Random Walks Conditioned to Stay Positive

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Let $S_n$ be a random walk formed by summing i.i.d. integer valued random variables $X_i, i \geq 1$: $S_n = X_1 + \cdots + X_n$. If the drift $EX_i$ is negative, then $S_n \to -\infty$ as $n \to \infty$. If $A_n$ is the event that $S_k \geq 0$ for $k = 1, \ldots, n$, then $P(A_n) \to 0$ as $n \to \infty$. In this talk we will consider conditional distributions for the random walk given $A_n$. The main result will show that finite dimensional distributions for the random walk given $A_n$ converge to those for a time homogeneous Markov chain on $\{0, 1, \ldots\}$. 
Exponential Families

Let $h$ denote the mass function for an integer valued random variable $X$ under $P_0$. Assume that $E_0 X = \sum x h(x) = 0$, and define

$$e^{\psi(\omega)} = E_0 e^{\omega X} = \sum_x e^{\omega x} h(x). \tag{1}$$

Then $f_\omega(x) = h(x) \exp[\omega x - \psi(\omega)]$, is a probability mass function whenever $\omega \in \Omega = \{\omega : \psi(\omega) < \infty\}$. Let $X, X_1, \ldots$ be i.i.d. under $P_\omega$ with marginal mass function $f_\omega$. Differentiating (1),

$$\psi'(\omega) e^{\psi(\omega)} = \sum_x x e^{\omega x} h(x),$$

and so

$$E_\omega X = \psi'(\omega).$$

Similarly, $\text{Var}_\omega(X) = \psi''(\omega)$. Note that since $\psi'' > 0$, $\psi'$ is increasing, and since $\psi'(0) = E_0 X = 0$, $\psi'(\omega) < 0$ when $\omega < 0$. 
Classical Large Deviation Theory

Exponential Tilting: Let \( S_n = X_1 + \cdots + X_n \) and \( \bar{X}_n = S_n/n \). Since \( X_1, \ldots, X_n \) have joint mass function

\[
\prod_{i=1}^{n} (h(x_i) \exp[\omega x_i - \psi(\omega)]) = \left[ \prod_{i=1}^{n} h(x_i) \right] \exp[\omega s_n - n\psi(\omega)],
\]

and \( E_{\omega} f(X_1, \ldots, X_n) = E_0 f(X_1, \ldots, X_n) \exp[\omega S_n - n\psi(\omega)] \).

In particular,

\[
P_{\omega}(S_n \geq 0) = e^{-n\psi(\omega)} E_0 [e^{\omega S_n}; S_n \geq 0].
\]

For notation, \( E[Y; A] \stackrel{\text{def}}{=} E[Y 1_A] \). Using this, it is easy to argue that if \( \omega < 0 \),

\[
\frac{1}{n} \log P_{\omega}(S_n \geq 0) \to -\psi(\omega), \quad \text{or} \quad P_{\omega}(S_n \geq 0) = e^{-n\psi(\omega)} \times e^{o(n)}.
\]

Also, for any \( \epsilon > 0 \),

\[
P_{\omega}(\bar{X}_n > \epsilon | \bar{X}_n \geq 0) \to 0,
\]

as \( n \to \infty \). [For regularity, need \( 0 \in \Omega^0 \).]
Refinements

Local Central Limit Theorem: If the distribution for \( X \) is lattice with span 1, then as \( n \to \infty \),

\[
P_0[S_n = k] \sim 1/\sqrt{2\pi n\psi''(0)}.
\]

Using this, for \( \omega < 0 \),

\[
P_\omega(S_n \geq 0) = e^{-n\psi(\omega)}E_0[e^{\omega S_n}; S_n \geq 0]
\]

\[
\sim e^{-n\psi(\omega)} \sum_{k=0}^{\infty} \frac{e^{\omega k}}{\sqrt{2\pi n\psi''(0)}}
\]

\[
= \frac{e^{-n\psi(\omega)}}{(1 - e^\omega)\sqrt{2\pi n\psi''(0)}},
\]

and

\[
P(S_n = k|S_n \geq 0) \to (1 - e^\omega)e^{\omega k},
\]

the mass function for a geometric distribution with success probability \( e^\omega \).
Goal, Sufficiency, and Notation

Let $\tau = \inf\{n : S_n < 0\}$ and note that

$$\tau > n \quad \text{if and only if} \quad S_j \geq 0, \quad j = 1, \ldots, n.$$

**Goal:** Study the behavior of the random walk given $\tau > n$ for large $n$.

**Sufficiency:** Under $P_\omega$, the conditional distribution for $X_1, \ldots, X_n$ given $S_n$ does not depend on $\omega$.

**New Measures:** Under $P_\omega^{(a)}$, the summands $X_1, X_2, \ldots$ are still i.i.d. with common mass function $f_\omega$, but

$$S_n = a + X_1 + \cdots + X_n.$$

Finally, $P_n^{(a,b)}$ denotes conditional probability under $P_\omega^{(a)}$ given $S_n = b$. Under this measure, $S_k$ goes from $a$ to $b$ in $n$ steps.
Positive Drift

If $\omega > 0$, then $E_{\omega}X > 0$. In this case, $P^{(a)}_{\omega}(\tau = \infty) > 0$, and conditioning on $\tau = \infty$ is simple. For $x \geq 0$,

$$P^{(a)}_{\omega}(S_1 = x | \tau = \infty) = \frac{P^{(a)}_{\omega}(S_1 = x, \tau = \infty)}{P^{(a)}_{\omega}(\tau = \infty)}$$

$$= P^{(a)}_{\omega}(S_1 = x) \frac{P^{(a)}_{\omega}(\tau = \infty)}{P^{(a)}_{\omega}(\tau = \infty)}.$$

Given $\tau = \infty$, the process $S_n$, $n \geq 0$, is a random walk, and the conditional transition kernel is an $h$-transform of the original transition kernel for $S_n$. 

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Simple Random Walk

If \( X_i = \pm 1 \) with \( p = P_{\omega}(X_i = 1) \), \( q = 1 - p = P_{\omega}(X_i = -1) \), and \( \omega = \frac{1}{2} \log(p/q) \), then \( S_n, n \geq 0 \), is a simple random walk.

**Theorem:** For a simple random walk, as \( n \to \infty \),

\[
P_n^{(a,b)}(\tau > n) \sim \frac{2(a + 1)(b + 1)}{n}.
\]

**Proof.** By the reflection principle,

\[
P_0^{(a)}(\tau < n, S_n = b) = P_0^{(a)}(S_n = -b - 2).
\]

Dividing by \( P_0(S_n = b) \) (a binomial probability),

\[
P_n^{(a,b)}(\tau < n) = \frac{\left(\frac{n+b-a}{2}\right)! \left(\frac{n-b+a}{2}\right)!}{\left(\frac{n-a-b-2}{2}\right)! \left(\frac{n+a+b+2}{2}\right)!}.
\]

Result follows from Stirling’s formula. \(\square\)
Result for Simple Random Walks

Let $Y_n, n \geq 0$, be a Markov chain on $\{0, 1, \ldots\}$ with $Y_0 = 0$ and transition matrix

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
1/4 & 0 & 3/4 & 0 & 0 & \cdots \\
0 & 2/6 & 0 & 4/6 & 0 & \cdots \\
0 & 0 & 3/8 & 0 & 5/8 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}
$$

**Theorem:** For a simple random walk with $p < 1/2$,

$$
P_\omega(S_1 = y_1, \ldots, S_j = y_j | \tau > n) \to P(Y_1 = y_1, \ldots, Y_j = y_j)
$$

as $n \to \infty$. 
Proof: Let \( B = \{S_1 = y_1, \ldots, S_j = y_j = y\} \) with \( y_1 = 1, y_i \geq 0, \) and \( |y_{i+1} - y_i| = 1. \) Then

\[
P_n^{(0,b)}(B, \tau > n) = \frac{P_0(B, \tau > n, S_n = b)}{P_0(S_n = b)}
= \frac{(1/2)^j P_n^{(y,b)}(\tau > n - j) P_0(S_n - j = b - y)}{P_0(S_n = b)}
\sim \frac{2(y + 1)(b + 1)}{2^j n}.
\]

Use this in

\[
P_\omega(B|\tau > n) = \frac{\sum_b P_n^{(0,b)}(B, \tau > n) P_\omega(S_n = b)}{\sum_b P_n^{(0,b)}(\tau > n) P_\omega(S_n = b)}.
\]

\( \Box \)
General Results

For the general case, let $Y_n, n \geq 0$, be a stationary Markov chain with $Y_0 = 0$ and

$$P(Y_{n+1} = z|Y_n = y) = P_0(X = z - y) \frac{E_0(z)(z - S_\tau)}{E_0(y)(y - S_\tau)}.$$

**Remark:** the transition kernel for $Y$ is an $h$-transform of the kernel for the random walk under $P_0$.

**Theorem:** For $\omega < 0$,

$$P_\omega(S_1 = y_1, \ldots, S_k = y_k|\tau > n) \rightarrow P(Y_1 = y_1, \ldots, Y_k = y_k)$$

as $n \rightarrow \infty$.

**Theorem:** Let $\tau^+(b) = \inf\{n : S_n > b\}$. For $\omega < 0$,

$$P_\omega(S_n = b|\tau > n) \rightarrow \frac{e^{b\omega} E_0 S_{\tau^+}(b)}{\sum_{k=0}^{\infty} e^{k\omega} E_0 S_{\tau^+}(k)}.$$
Approximate Reflection: $P_{n}^{(a,b)}(\tau > n)$, $b$ of order $\sqrt{n}$.

**Proposition:** If $0 \leq b = O(\sqrt{n})$,

$$P_{n}^{(a,b)}(\tau > n) = \frac{2b}{n\psi''(0)} E_{0}^{(a)}(a - S_{\tau}) + o(1/\sqrt{n}).$$

**Proof:** Let $g_{k}$ denote the $P_{0}$ mass function for $S_{k}$, and define

$$L_{k}(x) = \frac{P_{n}^{(a,b)}(S_{k} = x)}{P_{n}^{(a,-b)}(S_{k} = x)} = \frac{g_{n-k}(b-x)g_{n}(-b-a)}{g_{n-k}(-b-x)g_{n}(b-a)}.$$ 

Then $L_{k}(S_{k})$ is $dP_{n}^{(a,b)}/dP_{n}^{(a,-b)}$ restricted to $\sigma(S_{k})$ or $\sigma(X_{1}, \ldots, X_{k})$, and

$$P_{n}^{(a,b)}(\tau \leq n) = E_{n}^{(a,-b)} L_{\tau}(S_{\tau})$$

$$= E_{n}^{(a,-b)} \left[ 1 - \frac{2b}{n\psi''(0)}(a - S_{\tau}) + o(1/\sqrt{n}) \right].$$

Finish by arguing that $E_{n}^{(a,-b)} S_{\tau} \rightarrow E_{0}^{(a)} S_{\tau}$. □
Approximate Reflection: $P_n^{(a,b)}(\tau > n)$, $b$ of order one.

Corollary: As $n \to \infty$,

$$P_n^{(a,b)}(\tau > n) = \frac{2}{n\psi''(0)} E_0^{(a)}(a - S_\tau)E_0^{(b)}(b - S_\tau) + o(1/n).$$

Proof: Take $m = \lfloor n/2 \rfloor$. Then

$$P_n^{(a,b)}(\tau > n) = \sum_{c=0}^{\infty} P_n^{(a,b)}(S_m = c) P_m^{(a,c)}(\tau > m) P_{n-m}^{(b,c)}(\tau > n - m).$$

Result follows using the prior result since $S_m$ under $P_n^{(a,b)}$ is approximately normal. □
References


• Durrett (1980). Conditioned limit theorems for random walks with negative drift. *Z. Wahrsch. verw. Gebiete*