Random Walks Conditioned to Stay Positive

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Let S_n be a random walk formed by summing i.i.d. integer valued random variables X_i, $i \geq 1$: $S_n = X_1 + \cdots + X_n$. If the drift EX_i is negative, then $S_n \to -\infty$ as $n \to \infty$. If A_n is the event that $S_k \geq 0$ for $k = 1, \ldots, n$, then $P(A_n) \to 0$ as $n \to \infty$. In this talk we will consider conditional distributions for the random walk given A_n. The main result will show that finite dimensional distributions for the random walk given A_n converge to those for a time homogeneous Markov chain on $\{0, 1, \ldots\}$.

Exponential Families

Let h denote the mass function for an integer valued random variable X under P_0. Assume that $E_0X = \sum xh(x) = 0$, and define

$$e^{\psi(\omega)} = E_0 e^{\omega X} = \sum_x e^{\omega x} h(x). \tag{1}$$

Then $f_\omega(x) = h(x)\exp[\omega x - \psi(\omega)]$, is a probability mass function whenever $\omega \in \Omega = \{\omega : \psi(\omega) < \infty\}$.. Let $X, X_1, \ldots$ be i.i.d. under P_ω with marginal mass function f_ω. Differentiating (1),

$$\psi'(\omega)e^{\psi(\omega)} = \sum_x xe^{\omega x} h(x),$$

and so

$$E_\omega X = \psi'(\omega).$$

Similarly, $\mathrm{Var}_\omega(X) = \psi''(\omega)$. Note that since $\psi'' > 0$, ψ' is increasing, and since $\psi'(0) = E_0X = 0$, $\psi'(\omega) < 0$ when $\omega < 0$.

Classical Large Deviation Theory

Exponential Tilting: Let $S_n = X_1 + \cdots + X_n$ and $\overline{X}_n = S_n/n$. Since $X_1, \ldots, X_n$ have joint mass function

$$\prod_{i=1}^{n} \bigl(h(x_i) \exp[\omega x_i - \psi(\omega)]\bigr) = \left[\prod_{i=1}^{n} h(x_i)\right] \exp[\omega s_n - n\psi(\omega)],$$

and $$E_\omega f(X_1, \ldots, X_n) = E_0 f(X_1, \ldots, X_n) \exp[\omega S_n - n\psi(\omega)].$$

In particular,

$$P_\omega(S_n \geq 0) = e^{-n\psi(\omega)} E_0[e^{\omega S_n}; S_n \geq 0].$$

For notation, $E[Y; A] \stackrel{\text{def}}{=} E[Y 1_A]$. Using this, it is easy to argue that if $\omega < 0$,

$$\frac{1}{n} \log P_\omega(S_n \geq 0) \to -\psi(\omega), \quad \text{or} \quad P_\omega(S_n \geq 0) = e^{-n\psi(\omega)} \times e^{o(n)}.$$

Also, for any $\epsilon > 0$,

$$P_\omega(\overline{X}_n > \epsilon | \overline{X}_n \geq 0) \to 0,$$

as $n \to \infty$. [For regularity, need $0 \in \Omega^o$.]

Refinements

Local Central Limit Theorem: If the distribution for X is lattice with span 1, then as $n \to \infty$,

$$P_0[S_n = k] \sim 1/\sqrt{2\pi n\psi''(0)}.$$

Using this, for $\omega < 0$,

$$\begin{aligned} P_\omega(S_n \geq 0) &= e^{-n\psi(\omega)} E_0[e^{\omega S_n}; S_n \geq 0] \\ &\sim e^{-n\psi(\omega)} \sum_{k=0}^{\infty} \frac{e^{\omega k}}{\sqrt{2\pi n\psi''(0)}} \\ &= \frac{e^{-n\psi(\omega)}}{(1-e^{\omega})\sqrt{2\pi n\psi''(0)}}, \end{aligned}$$

and

$$P(S_n = k | S_n \geq 0) \to (1-e^{\omega})e^{\omega k},$$

the mass function for a geometric distribution with success probability e^{ω}.

Goal, Sufficiency, and Notation

Let $\tau = \inf\{n : S_n < 0\}$ and note that

$$\tau > n \quad \text{if and only if} \quad S_j \geq 0, \quad j = 1, \ldots, n.$$

Goal: Study the behavior of the random walk given $\tau > n$ for large n.

Sufficiency: Under P_ω, the conditional distribution for $X_1, \ldots, X_n$ given S_n does not depend on ω.

New Measures: Under $P_\omega^{(a)}$, the summands $X_1, X_2, \ldots$ are still i.i.d. with common mass function f_ω, but

$$S_n = a + X_1 + \cdots + X_n.$$

Finally, $P_n^{(a,b)}$ denotes conditional probability under $P_\omega^{(a)}$ given $S_n = b$. Under this measure, S_k goes from a to b in n steps.

Positive Drift

If $\omega > 0$, then $E_\omega X > 0$. In this case, $P_\omega^{(a)}(\tau = \infty) > 0$, and conditioning on $\tau = \infty$ is simple. For $x \geq 0$,

$$P_\omega^{(a)}(S_1 = x | \tau = \infty) = \frac{P_\omega^{(a)}(S_1 = x, \tau = \infty)}{P_\omega^{(a)}(\tau = \infty)}$$

$$= P_\omega^{(a)}(S_1 = x) \frac{P_\omega^{(x)}(\tau = \infty)}{P_\omega^{(a)}(\tau = \infty)}.$$

Given $\tau = \infty$, the process S_n, $n \geq 0$, is a random walk, and the conditional transition kernel is an h-transform of the original transition kernel for S_n.

Simple Random Walk

If $X_i = \pm 1$ with $p = P_\omega(X_i = 1)$, $q = 1 - p = P_\omega(X_i = -1)$, and $\omega = \frac{1}{2}\log(p/q)$, then S_n, $n \geq 0$, is a *simple random walk.*

Theorem: For a simple random walk, as $n \to \infty$,

$$P_n^{(a,b)}(\tau > n) \sim \frac{2(a+1)(b+1)}{n}.$$

Proof. By the reflection principle,

$$P_0^{(a)}(\tau < n, S_n = b) = P_0^{(a)}(S_n = -b - 2).$$

Dividing by $P_0(S_n = b)$ (a binomial probability),

$$P_n^{(a,b)}(\tau < n) = \frac{\left(\frac{n+b-a}{2}\right)!\left(\frac{n-b+a}{2}\right)!}{\left(\frac{n-a-b-2}{2}\right)!\left(\frac{n+a+b+2}{2}\right)!}.$$

Result follows from Stirling's formula. □

Result for Simple Random Walks

Let Y_n, $n \geq 0$, be a Markov chain on $\{0, 1, \ldots\}$ with $Y_0 = 0$ and transition matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 1/4 & 0 & 3/4 & 0 & 0 & \cdots \\ 0 & 2/6 & 0 & 4/6 & 0 & \cdots \\ 0 & 0 & 3/8 & 0 & 5/8 & \\ \vdots & \vdots & & \ddots & \ddots & \ddots \end{pmatrix}$$

Theorem: For a simple random walk with $p < 1/2$,

$$P_\omega(S_1 = y_1, \ldots, S_j = y_j | \tau > n) \to P(Y_1 = y_1, \ldots, Y_j = y_j)$$

as $n \to \infty$.

Proof: Let $B = \{S_1 = y_1, \ldots, S_j = y_j = y\}$ with $y_1 = 1$, $y_i \geq 0$, and $|y_{i+1} - y_i| = 1$. Then

$$
\begin{aligned}
P_n^{(0,b)}(B, \tau > n) &= \frac{P_0(B, \tau > n, S_n = b)}{P_0(S_n = b)} \\
&= \frac{(1/2)^j P_{n-j}^{(y,b)}(\tau > n - j) P_0(S_n - j = b - y)}{P_0(S_n = b)} \\
&\sim \frac{2(y+1)(b+1)}{2^j n}.
\end{aligned}
$$

Use this in

$$
P_\omega(B|\tau > n) = \frac{\sum_b P_n^{(0,b)}(B, \tau > n) P_\omega(S_n = b)}{\sum_b P_n^{(0,b)}(\tau > n) P_\omega(S_n = b)}.
$$

$\square$

General Results

For the general case, let Y_n, $n \geq 0$, be a stationary Markov chain with $Y_0 = 0$ and

$$P(Y_{n+1} = z | Y_n = y) = P_0(X = z - y) \frac{E_0^{(z)}(z - S_\tau)}{E_0^{(y)}(y - S_\tau)}.$$

Remark: the transition kernel for Y is an h-transform of the kernel for the random walk under P_0.

Theorem: For $\omega < 0$,

$$P_\omega(S_1 = y_1, \ldots, S_k = y_k | \tau > n) \to P(Y_1 = y_1, \ldots, Y_k = y_k)$$

as $n \to \infty$.

Theorem: Let $\tau^+(b) = \inf\{n : S_n > b\}$. For $\omega < 0$,

$$P_\omega(S_n = b | \tau > n) \to \frac{e^{b\omega} E_0 S_{\tau^+(b)}}{\sum_{k=0}^{\infty} e^{k\omega} E_0 S_{\tau^+(k)}}.$$

Approximate Reflection: $P_n^{(a,b)}(\tau > n)$, b of order $\sqrt{n}$.

Proposition: If $0 \le b = O(\sqrt{n})$,

$$P_n^{(a,b)}(\tau > n) = \frac{2b}{n\psi''(0)} E_0^{(a)}(a - S_\tau) + o\big(1/\sqrt{n}\big).$$

Proof: Let g_k denote the P_0 mass function for S_k, and define

$$L_k(x) = \frac{P_n^{(a,b)}(S_k = x)}{P_n^{(a,-b)}(S_k = x)} = \frac{g_{n-k}(b-x)g_n(-b-a)}{g_{n-k}(-b-x)g_n(b-a)}.$$

Then $L_k(S_k)$ is $dP_n^{(a,b)}/dP_n^{(a,-b)}$ restricted to $\sigma(S_k)$ or $\sigma(X_1, \ldots, X_k)$, and

$$\begin{aligned} P_n^{(a,b)}(\tau \le n) &= E_n^{(a,-b)} L_\tau(S_\tau) \\ &= E_n^{(a,-b)} \left[1 - \frac{2b}{n\psi''(0)}(a - S_\tau) + o\big(1/\sqrt{n}\big)\right]. \end{aligned}$$

Finish by arguing that $E_n^{(a,-b)} S_\tau \to E_0^{(a)} S_\tau$. $\square$

Approximate Reflection: $P_n^{(a,b)}(\tau > n)$, b of order one.

Corollary: As $n \to \infty$,

$$P_n^{(a,b)}(\tau > n) = \frac{2}{n\psi''(0)} E_0^{(a)}(a - S_\tau) E_0^{(b)}(b - S_\tau) + o(1/n).$$

Proof: Take $m = \lfloor n/2 \rfloor$. Then

$$P_n^{(a,b)}(\tau > n) = \sum_{c=0}^{\infty} P_n^{(a,b)}(S_m = c) P_m^{(a,c)}(\tau > m) P_{n-m}^{(b,c)}(\tau > n - m).$$

Result follows using the prior result since S_m under $P_n^{(a,b)}$ is approximately normal. $\square$

References

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