Extinction - 2 Case Studies

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Part I: Lifetime of a Branching-Annihilation Process
Part II: Distribution of Epidemic Sizes near Threshold
Extinction Rates - A Real-Space WKB Treatment

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A commentary / extension of Elgart–Kamenev (and Freidlin–Wentzell)
Parallel to Assaf and Meerson

Basic Birth-Death Process

Basic Processes:

\[ A \xrightarrow{\alpha} 2A \]
\[ A + A \xrightarrow{\beta} 0 \]

Rate Equation:

\[ \dot{n} = \alpha n - \beta n(n - 1) \]

- Logistic Growth
- Stable state at \( n = 1 + \alpha/\beta \)
- Unstable state at \( n = 0 \)

Stochastic Process Different:
- State at \( n = 1 + \alpha/\beta \) meta-stable
- Absorbing state at \( n = 0 \)

What is the extinction rate?
Consider the master equation:

\[ \dot{P}_n = \alpha (-nP_n + (n-1)P_{n-1}) + \frac{\beta}{2} (-n(n-1)P_n + (n+2)(n-1)P_{n+2}) \]

Long-time behavior given by \(-\Gamma_1\), negative eigenvalue closest to 0

For small \(\beta/\alpha\), \(\Gamma_1\) is exponentially small \(\Rightarrow\) very slow decay

Associated eigenvector called the quasi-stationary state

\(1/\Gamma_1\) is the mean first passage time for the quasi-stationary state

For small \(\beta/\alpha\), the mean first passage time starting from \(n\) is essentially independent of \(n\) (unless \(n\) is very small), and equal to \(1/\Gamma_1\) (implies all other eigenvalues much larger in magnitude)
We want to compute $\Gamma_1$ for small $\beta/\alpha$.

Strategy: Since $\Gamma_1$ exponentially small, we can solve *stationary* master equation:

$$0 = \alpha (-nP_n + (n - 1)P_{n-1}) + \frac{\beta}{2} (-n(n - 1)P_n + (n + 2)(n + 1)P_{n+2})$$

- $\Gamma_1$ determined by leakage to the absorbing state:

$$\Gamma_1 = \frac{\beta P_2}{\sum_n P_n}$$

Use Discrete WKB to solve master equation
Why not Fokker-Planck?

Width of peak near \( n = \alpha/\beta \) is \( O((\alpha/\beta)^{1/2}) \gg 1 \)

Approximating Finite-Differences by Derivatives should be good!

Define \( y \equiv \sqrt{\beta/\alpha}(n - \alpha/\beta) \)

Master equation reads:

\[
0 = \left( \frac{3}{2} P'' + (yP)' \right) + \frac{1}{2} \sqrt{\frac{\beta}{\alpha}} \left( P''' + 5yP'' + 8P' + 4yP + 2y^2 P \right) + O \left( \frac{\beta}{\alpha} \right)
\]

Obtain Fokker-Planck with small corrections!

What is bad???
Why not Fokker-Planck?

\[ y \equiv \sqrt{\frac{\beta}{\alpha}} \left( n - \frac{\alpha}{\beta} \right) \]

Leading order solution:

\[ P_0(y) \sim e^{-y^2/3} \]

Behavior of correction for large \( y \): \( y^3 \sqrt{\beta/\alpha} P_0(y) \)

Correction no longer small when \( y \sim (\alpha/\beta)^{1/6} \) so that \( n - \alpha/\beta \sim (\alpha/\beta)^{2/3} \)

Can’t use Fokker-Planck to get down to \( n \)’s of order 1.

Can’t calculate \( \Gamma_1 \) this way!!
Discrete WKB

WKB Ansatz: \( x \equiv \beta n / \alpha \)

\[
P_n = \exp \left( \frac{\alpha}{\beta} S_0(x) + S_1(x) + \cdots \right)
\]

To leading order: \( P_{n+k} \approx e^{kS_0'(x)} P_n \equiv \Lambda^k P_n \), where \( \Lambda(x) \equiv e^{S_0'(x)} \)

Leading Order WKB equation:

\[
0 = \alpha (-1 + \frac{1}{\Lambda}) + \frac{\alpha x}{2} (-1 + \Lambda^2)
\]

Solution:

\[
x = \frac{2}{\Lambda (\Lambda + 1)}
\]
Discrete WKB

Solution: \( x = 2/\left(\Lambda(\Lambda + 1)\right) \)

Connection to Fokker-Planck:

- For \( n \approx \alpha/\beta, \ x \approx 1 \)
- \( \Lambda \approx 1 + \frac{2}{3}(1 - x) \)
- \( S'_0 = \ln(\Lambda) \approx \frac{2}{3}(1 - x) \Rightarrow S_0 \approx -\frac{1}{3}(1 - x)^2 \)
- \( P \approx \exp\left(\frac{\alpha}{\beta}S_0\right) = e^{-y^2/3} \)

As \( x \to 0, \ \Lambda \to \infty \)

Scale of \( \Gamma_1 \):

\[
P_2/P_{\alpha/\beta} \approx \exp\left(\frac{\alpha}{\beta} \int_1^\infty \ln(\Lambda) \frac{dx}{d\Lambda} d\Lambda\right) = \exp\left(-\frac{\alpha}{\beta} \int_1^\infty \frac{x(\Lambda)}{\Lambda} d\Lambda\right)
\]
Connection to Elgart-Kamenev

\[ \Gamma_1 \sim P_2/P_{\alpha/\beta} \approx \exp \left( -\frac{\alpha}{\beta} \int_1^{\infty} \frac{x(\Lambda)}{\Lambda} d\Lambda \right) \]

Define coordinate \( q = n\Lambda \), momentum \( p = 1/\Lambda \)

\[ 0 \leq q \leq \alpha/\beta, \; 0 \leq p \leq 1, \]

\[ \Gamma_1 \sim \exp \left( -\int_0^1 qdp \right) \]

where

\[ q(p) = \frac{2\alpha}{\beta} \frac{p}{1 + p} \]

Metastable State

Extinction State

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To complete the calculation, we need $S_1(x)$ as well.

Answer:

$$Q(x) = Ae^{S_1(x)} = A \frac{\sqrt{\Lambda}(\Lambda + 1)^2}{\sqrt{2\Lambda + 1}}$$

This works as long as $n \gg 1$. Have to solve small-$n$ problem separately.

For small $n$, $P_n$ grows rapidly with $n$.

Leads to approximate master equation:

$$0 = \alpha (-nP_n + (n - 1)P_{n-1}) + \frac{\beta}{2} (-n(n - 1)P_n + (n + 2)(n + 1)P_{n+2})$$

Solution:

$$P_n \approx \left( \frac{\alpha}{\beta} \right)^{(n-1)/2} \frac{1}{n\Gamma((n + 1)/2)} P_1$$
Matching this to WKB solution yields \( A = P_1 \sqrt{\frac{\beta^3}{4\pi \alpha^3}} \)

Final Answer:

\[
\Gamma_1 = \sqrt{\frac{\alpha^3}{4\pi \beta}} e^{-2\alpha(1-\ln(2))}/\beta
\]
Quasistationary Distribution

The graph shows the quasistationary distribution for different approximations: Exact, WKB, Fokker-Planck, and Approx. Recursion. The x-axis represents $n$, and the y-axis represents $P_n / P_2$. The data points are plotted on a logarithmic scale.
Decay Rate

Decay Rate ($\Gamma$)

$1 / \beta$

$10^{0}$
$10^{-6}$
$10^{-12}$
$10^{-18}$
$10^{-24}$
$10^{-30}$

$0$ $20$ $40$ $60$ $80$ $100$

- Exact
- Small $\beta$ Approximation

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Numerics

Decay Rate

$\Gamma / e \Delta S$

$1 / \beta$

Exact

WKB

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Conclusion

Models with same rate equations yield very different extinction rates, i.e. different $\Delta S_0$.

- For 'parity' model with $A \overset{\alpha/2}{\rightarrow} 3A$,

$$\Gamma_1 \approx \sqrt{\frac{\alpha^3}{2\pi\beta}}e^{-\alpha/2\beta}$$

In ecology, for example, will never know 'real' model, extinction rate unknowable.

- Leigh (1981) knew the difference between the true extinction rate and the Fokker-Planck value — "equivalent to that accruing from mistaking $K$" (carrying capacity = metastable population)
- Implication is that since we don’t (can’t) know $K$ precisely, what does the exact answer matter?

Situation may be better for environmental stochasticity, where distribution is power-law and Fokker-Planck equation is valid.
And now for something completely different!
SIR Infection Model Near Threshold

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PRE (to appear)

Definition of SIR Model

One of the basic models of infection - Kermack and McKendrick (1927)

Well-Mixed Population of $N$ persons - divided into 3 classes
- $S \equiv$ Susceptible - Healthy, Infectible
- $I \equiv$ Infected - Sick, Infectious
- $R \equiv$ Recovered (Removed) - Non-Infectious, Non-Infectible

Fundamental Processes:

$$(S, I, R) \xrightarrow{\alpha S I / N} (S - 1, I + 1, R) \quad \text{Infection}$$

$$(S, I, R) \xrightarrow{1} (S, I - 1, R + 1) \quad \text{Recovery}$$

2 Parameters, $\alpha$ = infectivity, $N$

Standard Question: Start with 1 Infected, $(N - 1, 1, 0)$. Process ends when last Infected recovers, $(N - n, 0, n)$.

What is epidemic size, $n$?
The SIR Equations: (Kermack and McKendrick, 1927)

\[\begin{align*}
\dot{S} &= -\frac{\alpha}{N} SI \\
\dot{I} &= +\frac{\alpha}{N} SI - I \\
\dot{R} &= I
\end{align*}\]

\(N\) is conserved \(\Rightarrow\) Do not need to track \(R(t)\).

\(S\) decreases monotonically with time \(\Rightarrow\) Sufficient to consider \(I(S)\):

\[\frac{dI}{dS} = -1 + \frac{N}{\alpha S}\]

Solution is immediate: \(I = N - S + \frac{N}{\alpha} \ln \frac{S}{N-1}\)
Rate Equations (Cont’d)

\[
\frac{dI}{dS} = -1 + \frac{N}{\alpha S} \\
I = N - S + \frac{N}{\alpha} \ln \frac{S}{N - 1}
\]

Two cases:
- \(\alpha < \frac{N}{N-1} \approx 1\): I decreases monotonically to 0, i.e. infection immediately dies out.
- \(\alpha > 1\): I first increases, then decreases to 0.

\[
n = N - S|_{I=0} = rN
\]

where
\[
e^{-\alpha r} + r = 1
\]
\[
r \sim 2(\alpha - 1) \text{ as } \alpha \to 1^+,
\quad r \sim 1 - e^{-\alpha} \text{ as } \alpha \to \infty.
\]

Threshold at \(\alpha = 1\).
In principle, we could write down a master equation (or Fokker-Planck equation) for $P(S,I)$. Not stationary. Even to do WKB we would have to solve the semi-classical rate equations - not trivial.

Better idea: Do what we did to rate equations - eliminate time.

Define $T \equiv \text{number of transitions}$. In each transition, $I$ changes by $\pm 1$, Random Walk.

$S$ is given by $T$ and $I$:
- $I = 1 + T_+ - T_- = 1 + 2T_+ - T \Rightarrow T_+ = \frac{1}{2}(I + T - 1)$
- $S = N - 1 - T_+ = N - \frac{1}{2}(I + T + 1)$

Transition probabilities:

\[
p_- = \frac{1}{1 + \alpha S/N} = \frac{1}{1 + \alpha - \alpha(I + T + 1)/(2N)}
\]

\[
p_+ = 1 - p_-
\]
Stochastic Model, Infinite $N$ Limit

\[ p_- = \frac{1}{1 + \alpha S/N} = \frac{1}{1 + \alpha - \alpha (I + T + 1)/(2N)} \]

\[ p_+ = 1 - p_- \]

In initial stages, \( I + T + 1 \ll N \Rightarrow \) Simple Biased Random Walk, with Trap at Origin

For \( \alpha < 1 \):

- Walk biased toward origin.
- Avg. number of steps to hit origin from 1:

\[ 0 = 1 + \bar{T} \left( \frac{\alpha}{1 + \alpha} - \frac{1}{1 + \alpha} \right) \Rightarrow \bar{T} = \frac{1 + \alpha}{1 - \alpha} \]

Finite, so neglect of \( (I + T + 1)/N \) is always good.

- \( \bar{n} = \frac{1}{2}(\bar{T} + 1) = \frac{1}{1 - \alpha} \), Classic Result (Harris, 1989)
- Diverges as \( \alpha \to 1 \).
For $\alpha > 1$:

- Cannot use rate equations until $I$ is macroscopic.
- Finite chance that $I$ dies out before this.
- During this time, we can again neglect $(I + T + 1)/N$.
- Random Walk biased toward $\infty$.
- Probability of being absorbed at origin = $1/\alpha$.
- If $I$ escapes early extinction, then rate eqn. prediction is reliable. (Watson, 1980)

Thus,

$$\bar{n} = \left( 1 - \frac{1}{\alpha} \right) rN$$

- Always far from naive rate equation answer.
- $\bar{n} \sim 2(\alpha - 1)^2 N$ as $\alpha \to 1^+$. 
Clearly, both results $\alpha < 1 \ (n \rightarrow \infty)$, $\alpha > 1 \ (n \rightarrow 0)$ break down at threshold, $\alpha = 1$.

How wide is threshold region? (Ben-Naim and Krapivsky, 2004)

Define $\alpha - 1 \equiv \delta$.

$\bar{n} \sim O(1/\delta) \quad \alpha < 1$

$\bar{n} \sim O(\delta^2 N) \quad \alpha > 1$

$\Rightarrow \quad \delta \sim N^{-1/3} \quad \bar{n} \sim N^{1/3}$
Solution of Threshold Region

To analyze threshold region, we have to include \((I + T + 1)/N\) term in \(p_\pm\).

However, term is nevertheless small in threshold region, so

\[ p_\pm \approx \frac{1}{2} \mp \frac{1}{8N}(T + I - 2\delta N) \]

Extra drift is relevant for \(T \sim \delta N \sim N^{2/3}\)

For random walk, \(I \sim T^{1/2}\), so \(I\) term in drift is still irrelevant.

We are left with a random walk with a drift that increases linearly in "time".
Look at threshold \((\delta = 0)\) case first.

Solution without drift:

\[
P(T = 2k + 1) = 2^{-2k-1} \left( \binom{2k}{k} - \binom{2k}{k+1} \right) \sim \frac{1}{\sqrt{4\pi k^3}} \quad (k \to \infty)
\]

How does drift modify this result at long times?

Pass to Fokker-Planck equation:

\[
\frac{\partial P}{\partial T} = \frac{1}{2} \frac{\partial^2 P}{\partial I^2} + \frac{T}{4N} \frac{\partial P}{\partial I}
\]

Not separable!
Solution at Threshold (Cont’d)

Trick: Define

\[ P = e^{-IT/4N - T^3/(96N^2)} \psi \]

Then,

\[ \frac{\partial \psi}{\partial T} = \frac{1}{2} \frac{\partial^2 \psi}{\partial I^2} + \frac{I}{4N} \psi. \]

Boundary Conditions: \( \psi(0, T) = 0, \quad \psi(I, 0) = \delta(I - 1), \quad \psi(L, T) = 0 \)
(regularization)

After rescaling \( I \) by \( a \equiv (2N)^{1/3} \) and \( T \) by \( 2a^2 \), we get

\[ \frac{\partial \psi}{\partial T} = \frac{\partial^2 \psi}{\partial I^2} + I \psi \]

with \( \psi(I, 0) = \delta(I - 1/a)/a. \)

Eigenfunctions satisfy Airy equation!
Properties at Threshold

\[ P(n) = \frac{e^{-n^3/(12N^2)}}{\pi^2 a^3} \int_{-\infty}^{\infty} \frac{dE}{\text{Ai}^2(E) + \text{Bi}^2(E)} e^{En/a^2} \]

Asymptotics for small \( n \):
- Dominated by large negative \( E \), where \( \text{Ai}^2(E) + \text{Bi}^2(E) \approx (-E)^{-1/2} / \pi \)
  \[ P(n) \approx \frac{1}{\sqrt{4\pi n^{3/2}}} \]

Asymptotics for large \( n \):
- Dominated by \( \text{Bi}(E) \) giving maximum at \( E \sim n^2 \)
  \[ P(n) \approx \frac{1}{8\sqrt{\pi} N^2} n^{3/2} e^{-n^3/(16N^2)} \]
  - Strongly suppressed for \( n \gg N^{2/3} \)
  - Suppression found numerically by Ben-Naim & Krapivsky
Properties at Threshold

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Solution near Threshold

Return to $\delta \neq 0$.

Same Trick works:

$$P \equiv e^{-\frac{I(T-2\delta N)}{4N} - \frac{(T-2\delta N)^3-(2\delta N)^3}{96N^2}} \psi$$

so that

$$P_\delta(n) = e^{(n^2\delta N-n\delta^2 N^2)/(3N^2)} P_{\delta=0}(n)$$

For large $\delta$, this generates a second peak at $n \approx 2\delta N$, the "classical" rate equation answer.
Solution near Threshold

$n / N$

$P(n)$

$\delta = -1$
$\delta = 0$
$\delta = 1$
$\delta = 4$

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Consider $\bar{n}(\delta)$.

Asymptotics for large, negative $\delta$:
- Dominated by small $n$'s:
  $$\bar{n} \approx \int_0^\infty dn \ n e^{-n\delta^2/4} \frac{1}{2\sqrt{\pi}n^{3/2}} = -1/\delta$$
  - Matches on to subcritical result.

Asymptotics for large, positive $\delta$:
- Dominated by second, classical peak
  $$\bar{n} \approx 2\delta^2 N$$
  - Matches on to supercritical result.
Properties near Threshold

$$\frac{n}{N^{1/3}}$$

- $N = 10^3$
- $N = 10^4$
- $N = 10^5$

Large $N$

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THANK YOU
Does this work?

Calculated using exact 1-D master equation
Works as long as $\alpha$ not to close to threshold.
Does this work? (cont’d)

Hard to see, but again fails as $\alpha \to 1$. 

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Blow up region near $\alpha = 1$

Again, $N$-dependent boundary layer near threshold
Flux of $\psi$ to origin is:

$$F_\psi = \frac{1}{2} \frac{\partial \psi}{\partial I} \bigg|_{I=0} = \frac{1}{2a} \sum_n \phi'_n(0) \phi'_n(\frac{1}{a}) e^{E_n T}$$

$$\approx \frac{1}{2a^2} \sum_n (\phi'_n(0))^2 e^{E_n T}$$

(1)

Eigenfunctions:

$$\phi_n(I) = A_n \text{Ai}(-x + E_n) + B_n \text{Bi}(-x + E_n)$$

Boundary condition $\phi_n(0) = 0$ gives:

$$B_n = -\frac{A_n \text{Ai}(E_n)}{\text{Bi}(E_n)}$$

so,

$$\phi'_n(0) = -\frac{A_n}{\text{Bi}(E_n)} W_{r}(\text{Ai},\text{Bi}) = -\frac{A_n}{\pi \text{Bi}(E_n)}$$
Solution at Threshold (Cont’d)

\( A_n \) given by normalization:

\[
1 = \int_0^L \phi_n^2(I) dI = \left[ \phi_n' + (x - E) \phi_n \right]_0^L \\
= \left[ (\phi_n'(L))^2 - (\phi_n'(0))^2 \right] \approx \frac{(A_n^2 + B_n^2) L^{1/2}}{\pi} \tag{2}
\]

Change sum over \( n \) to integral over \( E \). Density of states:

\[
\frac{dn}{dE} \approx \frac{L^{1/2}}{\pi}
\]

Now, take away the cutoff \( L \) and undo the rescalings:

\[
P(n) = e^{-n^3/(12N^2)} \frac{\pi^2 a^3}{\int_{-\infty}^{\infty} \frac{dE}{\text{Ai}^2(E) + \text{Bi}^2(E)} e^{En/a^2}}
\]