac{c(\Phi,\alpha)}{T}.$$

Idea of the proof: Let $\mathcal{C}$ be the mapping obtained using the implicit function theorem (which gives the random modulations) then

$$\mathbb{P}\left(\tau_\alpha^\epsilon \leq T\right) \geq \mathbb{P}\left(\left|\mathcal{C}\left(u^{\epsilon,u_0}(T)\right) - c_0\right| > \alpha\right)$$

thus,

$$\underline{\lim}_{\epsilon \to 0} \epsilon^2 \log \mathbb{P}\left(\tau_\alpha^\epsilon \leq T\right) \geq -\inf\left\{\frac{\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}}{2},\ h :\ \left|\mathcal{C}\left(\mathbf{S}^{u_0}(h)(T)\right) - c_0\right| = \frac{3}{2}\alpha\right\}$$

we minimize on the smaller set where controls lead to a solution of the form

$$\varphi_{c(t)}\left(x - \int_0^t c(s)\right)$$

same as above but here boundary condition is $c(0) = c_0$ and $|c(T) - c_0| = \frac{3}{2}\alpha$.

Exit off neighborhood of randomly modulated soliton, lower bound

Proposition

For $T>0$, $\Phi=\left((1+x^2)I-\partial_x^2\right)^{-1/2}$ *and* $0<\alpha<\alpha_0$ *small enough,* $\exists c(\Phi,\alpha)$ *s.t.*

$$\underline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\tau_\alpha^\epsilon\leq T\right)\geq -\frac{c(\Phi,\alpha)}{T}.$$

Idea of the proof: Let $\mathcal{C}$ be the mapping obtained using the implicit function theorem (which gives the random modulations) then

$$\mathbb{P}\left(\tau_\alpha^\epsilon\leq T\right)\geq\mathbb{P}\left(\left|\mathcal{C}\left(u^{\epsilon,u_0}(T)\right)-c_0\right|>\alpha\right)$$

thus,

$$\underline{\lim}_{\epsilon\to 0}\epsilon^2\log\mathbb{P}\left(\tau_\alpha^\epsilon\leq T\right)\geq -\inf\left\{\frac{\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}}{2},\ h:\ \left|\mathcal{C}\left(\mathbf{S}^{u_0}(h)(T)\right)-c_0\right|=\frac{3}{2}\alpha\right\}$$

we minimize on the smaller set where controls lead to a solution of the form

$$\varphi_{c(t)}\left(x-\int_0^t c(s)\right)$$

same as above but here boundary condition is $c(0)=c_0$ and $|c(T)-c_0|=\frac{3}{2}\alpha$.

Annex
Blow up times for stochastic NLS, eg additive noise

- $\overline{\mathcal{T}^{-1}((T,\infty])} = \mathcal{E}_\infty$
- The LDP gives that

Proposition

For $u_0 \in \mathrm{H}^3$, $\mathrm{Ker}\Phi^* = \{0\}$ *and* $T \geq \mathcal{T}\left(\mathbf{S}(u_0, 0)\right)$,

$$\underline{\lim}_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(\mathcal{T}\left(u^{\epsilon, u_0}\right) > T\right) \geq -\frac{1}{2} \inf_{h \in \mathrm{L}^2(0,\infty;\mathrm{L}^2) : \mathcal{T}(\mathbf{S}(h)) > T} \left\{ \|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)} \right\} \in (-\infty, 0].$$

Proposition

If $T < \mathcal{T}\left(\mathbf{S}(u_0, 0)\right)$ *then*

$$\overline{\lim}_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(\mathcal{T}\left(u^{\epsilon, u_0}\right) \leq T\right) \leq -\frac{1}{2} \inf_{h \in \mathrm{L}^2(0,\infty;\mathrm{L}^2) : \mathcal{T}(\mathbf{S}(h)) \leq T} \left\{ \|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)} \right\} < 0$$

Annex

Blow up times for stochastic NLS, eg additive noise

- $\overline{\mathcal{T}^{-1}((T,\infty])} = \mathcal{E}_\infty$
- The LDP gives that

Proposition

For $u_0 \in \mathrm{H}^3$, $\mathrm{Ker}\Phi^* = \{0\}$ and $T \geq \mathcal{T}(\mathbf{S}(u_0, 0))$,

$$\underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathcal{T}\left(u^{\epsilon,u_0}\right) > T\right) \geq -\frac{1}{2}\inf_{h\in \mathrm{L}^2(0,\infty;\mathrm{L}^2):\mathcal{T}(\mathbf{S}(h))>T}\left\{\|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)}\right\} \in (-\infty, 0].$$

Proposition

If $T < \mathcal{T}(\mathbf{S}(u_0, 0))$ then

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathcal{T}\left(u^{\epsilon,u_0}\right) \leq T\right) \leq -\frac{1}{2}\inf_{h\in \mathrm{L}^2(0,\infty;\mathrm{L}^2):\mathcal{T}(\mathbf{S}(h))\leq T}\left\{\|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)}\right\} < 0$$

Annex

Blow up times for stochastic NLS, eg additive noise

- $\overline{\mathcal{T}^{-1}((T,\infty])} = \mathcal{E}_\infty$
- The LDP gives that

Proposition

For $u_0 \in \mathrm{H}^3$, $\mathrm{Ker}\Phi^* = \{0\}$ *and* $T \geq \mathcal{T}(\mathbf{S}(u_0,0))$,

$$\underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathcal{T}\left(u^{\epsilon,u_0}\right) > T\right) \geq -\frac{1}{2}\inf_{h\in \mathrm{L}^2(0,\infty;\mathrm{L}^2):\mathcal{T}(\mathbf{S}(h))>T}\left\{\|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)}\right\} \in (-\infty,0].$$

Proposition

If $T < \mathcal{T}(\mathbf{S}(u_0,0))$ *then*

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathcal{T}\left(u^{\epsilon,u_0}\right) \leq T\right) \leq -\frac{1}{2}\inf_{h\in \mathrm{L}^2(0,\infty;\mathrm{L}^2):\mathcal{T}(\mathbf{S}(h))\leq T}\left\{\|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)}\right\} < 0$$

Annex

Blow up times for stochastic NLS, eg additive noise

- $\overline{\mathcal{T}^{-1}((T,\infty])} = \mathcal{E}_\infty$
- The LDP gives that

Proposition

For $u_0 \in \mathrm{H}^3$, $\mathrm{Ker}\boldsymbol{\Phi}^* = \{0\}$ *and* $T \geq \mathcal{T}(\mathbf{S}(u_0,0))$,

$$\underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathcal{T}\left(u^{\epsilon,u_0}\right) > T\right) \geq -\frac{1}{2}\inf_{h\in \mathrm{L}^2(0,\infty;\mathrm{L}^2):\mathcal{T}(\mathbf{S}(h))>T}\left\{\|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)}\right\} \in (-\infty,0].$$

Proposition

If $T < \mathcal{T}(\mathbf{S}(u_0,0))$ *then*

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathcal{T}\left(u^{\epsilon,u_0}\right) \leq T\right) \leq -\frac{1}{2}\inf_{h\in \mathrm{L}^2(0,\infty;\mathrm{L}^2):\mathcal{T}(\mathbf{S}(h))\leq T}\left\{\|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)}\right\} < 0$$

Exit from a domain for weakly damped NLS equations

- We add the term $-\alpha u dt$ in the drift and consider subcritical nonlinearities ;
- We consider bounded measurable domains D, in L^2 and H^1, which contain 0 in their interior s.t. $\forall u_0 \in D,\ \forall t \geq 0,\ \mathbf{S}(u_0, 0)(t) \in D$;
- Some references on the study of this problem for SPDEs: Freidlin 88', Da Prato & Zabczyk 92', Chenal & Millet 97'
- Main difficulty compared to the SDE setting: D is not compact
- Main difficulties here:
  - Uniform continuity of the deterministic flow for bounded initial data would require smoother initial data ;
  - No global smoothing properties of the semi-group ;
  - No compact embeddings in the Sobolev spaces on $\mathbb{R}^d$;
  - Non locally Lipschitz nonlinearities when $d > 1$.
- But usual result holds: see *AOP* 07'.

Exit from a domain for weakly damped NLS equations

- We add the term $-\alpha u\mathrm{d}t$ in the drift and consider subcritical nonlinearities ;
- We consider bounded measurable domains D, in L^2 and H^1, which contain 0 in their interior s.t. $\forall u_0 \in D,\ \forall t \geq 0,\ \mathbf{S}(u_0, 0)(t) \in D$;
- Some references on the study of this problem for SPDEs: Freidlin 88', Da Prato & Zabczyk 92', Chenal & Millet 97'
- Main difficulty compared to the SDE setting: D is not compact
- Main difficulties here:
  - Uniform continuity of the deterministic flow for bounded initial data would require smoother initial data ;
  - No global smoothing properties of the semi-group ;
  - No compact embeddings in the Sobolev spaces on $\mathbb{R}^d$;
  - Non locally Lipschitz nonlinearities when $d > 1$.
- But usual result holds: see *AOP* 07'.

Exit from a domain for weakly damped NLS equations

- We add the term $-\alpha u dt$ in the drift and consider subcritical nonlinearities ;
- We consider bounded measurable domains D, in L^2 and H^1, which contain 0 in their interior s.t. $\forall u_0 \in D,\ \forall t \geq 0,\ \mathbf{S}(u_0, 0)(t) \in D$;
- Some references on the study of this problem for SPDEs: Freidlin 88', Da Prato & Zabczyk 92', Chenal & Millet 97'
- Main difficulty compared to the SDE setting: D is not compact
- Main difficulties here:
  - Uniform continuity of the deterministic flow for bounded initial data would require smoother initial data ;
  - No global smoothing properties of the semi-group ;
  - No compact embeddings in the Sobolev spaces on $\mathbb{R}^d$;
  - Non locally Lipschitz nonlinearities when $d > 1$.
- But usual result holds: see *AOP* 07'.

Exit from a domain for weakly damped NLS equations

- We add the term $-\alpha u dt$ in the drift and consider subcritical nonlinearities ;
- We consider bounded measurable domains D, in L^2 and H^1, which contain 0 in their interior s.t. $\forall u_0 \in D,\ \forall t \geq 0,\ \mathbf{S}(u_0, 0)(t) \in D$;
- Some references on the study of this problem for SPDEs: Freidlin 88', Da Prato & Zabczyk 92', Chenal & Millet 97'
- Main difficulty compared to the SDE setting: D is not compact
- Main difficulties here:
  - Uniform continuity of the deterministic flow for bounded initial data would require smoother initial data ;
  - No global smoothing properties of the semi-group ;
  - No compact embeddings in the Sobolev spaces on $\mathbb{R}^d$;
  - Non locally Lipschitz nonlinearities when $d > 1$.
- But usual result holds: see *AOP* 07'.

Exit from a domain for weakly damped NLS equations

- We add the term $-\alpha u dt$ in the drift and consider subcritical nonlinearities ;
- We consider bounded measurable domains D, in L^2 and H^1, which contain 0 in their interior s.t. $\forall u_0 \in D, \ \forall t \geq 0, \ \mathbf{S}(u_0, 0)(t) \in D$;
- Some references on the study of this problem for SPDEs: Freidlin 88', Da Prato & Zabczyk 92', Chenal & Millet 97'
- Main difficulty compared to the SDE setting: D is not compact
- Main difficulties here:
  - Uniform continuity of the deterministic flow for bounded initial data would require smoother initial data ;
  - No global smoothing properties of the semi-group ;
  - No compact embeddings in the Sobolev spaces on $\mathbb{R}^d$;
  - Non locally Lipschitz nonlinearities when $d > 1$.
- But usual result holds: see *AOP* 07'.

Exit from a domain for weakly damped NLS equations

- We add the term $-\alpha u dt$ in the drift and consider subcritical nonlinearities ;
- We consider bounded measurable domains D, in L^2 and H^1, which contain 0 in their interior s.t. $\forall u_0 \in D, \ \forall t \geq 0, \ \mathbf{S}(u_0, 0)(t) \in D$;
- Some references on the study of this problem for SPDEs: Freidlin 88', Da Prato & Zabczyk 92', Chenal & Millet 97'
- Main difficulty compared to the SDE setting: D is not compact
- Main difficulties here:
  - Uniform continuity of the deterministic flow for bounded initial data would require smoother initial data ;
  - No global smoothing properties of the semi-group ;
  - No compact embeddings in the Sobolev spaces on $\mathbb{R}^d$;
  - Non locally Lipschitz nonlinearities when $d > 1$.
- But usual result holds: see *AOP* 07'.

Exit from a domain for weakly damped NLS equations

- We add the term $-\alpha u dt$ in the drift and consider subcritical nonlinearities ;
- We consider bounded measurable domains D, in L^2 and H^1, which contain 0 in their interior s.t. $\forall u_0 \in D, \ \forall t \geq 0, \ \mathbf{S}(u_0, 0)(t) \in D$;
- Some references on the study of this problem for SPDEs: Freidlin 88', Da Prato & Zabczyk 92', Chenal & Millet 97'
- Main difficulty compared to the SDE setting: D is not compact
- Main difficulties here:
    - Uniform continuity of the deterministic flow for bounded initial data would require smoother initial data ;
    - No global smoothing properties of the semi-group ;
    - No compact embeddings in the Sobolev spaces on $\mathbb{R}^d$;
    - Non locally Lipschitz nonlinearities when $d > 1$.
- But usual result holds: see *AOP* 07'.

Exit from a domain for weakly damped NLS equations

- We add the term $-\alpha u dt$ in the drift and consider subcritical nonlinearities ;
- We consider bounded measurable domains D, in L^2 and H^1, which contain 0 in their interior s.t. $\forall u_0 \in D, \ \forall t \geq 0, \ \mathbf{S}(u_0, 0)(t) \in D$;
- Some references on the study of this problem for SPDEs: Freidlin 88', Da Prato & Zabczyk 92', Chenal & Millet 97'
- Main difficulty compared to the SDE setting: D is not compact
- Main difficulties here:
  - Uniform continuity of the deterministic flow for bounded initial data would require smoother initial data ;
  - No global smoothing properties of the semi-group ;
  - No compact embeddings in the Sobolev spaces on $\mathbb{R}^d$;
  - Non locally Lipschitz nonlinearities when $d > 1$.
- But usual result holds: see *AOP* 07'.

Exit from a domain for weakly damped NLS equations

- We add the term $-\alpha u\mathrm{d}t$ in the drift and consider subcritical nonlinearities ;
- We consider bounded measurable domains D, in L^2 and H^1, which contain 0 in their interior s.t. $\forall u_0 \in D,\ \forall t \geq 0,\ \mathbf{S}(u_0, 0)(t) \in D$;
- Some references on the study of this problem for SPDEs: Freidlin 88', Da Prato & Zabczyk 92', Chenal & Millet 97'
- Main difficulty compared to the SDE setting: D is not compact
- Main difficulties here:
  - Uniform continuity of the deterministic flow for bounded initial data would require smoother initial data ;
  - No global smoothing properties of the semi-group ;
  - No compact embeddings in the Sobolev spaces on $\mathbb{R}^d$;
  - Non locally Lipschitz nonlinearities when $d > 1$.
- But usual result holds: see *AOP* 07'.

Exit from a domain for weakly damped NLS equations

- We add the term $-\alpha u dt$ in the drift and consider subcritical nonlinearities ;
- We consider bounded measurable domains D, in L^2 and H^1, which contain 0 in their interior s.t. $\forall u_0 \in D, \ \forall t \geq 0, \ \mathbf{S}(u_0, 0)(t) \in D$;
- Some references on the study of this problem for SPDEs: Freidlin 88', Da Prato & Zabczyk 92', Chenal & Millet 97'
- Main difficulty compared to the SDE setting: D is not compact
- Main difficulties here:
  - Uniform continuity of the deterministic flow for bounded initial data would require smoother initial data ;
  - No global smoothing properties of the semi-group ;
  - No compact embeddings in the Sobolev spaces on $\mathbb{R}^d$;
  - Non locally Lipschitz nonlinearities when $d > 1$.
- But usual result holds: see *AOP* 07'.

Exit for stochastic weakly damped NLS equations

- Domains D are uniformly attracted to zero by $\mathbf{S}(.,0)$;
- The first exit time off D is defined by $\tau^{\epsilon,u_0} = \inf\{t \geq 0 : \ u(t) \in D^c\}$.
- We define
$$\overline{e} = \inf\left\{ I_0^T(w) : \ w(T) \in \overline{D}^c, \ T > 0 \right\},$$
and $\underline{e} = \lim_{\rho \to 0} e_\rho$
where $e_\rho = \inf\left\{ I_{u_0}^T(w) : \ \tilde{\mathbf{H}}(u_0) \leq \rho, \ w(T) \in (D_{-\rho})^c, \ T > 0 \right\}$,
with $\rho > 0$ small enough and $D_{-\rho} = D \setminus \mathcal{N}(\partial D, \rho)$;
- When N is a closed subset of ∂D, we define
$$e_{N,\rho} = \inf\left\{ I_{u_0}^T(w) : \ \tilde{\mathbf{H}}(u_0) \leq \rho, \ w(T) \in (D \setminus \mathcal{N}(N,\rho))^c, \ T > 0 \right\},$$
where $\rho > 0$ is small enough and
$$\underline{e}_N = \lim_{\rho \to 0} e_{N,\rho}.$$

Exit for stochastic weakly damped NLS equations

- Domains D are uniformly attracted to zero by $\mathbf{S}(.,0)$;
- The first exit time off D is defined by $\tau^{\epsilon,u_0} = \inf\{t \geq 0 : \ u(t) \in D^c\}$.
- We define
$$\overline{e} = \inf\left\{ I_0^T(w) : \ w(T) \in \overline{D}^c, \ T > 0 \right\},$$
and $\underline{e} = \lim_{\rho \to 0} e_\rho$
where $e_\rho = \inf\left\{ I_{u_0}^T(w) : \ \tilde{\mathbf{H}}(u_0) \leq \rho, \ w(T) \in (D_{-\rho})^c, \ T > 0 \right\}$,
with $\rho > 0$ small enough and $D_{-\rho} = D \setminus \mathcal{N}(\partial D, \rho)$;
- When N is a closed subset of ∂D, we define
$$e_{N,\rho} = \inf\left\{ I_{u_0}^T(w) : \ \tilde{\mathbf{H}}(u_0) \leq \rho, \ w(T) \in (D \setminus \mathcal{N}(N,\rho))^c, \ T > 0 \right\},$$
where $\rho > 0$ is small enough and
$$\underline{e}_N = \lim_{\rho \to 0} e_{N,\rho}.$$

Exit for stochastic weakly damped NLS equations

- Domains D are uniformly attracted to zero by $\mathbf{S}(.,0)$;
- The first exit time off D is defined by
  $\tau^{\epsilon,u_0} = \inf\{t \geq 0 : \; u(t) \in D^c\}$.
- We define

$$\overline{e} = \inf\left\{ I_0^T(w) : \; w(T) \in \overline{D}^c, \; T > 0 \right\},$$

and $\underline{e} = \lim_{\rho\to 0} e_\rho$

where $e_\rho = \inf\left\{ I_{u_0}^T(w) : \; \tilde{\mathbf{H}}(u_0) \leq \rho, \; w(T) \in (D_{-\rho})^c, \; T > 0 \right\}$,

with $\rho > 0$ small enough and $D_{-\rho} = D \setminus \mathcal{N}(\partial D, \rho)$;

- When N is a closed subset of ∂D, we define

$$e_{N,\rho} = \inf\left\{ I_{u_0}^T(w) : \; \tilde{\mathbf{H}}(u_0) \leq \rho, \; w(T) \in (D \setminus \mathcal{N}(N,\rho))^c, \; T > 0 \right\},$$

where $\rho > 0$ is small enough and

$$\underline{e}_N = \lim_{\rho\to 0} e_{N,\rho}.$$

Exit for stochastic weakly damped NLS equations

- Domains D are uniformly attracted to zero by $\mathbf{S}(.,0)$;
- The first exit time off D is defined by $\tau^{\epsilon,u_0}=\inf\{t\geq 0:\ u(t)\in D^c\}$.
- We define
$$\overline{e}=\inf\left\{I_0^T(w):\ w(T)\in\overline{D}^c,\ T>0\right\},$$
and $\underline{e}=\lim_{\rho\to 0}e_\rho$
where $e_\rho=\inf\left\{I_{u_0}^T(w):\ \tilde{\mathbf{H}}(u_0)\leq\rho,\ w(T)\in(D_{-\rho})^c,\ T>0\right\}$,
with $\rho>0$ small enough and $D_{-\rho}=D\setminus\mathcal{N}(\partial D,\rho)$;
- When N is a closed subset of ∂D, we define
$$e_{N,\rho}=\inf\left\{I_{u_0}^T(w):\ \tilde{\mathbf{H}}(u_0)\leq\rho,\ w(T)\in(D\setminus\mathcal{N}(N,\rho))^c,\ T>0\right\},$$
where $\rho>0$ is small enough and
$$\underline{e}_N=\lim_{\rho\to 0}e_{N,\rho}.$$

First exit time

Theorem

$\forall u_0 \in D, \ \forall \delta > 0, \ \exists L > 0:$

$$\overline{\lim}_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(\tau^{\epsilon, u_0} \notin \left(\exp\left(\frac{\underline{e} - \delta}{\epsilon}\right), \exp\left(\frac{\overline{e} + \delta}{\epsilon}\right)\right)\right) \leq -L, \tag{1}$$

and $\forall u_0 \in D$,

$$\underline{e} \leq \underline{\lim}_{\epsilon \to 0} \epsilon \log \mathbb{E}\left(\tau^{\epsilon, u_0}\right) \leq \overline{\lim}_{\epsilon \to 0} \epsilon \log \mathbb{E}\left(\tau^{\epsilon, u_0}\right) \leq \overline{e}. \tag{2}$$

Moreover, $\forall \delta > 0, \ \exists L > 0:$

$$\overline{\lim}_{\epsilon \to 0} \epsilon \log \sup_{u_0 \in D} \mathbb{P}\left(\tau^{\epsilon, u_0} \geq \exp\left(\frac{\overline{e} + \delta}{\epsilon}\right)\right) \leq -L, \tag{3}$$

and

$$\overline{\lim}_{\epsilon \to 0} \epsilon \log \sup_{u_0 \in D} \mathbb{E}\left(\tau^{\epsilon, u_0}\right) \leq \overline{e}. \tag{4}$$

Exit points

Theorem

If $\underline{e}_N > \overline{e}$, *then* $\forall u_0 \in D$, $\exists L > 0$:

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(u\left(\tau^{\epsilon,u_0}\right) \in N\right) \leq -L.$$

Corollary

Let $v^* \in \partial D$ *be s.t.* $\forall \delta > 0$ *and* $N = \{v \in \partial D : \ \|v - v^*\|_{L^2} \geq \delta\}$ *we have* $\underline{e}_N > \overline{e}$, *then*

$$\forall \delta > 0,\ \forall u_0 \in D,\ \exists L > 0 : \ \overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\left\|u\left(\tau^{\epsilon,u_0}\right) - v^*\right\|_{L^2} \geq \delta\right) \leq -L.$$

Exit points

Theorem

If $\underline{e}_N > \overline{e}$, *then* $\forall u_0 \in D$, $\exists L > 0$:

$$\overline{\lim}_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(u\left(\tau^{\epsilon, u_0}\right) \in N\right) \leq -L.$$

Corollary

Let $v^* \in \partial D$ *be s.t.* $\forall \delta > 0$ *and* $N = \{v \in \partial D : \ \|v - v^*\|_{L^2} \geq \delta\}$ *we have* $\underline{e}_N > \overline{e}$, *then*

$$\forall \delta > 0, \ \forall u_0 \in D, \ \exists L > 0 : \ \overline{\lim}_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(\left\|u\left(\tau^{\epsilon, u_0}\right) - v^*\right\|_{L^2} \geq \delta\right) \leq -L.$$