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Limit theorems with asymptotic expansions for stochastic processes.

There is a vast riches of limit theorems for sums of independent random variables: theorems about weak convergence, on large deviations, theorems with asymptotic expansions, etc. We can try to obtain the same kinds of theorems for families of stochastic processes.

If $X_1, X_2, ..., X_n, ...$ is a sequence of independent identically distributed random variables with expectation $EX_i = 0$ and variance $EX_i^2 = \sigma^2$, everybody knows the theorem about weak convergence of the distribution of the random variable $Z_n = (X_1 + ... + X_n)/\sqrt{n}$:

$$Ef(Z_n) \to Ef(Z_\infty) \qquad (n \to \infty) \tag{1}$$

for every bounded continuous function f, where Z_∞ is normal with parameters $(0, \sigma^2)$; or the same in terms of cumulative distribution functions:

$$F_{Z_n}(x) = P\{Z_n \le x\} \to F_{Z_\infty}(x) \quad (n \to \infty), \qquad -\infty < x < \infty. \tag{2}$$

If $E|X_i|^3 < \infty$, under some mild condition we have:

$$F_{Z_n}(x) = F_{Z_\infty}(x) - \frac{\gamma_3}{6\sqrt{n}} F_{Z_\infty}'''(x) + o(1/\sqrt{n}), \tag{3}$$

where $\gamma_3 = EX_i^3$; if $EX_i^4 < \infty$, under some stronger assumptions,

$$\begin{aligned} F_{Z_n}(x) = F_{Z_\infty}(x) - \frac{\gamma_3}{6\sqrt{n}} F_{Z_\infty}'''(x) \\ + \frac{1}{n} \cdot \text{some combination of } F_{Z_\infty}^{(4)}(x) \text{ and } F_{Z_\infty}^{(6)}(x) + o(1/n); \end{aligned} \tag{4}$$

etc.

For various families of stochastic processes $\xi^\varepsilon(t)$, $t \in [0, T]$, limit theorems about weak convergence of their function-space distributions have been obtained:

$$Ef(\xi^\varepsilon[0, T]) \to Ef(\xi^0[0, T]) \qquad (\varepsilon \to 0) \tag{5}$$

for every bounded continuous functional $f(x[0, T])$ on the space of functions on the interval $[0, T]$ (I show the interval in which the function is defined after the notation of the function), where $\xi^0(t)$, $t \in [0, T]$, is the limiting process.

Example: Suppose $\xi^\varepsilon(t)$ is, for every $\varepsilon > 0$, a Markov process with frequent, small jumps, making jumps of size $\varepsilon \cdot u$ according to the rate $\varepsilon^{-1} \cdot \mu_{t,x}(du)$, and staying at the same point between the jumps; that is, the process with generator

$$L_t^\varepsilon f(x) = \varepsilon^{-1} \int [f(x + \varepsilon u) - f(x)]\, \mu_{t,x}(du). \tag{6}$$

Let us introduce the notation $\alpha^j(t,\, x) = \displaystyle\int u^j\, \mu_{t,\, x}(du)$.

When $\varepsilon \downarrow 0$, the process $\xi^\varepsilon(t)$ with a certain initial value converges in probability to the solution of the equation $\dot{x}_*(t) = \alpha^1\bigl(t,\, x_*(t)\bigr)$ with the same initial condition (of course, the restriction of this solution being unique has to be introduced). This can be reformulated as a statement about weak convergence of the function-space distributions of $\xi^\varepsilon[0,\, T]$ to that of $x_*[0,\, T]$ (the function $x_*(t)$ being non-random, the right-hand side of (5) is nothing but $F(x_*[0,\, T])$).

The rate of convergence of $\xi^\varepsilon(t)$ towards $x_*(t)$ is $\varepsilon^{1/2}$; that is (under some more restrictions on $\mu_{t,\, x}(du)$) there exists a limit of the function-space distribution of the process $\zeta^\varepsilon(t) = \bigl(\xi^\varepsilon(t) - x_*(t)\bigr)/\varepsilon^{1/2}$:

$$Ef(\zeta^\varepsilon[0,\, T]) \to Ef(\zeta^0[0,\, T]), \tag{7}$$

the limiting process being the diffusion with generator

$$L_t^{\zeta^0} f(x) = \alpha_2^1\bigl(t,\, x_*(t)\bigr) \cdot x \cdot f'(x) + \frac{\alpha^2\bigl(t,\, x_*(t)\bigr)}{2}\, f''(x), \tag{8}$$

where the subscript $_2$ denotes differentiation in the second argument; this diffusion is Gaussian.

What form could asymptotic expansions be, making more precise the statement (5) of weak convergence? In the case of sums of independent random variables formulas (3), (4) suggest that the correction terms for the measure being the distribution of the random variable Z_n are – not measures, but rather countably additive set functions taking values of both signs (signed measures). However, there is little hope of getting something like this in the infinite-dimensional case.

Let us write the asymptotic expansion for $Ef(Z_n)$ corresponding to (3):

$$\begin{aligned} Ef(Z_n) &= \int_{-\infty}^{\infty} f(x)\, dF_{Z_n}(x) \\ &= \int_{-\infty}^{\infty} f(x)\, dF_{Z_\infty}(x) - \frac{\gamma_3}{6\sqrt{n}} \int_{-\infty}^{\infty} f(x)\, dF_{Z_\infty}'''(x) + o(1/\sqrt{n}) \end{aligned} \tag{9}$$

(under some restrictions). If the function $f(x)$ is three times continuously differentiable, we can integrate the second integral three times by parts, and write:

$$\begin{aligned} Ef(Z_n) &= \int_{-\infty}^{\infty} f(x)\, dF_{Z_\infty}(x) + \frac{\gamma_3}{6\sqrt{n}} \int_{-\infty}^{\infty} f'''(x)\, dF_{Z_\infty}(x) + o(1/\sqrt{n}) \\ &= E\bigl[f(Z_\infty) + \frac{\gamma_3}{6\sqrt{n}}\, f'''(Z_\infty)\bigr] + o(1/\sqrt{n}). \end{aligned} \tag{10}$$

This suggests that we can look for families of stochastic processes for limit theorems of the form

$$Ef(\xi^\varepsilon[0,\, T]) = E\bigl[f(\xi^0[0,\, T]) + k(\varepsilon) \cdot \boldsymbol{A}_1 f(\xi^0[0,\, T])\bigr] + o\bigl(k(\varepsilon)\bigr), \tag{11}$$

or

$$Ef(\xi^\varepsilon[0,\,T]) = E\bigl[f(\xi^0[0,\,T]) + k(\varepsilon)\cdot \boldsymbol{A}_1 f(\xi^0[0,\,T]) + k(\varepsilon)^2\cdot \boldsymbol{A}_2 f(\xi^0[0,\,T])\bigr] + o\bigl(k(\varepsilon)^2\bigr), \quad (12)$$

etc., where $k(\varepsilon) \to 0$ as $\varepsilon \to 0$, and $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, ... are linear operators acting on functionals: in the cases most similar to the case of sums of independent random variables, *differential* operators. These asymptotic expansions are likely to be obtained under some differentiability conditions on the functional $F(x[0,\,T])$.

In my papers: A refinement of the central limit theorem for stationary processes, *Teor. veroyatn. i primen.*, 1989, vol. 34, no. 3, pp. 451 – 464; Asymptotic expansions in limit theorems for stochastic processes. – I, *Probability Theory and Related Fields*, 1996, vol. 106, pp. 331 – 350; II, 1999, vol. 113, pp. 255 – 271; III, 2004, vol. 128, pp. 63 – 81, I obtained some results of this form – for Markov processes with frequent, small jumps, but also for some families of non-Markov processes (but for which the non-Markovness disappears in some sense in the limit).

In particular, a result of the form (11) is obtained for the process $\zeta^\varepsilon(t)$ in the above example, under the conditions of existence of the moment $\alpha^3(t,\,x)$ and some smoothness conditions (these conditions are, ultimately, imposed on the jump rate $\mu_{t,\,x}$).

Let us introduce differentiability conditions for the functional $f(x[0,\,T])$. We suppose that it has three continuous Gâteaux derivatives: the first derivative in the direction $\delta = \delta[0,\,T]$

$$Df(x[0,\,T])(\delta) = \lim_{h\to 0} \frac{f(x[0,\,T] + h\cdot\delta) - f(x[0,\,T])}{h}, \quad (13)$$

the second derivative in the directions δ_1, δ_2:

$$D^2 f(x[0,\,T])(\delta_1,\,\delta_2) = D\bigl(Df(x[0,\,T])(\delta_1)\bigr)(\delta_2), \quad (14)$$

etc.; and we suppose that the j-th derivative in the directions δ_1, ..., δ_j is represented in the form

$$D^j f(x[0,\,T])(\delta_1,\,...,\,\delta_j) = \underset{[0,\,T]^j}{\int\cdots\int} \delta_1(s_1)\cdot ...\cdot \delta_j(s_j)\; D^j f(x[0,\,T];\, ds_1 ... ds_j) \quad (15)$$

of an integral over the j-dimensional cube $[0,\,T]^j$, where $D^j f(x[0,\,T];\,\bullet)$ is a signed measure on $[0,\,T]^j$.

In typical examples, this signed measure, say, in the case of $j = 2$, consists of a part that has a two-dimensional density over the square $[0,\,T]^2$, plus a part that has a one-dimensional density on the sides $\{0\}\times[0,\,T]$, $\{1\}\times[0,\,T]$, $[0,\,T]\times\{0\}$, $[0,\,T]\times\{1\}$ and on the diagonal $\{(s,\,s):\ s\in[0,\,T]\}$ of the square, plus charges concentrated at the four vertices.

The factor $k(\varepsilon) = \varepsilon^{1/2}$, and the linear operator $\boldsymbol{A}_1$ is the differential operator

$$\boldsymbol{A}_1 f(x[0,\,T]) = \sum_{j=1}^{3} \int_{[0,\,T]^j} \Gamma_1^j(x[0,T];\, s_1,\,...,\,s_j)\; D^j f(x[0,\,T];\, ds_1 ... ds_j), \quad (16)$$

$$\Gamma_1^1(x[0,\,T];s_1) = \frac{1}{2}\int_0^{s_1} \alpha_{22}^1\big(t,\,x_*(t)\big)\,x(t)^2\,\exp\Big\{\int_0^{s_1} \alpha_2^1\big(v,x_*(v)\big)\,dv\Big\}\,dt, \tag{17}$$

$$\Gamma_1^2(x[0,\,T];\,s_1,\,s_2) = \frac{1}{2}\int_0^{\min(s_1,\,s_2)} \alpha_2^2\big(t,\,x_*(t)\big)\,x(t)\,\exp\Big\{\sum_{i=1}^{2}\int_0^{s_i} \alpha_2^1\big(v,\,x_*(v)\big)\,dv\Big\}\,dt, \tag{18}$$

$$\Gamma_1^3(x[0,\,T];\,s_1,\,s_2,\,s_3) = \frac{1}{6}\int_0^{\min(s_1,\,s_2,\,s_3)} \alpha^3\big(t,\,x_*(t)\big)\,\exp\Big\{\sum_{i=1}^{3}\int_0^{s_i} \alpha_2^1\big(v,\,x_*(v)\big)\,dv\Big\}\,dt. \tag{19}$$

Another kind of asymptotic expansions appears in the case when the "tails" of the jump distribution $\mu_{t,\,x}$ (not exactly a *distribution*: the total measure $\mu_{t,\,x}(-\infty,\,\infty)$ may be not equal to 1) do not decrease fast enough, and have power asymptotics.

Let us first consider again sums of independent random variables. If, say, $F_{X_i}(x) = P\{X_i \le x\} = c^-|x|^{-2.8} + o(|x|^{-3.4})$ as $x \to -\infty$, $F_{X_i}(x) = 1 - c^+x^{-2.8} + o(x^{-3.4})$ $(x \to \infty)$, the asymptotic expansion for $F_{Z_n}(x)$ has the form

$$F_{Z_n}(x) = F_{Z_\infty}(x) + \frac{1}{n^{0.4}}\,G(x) - \frac{\tilde{\gamma}_3}{6\sqrt{n}}\,F'''_{Z_\infty}(x) + o(1/n^{0.7}), \tag{20}$$

where $G(x)$ is a function depending on c^+ and c^-, $G(-\infty) = G(\infty) = 0$, and $\tilde{\gamma}_3$ is a constant: *not* the moment, because it does not exist: what we can call a *pseudomoment*:

$$\tilde{\gamma}_3 = \int_{-\infty}^{0} x^3\,d\big(F_{X_i}(x) - c^-|x|^{-2.8}\big) + \int_0^\infty x^3\,d\big(F_{X_i}(x) + c^+x^{-2.8}\big) \tag{21}$$

(in some papers, some different quantities are called pseudomoments). The result can be found (or almost can be found) in the paper: H.Cramér, On asymptotic expansions for sums of independent random variables with a limiting stable distribution, *Sankhyā*, 1963, A25, pp. 13 – 24.

A corresponding result for the stochastic process $\zeta^\varepsilon(t)$ of our example. Suppose, for simplicity, that the jump distribution $\mu_{t,\,x}$ does not depend on t, x: $\mu_{t,\,x} \equiv \mu$, and is concentrated on the right half-line ($\mu(-\infty,\,0] = 0$), so that $\xi^\varepsilon(t)$ is a process with independent increments with positive jumps only. Suppose this distribution has power "tails":

$$\mu(u,\,\infty) = c^+ \cdot u^{-2.8} + o(u^{-3.4}) \qquad (u \to \infty). \tag{22}$$

Then

$$Ef(\zeta^\varepsilon[0,\,T]) = E\big[f(\zeta^\varepsilon[0,\,T]) + \varepsilon^{0.4}\boldsymbol{A}_0 f(\zeta^\varepsilon[0,\,T]) + \varepsilon^{0.5}\boldsymbol{A}_1 f(\zeta^\varepsilon[0,\,T])\big] + o(\varepsilon^{0.7}), \tag{23}$$

where $\boldsymbol{A}_0$ is an integro-differential operator:

$$\boldsymbol{A}_0 f(x[0,\,T]) = c^+\int_0^T dt\int_0^\infty \big[f(x[0,T] + v\cdot I_{[t,\,T]}) - f(x[0,\,T]) - vDf(x[0,\,T])(I_{[t,\,T]}) - \frac{v^2}{2}\,D^2 f(x[0,\,T])(I_{[t,\,T]},\,I_{[t,\,T]})\big]\,d(-v^{-2.8}) \tag{24}$$

($I_{[t,\,T]}$ is the indicator function of the interval $[t,\,T]$), and $\boldsymbol{A}_1$ is a third-order differential operator, as in the case of finite third moment. (The results for stochastic processes are more transparent than Cramér's result for cumulative distribution functions: adding the summand $v \cdot I_{[t,\,T]}$ means making the sample function make a jump of size v at time $t \in (0,\,T]$.)

Also we can get an asymptotic expansion for the original process $\xi^\varepsilon(t)$ itself, with the limiting process $x_*(t) = x_0 + \alpha^1 \cdot t$ non-random; in this expansion we'll have no expectations in the right-hand side, but rather the values of the functionals at the function $x_*[0,\,T]$:

$$Ef(\xi^\varepsilon[0,\,T]) = f(x_*[0,\,T]) + \varepsilon \cdot \boldsymbol{A}_{-1} f(x_*[0,\,T]) + \varepsilon^{1.8} \cdot \boldsymbol{A}_0 f(x_*[0,\,T]) + \varepsilon^2 \cdot \boldsymbol{A}_1 f(x_*[0,\,T]) + o(\varepsilon^{2.4}), \tag{25}$$

where $\boldsymbol{A}_{-1}$ is a second-order differential operator.

Now, in my abstract I promised to speak also of large deviations.

Suppose for a family $\xi^\varepsilon(t)$, $0 \le t \le T$, of stochastic processes a law-of-large-numbers result holds: $\xi^\varepsilon(t)$ converges in probability to a non-random function $x_*(t)$. The results on large deviations are those about the asymptotics of probabilities $P\{\xi^\varepsilon[0,\,T] \in A\}$ for sets A at a positive distance from $x_*[0,\,T]$. Here belong also results about asymptotics of the expectations $Ef^\varepsilon(\xi^\varepsilon[0,\,T])$ for families of functionals $f^\varepsilon(x[0,\,T])$ if an essential part of the expectation is due to sample functions $\xi^\varepsilon[0,\,T]$ that are far from $x_*[0,\,T]$.

There are two opposite types of large-deviation results, which are clearly seen in the example of families of Markov processes with frequent, small jumps: those in which the probability of a large deviation is due mainly to sample functions $\xi^\varepsilon[0,\,T]$ that are close to some smooth functions $\phi(t)$, $0 \le t \le T$ (and the asymptotics in this case is exponential); and those in which the probabilities of large deviations are due mainly to sample functions with one or more large jumps. Possible are also results that are intermediate between these two types.

Large-deviation results of the first type have been extensively studied, while the second type has attracted but little interest. I think that results of this type are worth studying. Some results were obtained by a student of mine: V.V.Godovan'chuk, Asymptotic probabilities of large deviations due to large jumps of a Markov process, *Teor. veroyatn. i primen.*, 1981, vol. 26, no. 2, pp. 314 – 327.

Under some different conditions, such results can be obtained from limit theorems with asymptotic expansions of the type (22).

If the functional $f(x[0,\,T])$ is equal to 0 in some neighborhood of the (limiting) function $x_*[0,\,T]$, the problem of finding the asymptotics of $Ef(\xi^\varepsilon[0,\,T])$ is one about large deviations. For such functionals $f(x[0,\,T])$, $\boldsymbol{A}_{-1} f(x[0,\,T])$, $\boldsymbol{A}_1 f(x[0,\,T]) = 0$ ($\boldsymbol{A}_{-1}$ and $\boldsymbol{A}_1$ being *differential* operators), and formula (25) turns into

$$Ef(\xi^\varepsilon[0,\,T]) = \varepsilon^{1.8} \cdot c^+ \int_0^\infty dt \int_0^\infty f(x_*[0,\,T] + v \cdot I_{[t,\,T]})\; d(-v^{-2.8}) + o(\varepsilon^{2.4}). \tag{26}$$

I am sorry a little for having formulated no result precisely, and even not mentioning what kind of metric we are considering in our function space. True, for the Gâteaux derivatives this metric is not important.