LDP for SPDEs: a Martingale Approach

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SPDEs: Setting

$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, $u = u(t,x)$, $0 \leq t \leq T$, $x \in G$,
G is d-dimensional (can be $\mathbb{R}^d$);
$\{h_k,\, k \geq 1\}$ — CONS in $L_2(G)$,
w_k — independent Wiener processes; Itô Integral.

$$du = (\mathbf{A}u + f)dt + \sum_{k\geq 1}(\mathbf{M}_k u + g_k)dw_k(t),\; u(0,x) = u_0(x).$$

- **Additive noise:** $\mathbf{M}_k = 0$ for all k.
- **Multiplicative noise:** otherwise.

- **Space-time white noise:**

$$dW(t,x) = \sum_{k\geq 1} h_k(x)dw_k(t).$$

$\mathbf{M}_k u = h_k \mathbf{M}u$,
$g_k(t,x) = h_k(x)g(t,x)$:

$$du = (\mathbf{A}u + f)dt + (\mathbf{M}u + g)dW(t,x).$$

- **colored in space, white in time noise:**

$$dW_Q(t,x) = \sum_{k\geq 1} q_k h_k(x)dw_k(t),\; \lim_{k\to\infty} q_k = 0.$$

- **Solution** — Strong in every sense of the word:

$$(u,\varphi)(t) = (u_0,\varphi) + \int_0^t (\mathbf{A}u + f, \varphi)(s)ds$$
$$+ \sum_{k\geq 1}\int_0^t (\mathbf{M}_k u + g_k, \varphi)(s)dw_k(s).$$

Examples

- **Linear second-order equation**

$$du = (a_{ij}D_iD_ju + b_iD_i + cu + f)dt$$
$$+ (\sigma_{ik}D_iu + \nu_k u + g_k)dw_k.$$

$D_i = \partial/\partial x_i$,

$a, b, c, \sigma, \nu, f, g$ can be functions of t, x, ω.

- **Reaction-diffusion equation**

$du = (au_{xx} + f(u))dt + g(u)dW(t,x)$, $x \in \mathbb{R}$.

$f = 0$, $g(u) = \sqrt{u}$ — super-Brownian motion.

$f = 0$, $g(u) = \sqrt{u(1-u)}$ — Fleming-Viot model.

$f(u) = u(1-u)$, $g = 1$ — KPP-Fisher.

- **Deterministic XYZ equation + "noise" = stochastic XYZ equation.**

Additive space-time white noise is often the first choice.

SPDEs: Problems

Forward problems

- Existence, uniqueness, regularity of the solution

Inverse problems

- Statistical inference

Here, space-time white noise is more difficult.

Fun problems

- Averaging and homogenization
- Invariant measures
- Large deviations

Here, space-time white noise can help.

Example

Time-only noise is easier than space-time.

Time only: $du = \Delta u\, dt + u\, dw(t)$;

$$u(t,x) = \left(\frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-|x-y|^2/(4t)} u_0(y)dy \right) e^{w(t)-(t/2)}.$$

Compare this with

Space-time: $du = \Delta u\, dt + u\, dW(t,x)$;

- no closed-form solution;
- u Holder $< 1/2$ in space;
- u Holder $< 1/4$ in time;
- only if $d = 1$.

A Different Example

Time-only noise is harder than space-time.

$du = u_{xx}dt + \sqrt{\varepsilon}\, u\, dW(t,x),\ \ t \in (0,T),\ \ x \in (0,1)$:

Have LDP:

$$J(\psi) = \frac{1}{2}\int_0^T \int_0^1 \left(\frac{\psi_t(t,x) - \psi_{xx}(t,x)}{\psi(t,x)}\right)^2 dx\, dt$$

Ioffe (1991), Sowers (1992),
Chenal and Millet (1997), Cardon-Weber (1999)

Compare this with

$$du = u_{xx}dt + \sqrt{\varepsilon}\, u\, dw(t).$$

$$u(t,x) = u_{heat}(t,x)e^{\sqrt{\varepsilon}\, w(t) - (\varepsilon\, t/2)}.$$

Do we have an LDP here?

Spatial Effects of Time-Only Noise

$$du(t,x) = a\,u_{xx}\,dt + \sigma\,u_x\,dw(t),\ x \in \mathbb{R}.$$

$$\widehat{u}(t,y) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} u(t,x)e^{-\mathfrak{i}xy}dx,\ \mathfrak{i} = \sqrt{-1}.$$

$$d\widehat{u}(t,y) = -ay^2\widehat{u}(t,y)\,dt - \mathfrak{i}\sigma\,y\,\widehat{u}(t,y)\,dw(t)$$

$$\widehat{u}(t,y) = \widehat{u}(0,y)e^{-\left(a-(\sigma^2/2)\right)y^2t-\mathfrak{i}y\sigma w(t)}.$$

Parabolicity condition: $\sigma^2 < 2a$

Connection with physics: stochastic transport equation; $a - (\sigma^2/2)$ is the effective viscosity.

Proving LDP

$(\Omega, \mathcal{F}, \mathbb{P})$, a probability space.
(X, ρ), a metric space.
$(\xi_\varepsilon, \varepsilon > 0)$, a family of X-valued random elements.
$\exists\ \psi^\circ \in X :\ \lim_{\varepsilon \to 0} \mathbb{P}\Big(\rho(\xi_\varepsilon, \psi^\circ) > \delta\Big) = 0,\ \delta > 0.$
$G \subset X$, $\psi^\circ \notin G$.

Goal—LDP:

$$\mathbb{P}(\xi_\varepsilon \in G) \asymp e^{-\varepsilon^{-1} \inf\limits_{\psi \in G} J(\psi)}.$$

Need

Exponential Tightness:

$$\forall\ \psi \in X\ \exists\ K,\ \text{compact in } X :$$
$$\limsup_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}(\xi_\varepsilon \notin K) < -C.$$

AND

Local LDP:

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\Big(\rho(\xi_\varepsilon, \psi) < \delta\Big) = -J(\psi).$$

Local LDP, Upper Bound

- **Stochastic exponential:**

Find $\mathsf{H} = \mathsf{H}(\lambda, \psi) : X_1 \times X_2 \to \mathbb{R}$ so that X_1, X_2 are sufficiently large subsets of X, $\sup_\lambda \mathsf{H}(\lambda, \psi) = \mathsf{H}(\lambda^*, \psi)$, $\lambda^* = \lambda(\psi)$, and

$$\mathbb{E}e^{\varepsilon^{-1}\mathsf{H}(\lambda,\xi_\varepsilon)} = 1, \quad \lambda \in X_1.$$

- **Upper bound**:

$$1 \geq \mathbb{E}\Big(\big(\mathbf{1}_{\rho(\xi_\varepsilon,\psi)<\delta}\big)\ e^{\varepsilon^{-1}\mathsf{H}(\lambda,\xi_\varepsilon)}\Big) \gtrsim e^{\varepsilon^{-1}\mathsf{H}(\lambda,\psi)}\mathbb{P}\Big(\rho(\xi_\varepsilon,\psi) < \delta\Big);$$

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\Big(\rho(\xi_\varepsilon,\psi) < \delta\Big) \leq -\sup_\lambda \mathsf{H}(\lambda,\psi).$$

Reasonable guess: $J(\psi) = \sup_\lambda \mathsf{H}(\lambda,\psi) = \mathsf{H}(\lambda^*,\psi)$.

Local LDP, Lower Bound

- **Change of measure:**

$$d\widetilde{\mathbb{P}} = e^{\varepsilon^{-1}\mathsf{H}(\lambda^*,\xi_\varepsilon)}\ d\,\mathbb{P}$$

- **Shift of the limit:**

$$\lim_{\varepsilon\to 0}\widetilde{\mathbb{P}}\Big(\rho(\xi_\varepsilon,\psi)<\delta\Big)=1,\ \delta>0.$$

- **Lower bound:**

$$\mathbb{P}\Big(\rho(\xi_\varepsilon,\psi)<\delta\Big)=\widetilde{\mathbb{E}}\Bigg(\Big(\mathbf{1}_{\rho(\xi_\varepsilon,\psi)<\delta}\Big)\ e^{-\varepsilon^{-1}\mathsf{H}(\lambda^*,\xi_\varepsilon)}\Bigg)$$

$$\gtrsim e^{-\varepsilon^{-1}\mathsf{H}(\lambda^*,\psi)}\widetilde{\mathbb{P}}\Big(\rho(\xi_\varepsilon,\psi)<\delta\Big);$$

$$\lim_{\delta\to 0}\liminf_{\varepsilon\to 0}\varepsilon\ \ln\mathbb{P}\Big(\rho(\xi_\varepsilon,\psi)<\delta\Big)\ge -J(\psi).$$

Reference (SODEs): Liptser–Puhalskii (1992)

Example 1. $X=\mathcal{C}(0,T)$, $\xi_\varepsilon=\sqrt{\varepsilon}\,w(t)$, $\psi(0)=0$.

$$\mathrm{H}(\lambda,\psi)=\int_0^T \lambda(t)\psi'(t)dt-\frac{1}{2}\int_0^T \lambda^2(t)dt.$$

$$\lambda^*=\psi'=\frac{d\psi}{dt},\quad J(\psi)=\frac{1}{2}\int_0^T |\psi'(t)|^2dt.$$

Change of measure is Girsanov's Theorem

Example 2. $X=\mathcal{C}((0,T)\times\mathbb{R})$, $\xi_\varepsilon=u_\varepsilon(t,x)$,

$$du_\varepsilon=(u_\varepsilon)_{xx}\,dt+\sqrt{\varepsilon}\,u_\varepsilon\,dW(t,x).$$

$$\mathrm{H}(\lambda,\psi)=\int_0^T\int_{\mathbb{R}}\lambda(t,x)\Big(\psi_t(t,x)-\psi_{xx}(t,x)\Big)dx\,dt \\ -\frac{1}{2}\int_0^T\int_{\mathbb{R}}\psi^2(t,x)\lambda^2(t,x)\,dx\,dt.$$

$$\lambda^*(t,x)=\frac{\psi_t(t,x)-\psi_{xx}(t,x)}{\psi^2(t,x)},$$

$$J(\psi)=\frac{1}{2}\int_0^T\int_{\mathbb{R}}\left(\frac{\psi_t(t,x)-\psi_{xx}(t,x)}{\psi(t,x)}\right)^2dx\,dt.$$

Example 3. $X = \mathcal{C}((0,T)\times\mathbb{R})$, $\xi_\varepsilon = u_\varepsilon(t,x)$,

$$du_\varepsilon = (u_\varepsilon)_{xx}\,dt + \sqrt{\varepsilon}\,(u_\varepsilon)_x\,dw(t).$$

$$\mathsf{H}(\lambda,\psi) = \int_0^T\int_{\mathbb{R}}\lambda(t,x)\Big(\psi_t(t,x)-\psi_{xx}(t,x)\Big)dx\,dt \\ -\frac{1}{2}\int_0^T\left(\int_{\mathbb{R}}\psi_x(t,x)\lambda(t,x)\,dx\right)^2 dt.$$

If $\psi_t(t,x)-\psi_{xx}(t,x) = f(t)\psi_x(t,x)$, then

$$\lambda^*(t,x) = \lambda^*(t) = \frac{f(t)}{\int_{\mathbb{R}}\psi_x(t,x)dx},$$

AND

$$J(\psi) = \frac{1}{2}\int_0^T f^2(t)dt;$$

otherwise, $J(\psi) = +\infty$.

In general, consider

$$du_\varepsilon(t,x) = \mathbf{A}u_\varepsilon(t,x)dt + \sqrt{\varepsilon}\sum_{k\geq 1}\mathbf{M}_k u_\varepsilon(t,x)dw_k(t).$$

(Expected) Theorem

If $\psi_t(t,x) - \mathbf{A}\psi(t,x) = \sum_{k\geq 1} f_k(t)\mathbf{M}_k\psi(t,x)$, then

$$J(\psi) = \frac{1}{2}\inf\sum_{k\geq 1}\int_0^T f_k^2(t)dt.$$

Otherwise, $J(\psi) = +\infty$.

$$\lambda^* : \int_{\mathbb{R}} \lambda^*(t,x)\mathbf{M}_k\psi(t,x)dx = f_k(t).$$

Conclusion

LDP for SPDEs:

Easy answers, hard proofs.

What about Huge Deviations?