BLACK HOLES AND QUBITS

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Abstract

- We establish a correspondence between the entanglement measures of qubits in quantum information theory and the Bekenstein-Hawking entropy of black holes in string theory.

- M. J. Duff, "String triality, black hole entropy and Cayley's hyperdeterminant," Phys. Rev. D **76**, 025017 (2007) [arXiv:hep-th/0601134]
- R. Kallosh and A. Linde, "Strings, black holes, and quantum information," Phys. Rev. D **73**, 104033 (2006) [arXiv:hep-th/0602061].
- P. Levay, "Stringy black holes and the geometry of entanglement," Phys. Rev. D **74**, 024030 (2006) [arXiv:hep-th/0603136].
- M. J. Duff and S. Ferrara, "E_7 and the tripartite entanglement of seven qubits," Phys. Rev. D **76**, 025018 (2007) [arXiv:quant-ph/0609227].
- P. Levay, "Strings, black holes, the tripartite entanglement of seven qubits and the Fano plane," Phys. Rev. D **75**, 024024 (2007) [arXiv:hep-th/0610314].

- M. J. Duff and S. Ferrara, "E_6 and the bipartite entanglement of three qutrits," Phys. Rev. D **76**, 124023 (2007) [arXiv:0704.0507 [hep-th]].
- P. Levay, "A three-qubit interpretation of BPS and non-BPS *STU* black holes," Phys. Rev. D **76**, 106011 (2007) [arXiv:0708.2799 [hep-th]].
- L. Borsten, D. Dahanayake, M. J. Duff, W. Rubens and H. Ebrahim, "Wrapped branes as qubits," Phys.Rev.Lett.100:251602,2008 arXiv:0802.0840 [hep-th].
- P. Levay, M. Saniga and P. Vrana, "Three-Qubit Operators, the Split Cayley Hexagon of Order Two and Black Holes," arXiv:0808.3849 [quant-ph].

Black holes and entanglement

1) BLACK HOLES AND ENTANGLEMENT

BH/qubit correspondence

- Quantum entanglement lies at the heart of quantum information theory, with applications to quantum computing, teleportation, cryptography and communication. In the apparently separate world of quantum gravity, the Bekenstein-Hawking entropy of black holes has also occupied center stage.
- Despite their apparent differences, recent work has established a correspondence between the tripartite entanglement measure of three qubits and the macroscopic entropy of the four-dimensional 8-charge *STU* black hole of $N = 2$ supergravity.

BH/qubit correspondence

- The measure of tripartite entanglement of three qubits (Alice, Bob and Charlie), known as the 3-tangle τ_{ABC}, and the entropy S of the 8-charge STU black hole of supergravity are related by:

$$S = \frac{\pi}{2}\sqrt{\tau_{ABC}}$$

Duff: hep-th/0601134

Further developments

- Further papers have written a more complete dictionary, which translates a variety of phenomena in one language to those in the other:
- For example, one can relate the classification of three-qubit entanglements to the classification of supersymmetric black holes as in the following table:

Table

Class	S_A	S_B	S_C	Det a	Black hole	Susy
A-B-C	0	0	0	0	small	1/2
A-BC	0	>0	>0	0	small	1/4
B-CA	>0	0	>0	0	small	1/4
C-AB	>0	>0	0	0	small	1/4
W	>0	>0	>0	0	small	1/8
GHZ	>0	>0	>0	<0	large	1/8
GHZ	>0	>0	>0	>0	large	0

Table: Classification of three-qubit entanglements and their corresponding $D = 4$ black holes.

Further developments continued

- The attractor mechanism on the black hole side is related to optimal local distillation protocols on the QI side; the supersymmetric and non-supersymmetric cases corresponding to the suppression or non-suppression of bit-flip errors .

Abstract

- There is also quantum information theoretic interpretation of the 56 charge $N = 8$ black hole in terms of a Hilbert space consisting of seven copies of the three-qubit Hilbert space.
- It relies on the decomposition $E_{7(7)} \supset [SL(2)]^7$ and admits the interpretation, via the Fano plane, of a tripartite entanglement of seven qubits, with the entanglement measure given by Cartan's quartic $E_{7(7)}$ invariant.
- Since the Fano plane provides the multiplication table of the octonions, this means that the octonions, often written off as a lost cause in physics (Penrose, Streater), may actually be testable in the laboratory.

Microscopic origin

- Nevertheless, we still do not know whether there are any physical reasons underlying these mathematical coincidences.
- With this in mind, we also turn our attention to connecting the qubits to the *microscopic* origin of the black hole entropy.

Further developments continued

- We consider the configurations of intersecting D3-branes, whose wrapping around the six compact dimensions T^6 provides the microscopic string-theoretic interpretation of the charges, and associate the three-qubit basis vectors $|ABC\rangle$, $(A, B, C = 0$ or $1)$ with the corresponding 8 wrapping cycles.
- To wrap or not to wrap: that is the qubit

Further developments continued

- In particular, we relate a well-known fact of quantum information theory, that the most general real three-qubit state can be parameterized by four real numbers and an angle, to a well-known fact of string theory, that the most general *STU* black hole can be described by four D3-branes intersecting at an angle.

Qubits

2) QUBITS

Two qubits

- The two qubit system Alice and Bob (where $A, B = 0, 1$) is described by the state

$$\begin{aligned} |\Psi\rangle &= a_{AB}|AB\rangle \\ &= a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle. \end{aligned}$$

- The bipartite entanglement of Alice and Bob is given by

$$\tau_{AB} = 4|\det \rho_A| = 4|\det a_{AB}|^2,$$

where

$$\rho_A = Tr_B|\Psi\rangle\langle\Psi|$$

- It is invariant under $SL(2)_A \times SL(2)_B$, with a_{AB} transforming as a $(2,2)$, and under a discrete duality that interchanges A and B.

Two qubits

- Example, separable state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|01\rangle$$

$$\tau_{AB} = 0$$

- Example, Bell state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

$$\tau_{AB} = 1$$

- EPR “paradox”

Three qubits

The three qubit system Alice, Bob and Charlie (where $A, B, C = 0, 1$) is described by the state

$$\begin{aligned}|\Psi\rangle &= a_{ABC}|ABC\rangle \\ &= a_{000}|000\rangle + a_{001}|001\rangle + a_{010}|010\rangle + a_{011}|011\rangle \\ &+ a_{100}|100\rangle + a_{101}|101\rangle + a_{110}|110\rangle + a_{111}|111\rangle.\end{aligned}$$

Cayley's hyperdeterminant

- The tripartite entanglement of Alice, Bob and Charlie is given by

$$\tau_{ABC} = 4|\mathrm{Det}\, a_{ABC}|,$$

Coffman et al: quant-ph/9907047

- $\mathrm{Det}\, a_{ABC}$ is Cayley's hyperdeterminant

$$\mathrm{Det}\, a_{ABC} = -\frac{1}{2}\varepsilon^{A_1A_2}\varepsilon^{B_1B_2}\varepsilon^{C_1C_4}\varepsilon^{C_2C_3}\varepsilon^{A_3A_4}\varepsilon^{B_3B_4}$$
$$\cdot\, a_{A_1B_1C_1}a_{A_2B_2C_2}a_{A_3B_3C_3}a_{A_4B_4C_4}$$

Symmetry

- Explicitly

$$
\begin{aligned}
&\text{Det}\ a_{ABC} = \\
&a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{100}^2 a_{011}^2 \\
&- 2(a_{000}a_{001}a_{110}a_{111} + a_{000}a_{010}a_{101}a_{111} \\
&+ a_{000}a_{100}a_{011}a_{111} + a_{001}a_{010}a_{101}a_{110} \\
&+ a_{001}a_{100}a_{011}a_{110} + a_{010}a_{100}a_{011}a_{101}) \\
&+ 4(a_{000}a_{011}a_{101}a_{110} + a_{001}a_{010}a_{100}a_{111}).
\end{aligned}
$$

- It is invariant under $SL(2)_A \times SL(2)_B \times SL(2)_C$, with a_{ABC} transforming as a $(2,2,2)$, and under a discrete triality that interchanges A, B and C.

Local entropy

Another useful quantity is the local entropy S_A, which is a measure of how entangled A is with the pair BC:

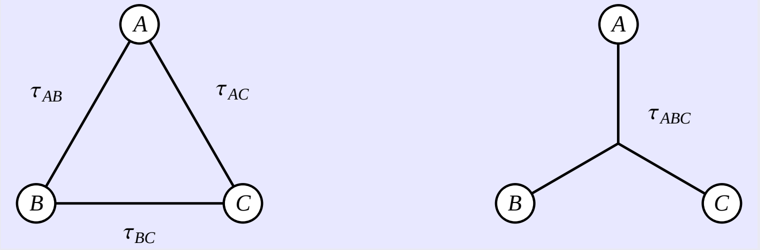
$$S_A = 4\det \rho_A \equiv \tau_{A(BC)}$$

where ρ_A is the reduced density matrix

$$\rho_A = \mathrm{Tr}_{BC}|\Psi\rangle\langle\Psi|,$$

and with similar formulae for B and C.

Tangles



- 2-tangles τ_{AB}, τ_{BC}, and τ_{CA} give bipartite entanglements between pairs in 3-qubit system
- 3-tangle τ_{ABC} is a measure of the genuine 3-way entanglement:

$$\tau_{ABC} = \tau_{A(BC)} - \tau_{AB} - \tau_{CA}$$

Entanglement classes

Entanglement classes

Class	$\tau_{A(BC)}$	$\tau_{B(AC)}$	$\tau_{(AB)C}$	τ_{ABC}
A-B-C	0	0	0	0
A-BC	0	> 0	> 0	0
B-CA	> 0	0	> 0	0
C-AB	> 0	> 0	0	0
W	> 0	> 0	> 0	0
GHZ	> 0	> 0	> 0	$\neq 0$

Complex qubit parameters

- Two states of a composite quantum system are regarded as equivalent if they are related by a unitary transformation which factorizes into separate transformations on the component parts, so-called *local unitaries*. The Hilbert space decomposes into equivalence classes, or *orbits* under the action of the group of local unitaries.

Complex qubit parameters

- For unnormalized three-qubit states, the number of parameters[Linden and Popescu: quant-ph/9711016] needed to describe inequivalent states or, what amounts to the same thing, the number of algebraically independent invariants [Sudbery: quant-ph/0001116] is given by the dimension of the space of orbits

$$\frac{\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2}{U(1) \times SU(2) \times SU(2) \times SU(2)}$$

namely, $16 - 10 = 6$.

Real qubit parameters

- However, for subsequent comparison with the *STU* black hole [Duff, Liu and Rahmfeld: hep-th/9508094; Behrndt et al: hep-th/9608059], we restrict our attention to states with *real* coefficients a_{ABC}.
- In this case, one can show that there are five algebraically independent invariants: $\mathrm{Det}\ \mathrm{a}$, S_A, S_B, S_C and the norm $\langle\Psi|\Psi\rangle$, corresponding to the dimension of

$$\frac{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2}{SO(2) \times SO(2) \times SO(2)}$$

namely, $8 - 3 = 5$.

5 parameter state

- Hence, the most general real three-qubit state can be described by just five parameters.
  Acin et al: quant-ph/0009107
- It may conveniently be written

$$|\Psi\rangle = -N_3\cos^2\theta|001\rangle - N_2|010\rangle + N_3\sin\theta\cos\theta|011\rangle - N_1|100\rangle - N_3\sin\theta\cos\theta|101\rangle + (N_0 + N_3\sin^2\theta)|111\rangle.$$

Representatives

Representatives from each class are:

- Class A-B-C (product states):

$$N_0|111\rangle.$$

- Classes A-BC, (bipartite entanglement):

$$N_0|111\rangle - N_1|100\rangle,$$

  and similarly B-CA, C-AB.

- Class W (maximizes bipartite entanglement):

$$-N_1|100\rangle - N_2|010\rangle - N_3|001\rangle.$$

- Class GHZ (genuine tripartite entanglement):

$$N_0|111\rangle - N_1|100\rangle - N_2|010\rangle - N_3|001\rangle.$$

STU black holes

3) STU BLACK HOLES

The *STU* model consists of $N = 2$ supergravity coupled to three vector multiplets interacting through the special Kahler manifold $[SL(2)/SO(2)]^3$:

$$\mathcal{S}_{\mathrm{STU}} = \frac{1}{16\pi G}\int \mathrm{e}^{-\eta}\Big[$$

$$\left(R + \frac{1}{4}\left(\mathrm{Tr}\left[\partial\mathcal{M}_T^{-1}\partial\mathcal{M}_T\right] + \mathrm{Tr}\left[\partial\mathcal{M}_U^{-1}\partial\mathcal{M}_U\right]\right)\right)\star 1$$

$$+\star\,\mathrm{d}\eta\wedge\mathrm{d}\eta - \frac{1}{2}\star H_{[3]}\wedge H_{[3]} - \frac{1}{2}\star F_{S[2]}^{\mathrm{T}}\wedge(\mathcal{M}_T\otimes\mathcal{M}_U)\,F_{S[2]}\Big]$$

$$\mathcal{M}_S = \frac{1}{\Im(S)}\begin{pmatrix} 1 & \Re(S) \\ \Re(S) & |S|^2 \end{pmatrix} \quad \text{etc.}$$

STU parameters

- A general static spherically symmetric black hole solution depends on 8 charges denoted $q_0, q_1, q_2, q_3, p^0, p^1, p^2, p^3$, but the generating solution depends on just $8 - 3 = 5$ parameters [Cvetic and Youm: hep-th/9512127; Cvetic and Hull: hep-th/9606193], after fixing the action of the isotropy subgroup $[SO(2)]^3$.

Black hole entropy

The STU black hole entropy is a complicated function of the 8 charges :

$$(S/\pi)^2 = -(p \cdot q)^2$$
$$+4\Big[(p^1 q_1)(p^2 q_2) + (p^1 q_1)(p^3 q_3) + (p^3 q_3)(p^2 q_2)$$
$$+q_0 p^1 p^2 p^2 - p^0 q_1 q_2 q_3\Big]$$

Behrndt et al: hep-th/9608059

Qubit correspondence

- By identifying the 8 charges with the 8 components of the three-qubit hypermatrix a_{ABC},

$$\begin{bmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \\ q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} a_{000} \\ -a_{001} \\ -a_{010} \\ -a_{100} \\ a_{111} \\ a_{110} \\ a_{101} \\ a_{011} \end{bmatrix}$$

one finds that the black hole entropy is related to the 3-tangle as in

$$S = \pi\sqrt{|\mathrm{Det}\ a_{ABC}|} = \frac{\pi}{2}\sqrt{\tau_{ABC}}$$

Duff: hep-th/0601134

Embeddings

The solution can usefully be embedded in

- $N = 4$ supergravity with symmetry $SL(2) \times SO(6, 22)$, the low-energy limit of the heterotic string compactified on T^6, where the charges transform as a $(2, 28)$.
- $N = 8$ supergravity with symmetry $E_{7(7)}$, the low-energy limit of the Type IIA or Type IIB strings, compactified on T^6 or M-theory on T^7, where the charges transform as a 56.

Remarkably, the same five parameters suffice to describe these 56-charge black holes.

Table

Class	S_A	S_B	S_C	Det a	Black hole	Susy
A-B-C	0	0	0	0	small	1/2
A-BC	0	> 0	> 0	0	small	1/4
B-CA	> 0	0	> 0	0	small	1/4
C-AB	> 0	> 0	0	0	small	1/4
W	> 0	> 0	> 0	0	small	1/8
GHZ	> 0	> 0	> 0	< 0	large	1/8
GHZ	> 0	> 0	> 0	> 0	large	0

Table: Classification of three-qubit entanglements and their corresponding $D = 4$ black holes.

$N = 8$ CASE

4) $N = 8$ CASE

$E_{7(7)}$

- There is, in fact, a quantum information theoretic interpretation of the 56 charge $N = 8$ black hole in terms of a Hilbert space consisting of seven copies of the three-qubit Hilbert space. It relies on the decomposition $E_{7(7)} \supset [SL(2)]^7$

Decomposition of the 56

- Under

$$E_{7(7)} \supset$$

$$SL(2)_A \times SL(2)_B \times SL(2)_C \times SL(2)_D \times SL(2)_E \times SL(2)_F \times SL(2)_G$$

the 56 decomposes as

$$56 \to$$

$$(2, 2, 1, 2, 1, 1, 1)$$
$$+(1, 2, 2, 1, 2, 1, 1)$$
$$+(1, 1, 2, 2, 1, 2, 1)$$
$$+(1, 1, 1, 2, 2, 1, 2)$$
$$+(2, 1, 1, 1, 2, 2, 1)$$
$$+(1, 2, 1, 1, 1, 2, 2)$$
$$+(2, 1, 2, 1, 1, 1, 2)$$

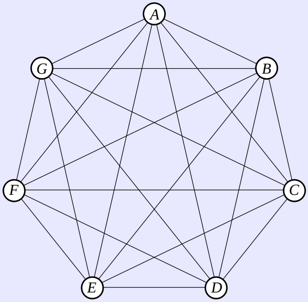
Seven qubits

- It admits the interpretation of a tripartite entanglement of seven qubits, Alice, Bob, Charlie, Daisy, Emma, Fred and George:

$$
\begin{aligned}
|\psi\rangle = \quad & a_{ABD}|ABD\rangle \\
& + b_{BCE}|BCE\rangle \\
& + c_{CDF}|CDF\rangle \\
& + d_{DEG}|DEG\rangle \\
& + e_{EFA}|EFA\rangle \\
& + f_{FGB}|FGB\rangle \\
& + g_{GAC}|GAC\rangle
\end{aligned}
$$

The following diagram may help illustrate the tripartite entanglement between the 7 qubits

E_7 Entanglement



Cartan invariant

- The entanglement measure given by Cartan's quartic $E_{7(7)}$ invariant.

$$I_4 = -\mathrm{Tr}((xy)^2) + \frac{1}{4}\mathrm{Tr}(xy)^2 - 4\left(\mathrm{Pf}(x) + \mathrm{Pf}(y)\right)$$

x^{IJ} and y_{IJ} are again 8×8 antisymmetric charge matrices
Duff and Ferrara: quant-ph/0609227
Levay: hep-th/0610314

x^{IJ}

$$x^{IJ} =$$

$$\begin{pmatrix}
0 & -a_{111} & -b_{111} & -c_{111} & -d_{111} & -e_{111} & -f_{111} & -g_{111} \\
a_{111} & 0 & f_{001} & d_{100} & -c_{010} & g_{010} & -b_{100} & -e_{001} \\
b_{111} & -f_{001} & 0 & g_{001} & e_{100} & -d_{010} & a_{010} & -c_{100} \\
c_{111} & -d_{100} & -g_{001} & 0 & a_{001} & f_{100} & -e_{010} & b_{010} \\
d_{111} & c_{010} & -e_{100} & -a_{001} & 0 & b_{001} & g_{100} & -f_{010} \\
e_{111} & -g_{010} & d_{010} & -f_{100} & -b_{001} & 0 & c_{001} & a_{100} \\
f_{111} & b_{100} & -a_{010} & e_{010} & -g_{100} & -c_{001} & 0 & d_{001} \\
g_{111} & e_{001} & c_{100} & -b_{010} & f_{010} & -a_{100} & -d_{001} & 0
\end{pmatrix}$$

y_{IJ}

$$y_{IJ} =$$

$$\begin{pmatrix}
0 & -a_{000} & -b_{000} & -c_{000} & -d_{000} & -e_{000} & -f_{000} & -g_{000} \\
a_{000} & 0 & f_{110} & d_{011} & -c_{101} & g_{101} & -b_{011} & -e_{110} \\
b_{000} & -f_{110} & 0 & g_{110} & e_{011} & -d_{101} & a_{101} & -c_{011} \\
c_{000} & -d_{011} & -g_{110} & 0 & a_{110} & f_{011} & -e_{101} & b_{101} \\
d_{000} & c_{101} & -e_{011} & -a_{110} & 0 & b_{110} & g_{011} & -f_{101} \\
e_{000} & -g_{101} & d_{101} & -f_{011} & -b_{110} & 0 & c_{110} & a_{011} \\
f_{000} & b_{011} & -a_{101} & e_{101} & -g_{011} & -c_{110} & 0 & d_{110} \\
g_{000} & e_{110} & c_{011} & -b_{101} & f_{101} & -a_{011} & -d_{110} & 0
\end{pmatrix}$$

I_4

Schematically,

$$\begin{aligned} I_4 = a^4 + b^4 + c^4 + d^4 + e^4 + f^4 + g^4 \\ + 2\Big[a^2b^2 + a^2c^2 + a^2d^2 + a^2e^2 + a^2f^2 + a^2g^2 \\ + b^2c^2 + b^2d^2 + b^2e^2 + b^2f^2 + b^2g^2 \\ + c^2d^2 + c^2e^2 + c^2f^2 + c^2g^2 \\ + d^2e^2 + d^2f^2 + d^2g^2 \\ + e^2f^2 + e^2g^2 \\ + f^2g^2\Big] \\ + 8\,[abce + bcdf + cdeg + defa + efgb + fgac + gabd]\,, \end{aligned}$$

where a^4 is Cayley's hyperdeterminant etc

$N = 8$ case

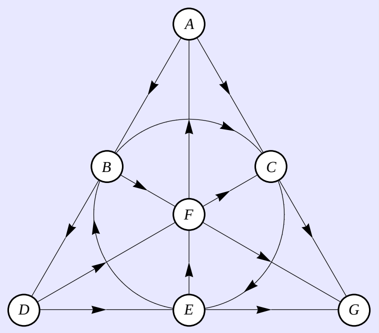
- Remarkably, because the generating solution depends on the same five parameters as the *STU* model, its classification of states will exactly parallel that of the usual three qubits. Indeed, the Cartan invariant reduces to Cayley's hyperdeterminant in a canonical basis.

Kallosh and Linde: hep-th/0602061

Fano plane

An alternative description is provided by the Fano plane which has seven points, representing the seven qubits, and seven lines (the circle counts as a line) with three points on every line, representing the tripartite entanglement, and three lines through every point.

Fano plane



Octonions

The Fano plane also provides the multiplication for the imaginary octonions:

	A	B	C	D	E	F	G
A		D	G	$-B$	F	$-E$	$-C$
B	$-D$		E	A	$-C$	G	$-F$
C	$-G$	$-E$		F	B	$-D$	A
D	B	$-A$	$-F$		G	C	$-E$
E	$-F$	C	$-B$	$-G$		A	D
F	E	$-G$	D	$-C$	$-A$		B
G	C	F	$-A$	E	$-D$	$-B$	

Wrapped D3-branes and 3 qubits

5) WRAPPED D3-BRANES AND 3 QUBITS

Microscopic analysis

- This is not unique since there are many ways of embedding the *STU* model in string/M-theory, but a useful one from our point of view is that of four D3-branes wrapping the $(579), (568), (478), (469)$ cycles of T^6 with wrapping numbers N_0, N_1, N_2, N_3 and intersecting over a string. Klebanov and Tseytlin: hep-th/9604166
- The wrapped circles are denoted by a cross and the unwrapped circles by a nought as shown in the following table:

4	5	6	7	8	9	macro charges	micro charges	$\|ABC\rangle$
x	o	x	o	x	o	p^0	0	$\|000\rangle$
o	x	o	x	x	o	q_1	0	$\|110\rangle$
o	x	x	o	o	x	q_2	$-N_3\sin\theta\cos\theta$	$\|101\rangle$
x	o	o	x	o	x	q_3	$N_3\sin\theta\cos\theta$	$\|011\rangle$
o	x	o	x	o	x	q_0	$N_0+N_3\sin^2\theta$	$\|111\rangle$
x	o	x	o	o	x	$-p^1$	$-N_3\cos^2\theta$	$\|001\rangle$
x	o	o	x	x	o	$-p^2$	$-N_2$	$\|010\rangle$
o	x	x	o	x	o	$-p^3$	$-N_1$	$\|100\rangle$

Table: Three qubit interpretation of the 8-charge $D=4$ black hole from four D3-branes wrapping around the lower four cycles of T^6 with wrapping numbers N_0, N_1, N_2, N_3.

Fifth parameter

- The fifth parameter θ is obtained by allowing the N_3 brane to intersect at an angle which induces additional effective charges on the $(579), (569), (479)$ cycles [Balasubramanian and Larsen: hep-th/ 9704143; Balasubramanian: hep-th/9712215; Bertolini and Trigiante: hep-th/0002191].
- The microscopic calculation of the entropy consists of taking the logarithm of the number of microstates and yields the same result as the macroscopic one [Bertolini and Trigiante: hep-th/0008201].

Qubit interpretation

- To make the black hole/qubit correspondence we associate the three T^2 with the $SL(2)_A \times SL(2)_B \times SL(2)_C$ of the three qubits Alice, Bob, and Charlie. The 8 different cycles then yield 8 different basis vectors $|ABC\rangle$ as in the last column of the Table, where $|0\rangle$ corresponds to xo and $|1\rangle$ to ox.
- We see immediately that we reproduce the five parameter three-qubit state $|\Psi\rangle$:

$$|\Psi\rangle = -N_3 \cos^2\theta|001\rangle - N_2|010\rangle + N_3 \sin\theta\cos\theta|011\rangle - N_1|100\rangle - N_3 \sin\theta\cos\theta|101\rangle + (N_0 + N_3 \sin^2\theta)|111\rangle.$$

- Note from the Table that the GHZ state describes four D3-branes intersecting over a string, or groups of 4 wrapping cycles with just one cross in common.

IIA and IIB

- Performing a T-duality transformation, one obtains a Type IIA interpretation with zero D6-branes, N_0 D0-branes, N_1, N_2, N_3 D4-branes plus effective D2-brane charges, where $|0\rangle$ now corresponds to xx and $|1\rangle$ to oo.

Wrapped D3-branes and 3 qubits

6) WRAPPED M2-BRANES AND 2 QUTRITS

Qutrit interpretation

- All this suggests that the analogy between $D = 5$ black holes and three-state systems (0 or 1 or 2), known as qutrits [Duff and Ferrara: 0704.0507 [hep-th]], should involve the choice of wrapping a brane around one of three circles in T^3. This is indeed the case, with the number of qutrits being two.
- The two-qutrit system (where $A, B = 0, 1, 2$) is described by the state

$$|\Psi\rangle = a_{AB}|AB\rangle,$$

and the Hilbert space has dimension $3^2 = 9$.

2-tangle

- The bipartite entanglement of Alice and Bob is given by the 2-tangle
$$\tau_{AB} = 27 \det \rho_A = 27 |\det a_{AB}|^2,$$
where ρ_A is the reduced density matrix
$$\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi|.$$
- The determinant is invariant under $SL(3)_A \times SL(3)_B$, with a_{AB} transforming as a $(3, 3)$, and under a discrete duality that interchanges A and B.

$D = 5$ black hole

- For subsequent comparison with the $D = 5$ black hole, we restrict our attention to unnormalized states with real coefficients a_{AB}.
- There are three algebraically independent invariants : τ_{AB}, C_2 (the sum of the squares of the absolute values of the minors of ρ_{AB}) and the norm $\langle\Psi|\Psi\rangle$, corresponding to the dimension of

$$\frac{\mathbb{R}^3 \times \mathbb{R}^3}{SO(3) \times SO(3)}$$

namely, $9 - 6 = 3$.

$D = 5$ black hole

- Hence, the most general two-qutrit state can be described by just three parameters, which may conveniently taken to be three real numbers N_0, N_1, N_2,.

$$|\Psi\rangle = N_0|00\rangle + N_1|11\rangle + N_2|22\rangle$$

- A classification of two-qutrit entanglements, depending on the rank of the density matrix, is given in the following table:

$D = 5$ table

Class	C_2	τ_{AB}	Black hole	Susy
A-B	0	0	small	1/2
Rank 2 Bell	> 0	0	small	1/4
Rank 3 Bell	> 0	> 0	large	1/8

Table: Classification of two-qutrit entanglements and their corresponding $D = 5$ black holes.

$D = 5$ black hole

- The 9-charge $N = 2$, $D = 5$ black hole may also be embedded in the $N = 8$ theory in different ways. The most convenient microscopic description is that of three M2-branes wrapping the (58), (69), (710) cycles of the T^6 compactification of $D = 11$ M-theory, with wrapping numbers N_0, N_1, N_2 and intersecting over a point [Papadopoulos and Townsend: hep-th/9603087; Klebanov and Tseytlin:hep-th/9604166].
- To make the black hole/qutrit correspondence we associate the two T^3 with the $SL(3)_A \times SL(3)_B$ of the two qutrits Alice and Bob, where $|0\rangle$ corresponds to xoo, $|1\rangle$ to oxo and $|2\rangle$ to oox.The 9 different cycles then yield the 9 different basis vectors $|AB\rangle$ as in the last column of the following Table:

$D = 5$ table

5	6	7	8	9	10	macro charges	micro charges	$\|AB\rangle$
x	o	o	x	o	o	p^0	N_0	$\|00\rangle$
o	x	o	o	x	o	p^1	N_1	$\|11\rangle$
o	o	x	o	o	x	p^2	N_2	$\|22\rangle$
x	o	o	o	x	o	p^3	0	$\|01\rangle$
o	x	o	o	o	x	p^4	0	$\|12\rangle$
o	o	x	x	o	o	p^5	0	$\|20\rangle$
x	o	o	o	o	x	p^6	0	$\|02\rangle$
o	x	o	x	o	o	p^7	0	$\|10\rangle$
o	o	x	o	x	o	p^8	0	$\|21\rangle$

- We see immediately that we reproduce the three parameter two-qutrit state $|\Psi\rangle$:

$$|\Psi\rangle = N_0|00\rangle + N_1|11\rangle + N_2|22\rangle$$

- The black hole entropy, both macroscopic and microscopic, turns out to be given by the 2-tangle

$$S = 2\pi\sqrt{|\det a_{AB}|},$$

  and the classification of the two-qutrit entanglements matches that of the black holes .

- Note that the non-vanishing cubic combinations appearing in $\det a_{AB}$ correspond to groups of 3 wrapping cycles with no crosses in common, i.e. that intersect over a point.

Embeddings

- There is, in fact, a quantum information theoretic interpretation of the 27 charge $N = 8, D = 5$ black hole in terms of a Hilbert space consisting of three copies of the two-qutrit Hilbert space. It relies on the decomposition $E_{6(6)} \supset [SL(3)]^3$ and admits the interpretation of a bipartite entanglement of three qutrits, with the entanglement measure given by Cartan's cubic $E_{6(6)}$ invariant.
  Duff and Ferrara: 0704.0507 [hep-th]
- Once again, however, because the generating solution depends on the same three parameters as the 9-charge model, its classification of states will exactly parallel that of the usual two qutrits. Indeed, the Cartan invariant reduces to $\det a_{AB}$ in a canonical basis.
  Ferrara and Maldacena: hep-th/9706097

SUMMARY

Summary

- Our Type IIB microscopic analysis of the $D = 4$ black hole has provided an explanation for the appearance of the qubit two-valuedness (0 or 1) that was lacking in the previous treatments: The brane can wrap one circle or the other in each T^2.
- The number of qubits is three because of the number of extra dimensions is six.
- The five parameters of the real three-qubit state are seen to correspond to four D3-branes intersecting at an angle.

Summary

- Our M-theory analysis of the $D = 5$ black hole has provided an explanation for the appearance of the qutrit three-valuedness (0 or 1 or 2) that was lacking in the previous treatments: The brane can wrap one of the three circles in each T^3.
- The number of qutrits is two because of the number of extra dimensions is six.
- The three parameters of the real two-qutrit state are seen to correspond to three intersecting M2-branes.

Summary

- It would be interesting to see whether we can now find an underlying physical justification