Hidden symmetry of correlation functions and amplitudes in $\mathcal{N} = 4$ SYM, part II

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Dualities in planar $\mathcal{N} = 4$ SYM

- Natural observables in a (conformal) gauge theory:

- Correlation functions: \[ G_n(x_i) = \langle \mathcal{O}(x_1) \mathcal{O}(x_2) \ldots \mathcal{O}(x_n) \rangle \]

- Scattering amplitudes: \[ A_n(p_i) = \langle p_1, p_2, \ldots, p_n | S | 0 \rangle \]

- Carry different/supplementary information about gauge theory:

\[ G_n = \text{off-shell (anomalous dimensions, structure constants of OPE)} \]
\[ A_n = \text{on-shell (S-matrix)} \]

- They are related to each other in planar $\mathcal{N} = 4$ SYM:

\[ \lim_{x_{i,i+1}^2 \to 0} \ln G_n(x_i) \sim 2 \ln A_n(p_i), \quad p_i = x_i - x_{i+1} \]

- Have a new hidden symmetry (ultimately related to integrability of $\mathcal{N} = 4$ SYM)

- Allows us to predict correlators/amplitudes at higher loops without any Feynman graph calculations!
Gluon amplitudes in $\mathcal{N} = 4$ SYM

- Four-gluon amplitude in $\mathcal{N} = 4$ SYM at weak coupling $\alpha = g^2 N_c / (8\pi^2)$

$$A^{+++−−}_4 / A^{(tree)}_4 = 1 + \alpha \, s t I^{(1)}(s, t) + O(\alpha^2),$$

Scalar box in the dimensional regularization (for IR divergences) with $D = 4 - 2\epsilon$

$$I^{(1)}(s, t) = x_2 x_3 x_4 x_5 \sim \int \frac{d^D x_5}{x_1^2 x_2^2 x_3^2 x_4^2 x_5^2}, \quad (x_{12}^2 = x_{23}^2 = x_{34}^2 = x_{41}^2 = 0)$$

Dual variables $p_i = x_i - x_{i+1}$ with $p_i^2 = x_{i,i+1}^2 = 0$

- (Broken) dual conformal symmetry
- All-loop BDS ansatz / AdS prediction / Wilson loop duality
- Explicit expressions for loop integrands up to 5 loops ... and even 7 loops [Marcus talk]
- Seemingly increasing complexity of diagrams at higher loops
Correlation functions

✔ Protected superconformal operators made from six real scalars $\Phi^I$

\[ \mathcal{O}(x) = \text{Tr}(ZZ), \quad \tilde{\mathcal{O}}(x) = \text{Tr}(\tilde{Z}\tilde{Z}), \quad Z = \Phi^1 + i\Phi^2 \]

✗ All-loop scaling dimension = tree level dimension

✗ Two- and three-point correlation functions do not receive quantum corrections

✔ Simplest correlation function

\[ G_4 = \langle \mathcal{O}(x_1)\tilde{\mathcal{O}}(x_2)\mathcal{O}(x_3)\tilde{\mathcal{O}}(x_4) \rangle = G_4^{(0)} \left[ 1 + 2a x_{13}^2 x_{24}^2 g(1, 2, 3, 4) + O(a^2) \right] \]

One-loop ‘cross’ integral

\[ g(1, 2, 3, 4) = \frac{1}{4\pi^2} \int \frac{d^4x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}, \quad (x_{12}^2, x_{23}^2, x_{34}^2, x_{41}^2 \neq 0) \]

✔ Loop corrections to the amplitude and to the correlator involve the same integral $g(1, 2, 3, 4)$ but for different kinematics: on-shell $x_{i,i+1}^2 = 0$ for $A_4$ and off-shell $x_{i,i+1}^2 \neq 0$ for $G_4$

✔ Amplitude/correlation function duality

\[ \lim_{x_{i,i+1}^2 \to 0} \ln \left( \frac{G_4}{G_4^{(0)}} \right) = \ln \left( \frac{A_4}{A_4^{(\text{tree})}} \right) \]

Understood at the level of integrands in planar $\mathcal{N} = 4$ SYM
A hidden symmetry

Examine one-loop correction to the correlator

\[ G_{4}^{(1)} \sim \int \frac{d^{4}x_{5}}{x_{15}^{2}x_{25}^{2}x_{35}^{2}x_{45}^{2}} \]

The corresponding integrand

\[ [G_{4}^{(1)}]_{\text{Integrand}} \sim \frac{1}{x_{15}^{2}x_{25}^{2}x_{35}^{2}x_{45}^{2}} \]

The r.h.s. has \( S_{4} \) permutation symmetry w.r.t. exchange of the external points 1, 2, 3, 4

Equivalent form of writing

\[ [G_{4}^{(1)}]_{\text{Integrand}} \sim x_{12}^{2}x_{13}^{2}x_{14}^{2}x_{23}^{2}x_{24}^{2}x_{34}^{2} \times \left[ \prod_{i<j} \frac{1}{x_{i,j}^{2}} \right] \]

The second factor in the r.h.s. has the complete \( S_{5} \) permutation symmetry!
Two loops

✔ Explicit two-loop calculation:

\[
G^{(2)} = h(1, 2; 3, 4) + h(3, 4; 1, 2) + h(2, 3; 1, 4) + h(1, 4; 2, 3) \\
+ h(1, 3; 2, 4) + h(2, 4; 1, 3) + \frac{1}{2} (x^2_{12} x^2_{34} + x^2_{13} x^2_{24} + x^2_{14} x^2_{23}) [g(1, 2, 3, 4)]^2
\]

\[h(1, 2; 3, 4)\] – ‘double’ scalar box integral

✔ Go to a common denominator

\[
G^{(2)}_4 = x^2_{12} x^2_{13} x^2_{14} x^2_{23} x^2_{24} x^2_{34} \int d^4 x_5 d^4 x_6 f^{(2)}(x_1, \ldots, x_6),
\]

✔ 7 integrals in \(G^{(2)}_4\) are described by a single \(f\)–function

\[
f^{(2)}(x_1, \ldots, x_6) = \frac{1}{48} \sum_{\sigma \in S_6} \frac{x^2_{\sigma_1 \sigma_2} x^2_{\sigma_3 \sigma_4} x^2_{\sigma_5 \sigma_6}} {\prod_{1 \leq i < j \leq 6} x^2_{ij}}
\]

Has the complete \(S_6\) permutation symmetry!

✔ Integrand of all-loop correlator has the complete permutation symmetry exchanging the external 1, 2, 3, 4 and internal, integration points (no need for the planar limit!)
Loop corrections to 4-point correlator

\[ G_4^{(\ell)}(1, 2, 3, 4) = x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2 \int d^4 x_5 \ldots d^4 x_{4+\ell} f^{(\ell)}(x_1, \ldots, x_{4+\ell}) , \]

✔ General form of \( f^{(\ell)} \) for arbitrary \( \ell \):

\[ f^{(\ell)}(x_1, \ldots, x_{4+\ell}) = \frac{P^{(\ell)}(x_1, \ldots, x_{4+\ell})}{\prod_{1 \leq i < j \leq 4+\ell} x_{ij}^2} , \]

Can be deduced from the OPE analysis of the tree-level correlator

✔ The polynomial \( P^{(\ell)} \) should satisfy the conditions:

✗ be invariant under \( S_{4+\ell} \) permutations of \( x_1, \ldots, x_{4+\ell} \)

✗ have a uniform conformal weight \( (1 - \ell) \) at each point, both external and internal

\[ P^{(\ell)}(x_i^{-1}) = \prod_{i=1}^{4+\ell} (x_i^2)^{-\ell+1} P^{(\ell)}(x_i) \]

✔ Graph theory solution:

\[ P^{(\ell)} \mapsto \text{Multi-graph with } (4 + \ell) \text{ vertices of degree } (\ell - 1) \]
Three loops

\[ P^{(3)} \rightarrow \text{Multi-graph with 7 vertices of degree 2} \]

(a) heptagon: \[ x_{12}^2 x_{23}^2 x_{34}^2 x_{45}^2 x_{56}^2 x_{67}^2 x_{71}^2 + S_7 \text{ permutations} \]

(b) 2-gon \times\text{pentagon}: \[ (x_{12}^4)(x_{34}^2 x_{45}^2 x_{56}^2 x_{67}^2 x_{73}^2) + S_7 \text{ permutations} \]

(c) triangle \times\text{square}: \[ (x_{12}^2 x_{23}^2 x_{31}^2)(x_{45}^2 x_{56}^2 x_{67}^2 x_{74}^2) + S_7 \text{ permutations} \]

(d) 2-gon \times\text{2-gon} \times\text{triangle}: \[ (x_{12}^4)(x_{34}^4)(x_{56}^2 x_{67}^2 x_{75}^2) + S_7 \text{ permutations} \]

\[ P^{(3)} = \text{linear combination of four terms with arbitrary coefficients} \]
Three loops II

From polynomial to integrand

\[
f^{(3)}(x_1, \ldots, x_7) = \sum_{i=a,b,c,d} C_i \frac{P_i^{(3)}(x_1, \ldots x_7)}{\prod_{1 \leq i < j \leq 7} x_{ij}^2}
\]

Contributions corresponding to four graphs for \( f^{(3)} \) (solid line = \( 1/x_{\sigma_i \sigma_j}^2 \), dashed line = \( x_{\sigma_i \sigma_j}^2 \)):

\( f^{(3)} = \text{linear combination of four diagrams with arbitrary coefficients } C_a, C_b, C_c, C_d \)

This holds in \( \mathcal{N} = 4 \) SYM for arbitrary gauge group \( SU(N_c) \)

All topologies except (b) are nonplanar, \( C_a = C_c = C_d = O(1/N_c^2) \), but how to fix \( C_b \)?

Number of coefficients to determine in the planar limit:

3 loops: 1, 4 loops: 3, 5 loops: 7, 6 loops: 36
OPE constraints

✔ Correlation function in the like-cone limit $x_{12}^2, x_{23}^2, x_{34}^2, x_{41}^2 \to 0$

$$\ln G_4(1, 2, 3, 4) \sim \Gamma_{\text{cusp}}(a) \ln u \ln v, \quad u, v \to 0$$

Conformal cross-ratios $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, v = \frac{x_{23}^2 x_{41}^2}{x_{13}^2 x_{24}^2}$

✔ Examine two-loop integrand

$$\ln G_4 \sim a G_4^{(1)} + a^2 \left[ G_4^{(2)} - \frac{1}{2} (G_4^{(1)})^2 \right]$$

$$= a \frac{x_{13}^2 x_{24}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} + a^2 \frac{x_{13}^2 x_{24}^2 [x_{13}^2 (x_{25}^2 x_{46}^2 + x_{45}^2 x_{26}^2) + x_{24}^2 (x_{36}^2 x_{15}^2 + x_{16}^2 x_{35}^2) - x_{13}^2 x_{24}^2 x_{56}^2]}{2 x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2 x_{16}^2 x_{26}^2 x_{36}^2 x_{46}^2 x_{56}^2}$$

Divergences come from integration over $x_5$ and $x_6$ approaching the light-like edges, e.g.

$x_5 \to x_1 - \alpha x_{12}$

$$x_{5i}^2 \to \alpha x_{1i}^2 + (1 - \alpha) x_{2i}^2, \quad 0 \leq \alpha \leq 1$$

✔ For the integral to have at most double-log asymptotics $\sim \ln u \ln v$ the polynomial in the numerator should vanish in this limit

✔ This condition alone fixes all the coefficients $C_i$. Checked to 2-, 3-, 4-, 5- and 6-loops.

Permutation symmetry + OPE constraints allow us to construct the integrand of $G_4$ in the planar limit up to 6 loops!
Four-point correlator at three loops

Our result for 4-point correlation function in planar $\mathcal{N} = 4$ SYM

$$G_4(1, 2, 3, 4) = G_4^{(0)} + aG^{(1)} + a^2G^{(2)} + a^3G^{(3)} + O(a^4)$$

$G^{(\ell)}$ are given by the sum of scalar Feynman integrals

$$G^{(1)} = g(1, 2, 3, 4),$$

$$G^{(2)} = [h(1, 2; 3, 4) + 5 \text{ perms}] + \frac{1}{2} \left[ x_{12}^2 x_{34}^2 (g(1, 2, 3, 4))^2 + 3 \text{ perms} \right],$$

$$G^{(3)} = [T(1, 3; 2, 4) + 11 \text{ perms}] + [E(2; 1, 3; 4) + 11 \text{ perms}] + [L(1, 3; 2, 4) + 5 \text{ perms}]$$

$$+ [(g \times h)(1, 3; 2, 4) + 5 \text{ perms}] + \frac{1}{2} [H(1, 3; 2, 4) + 11 \text{ perms}],$$

2- and 3-loop topologies:

$$h(1, 2; 3, 4) \quad T(1, 3; 2, 4) \quad E(1; 2, 4; 3) \quad L(1, 3; 2, 4) \quad g \times h(1, 3; 2, 4) \quad H(1, 2; 3, 4)$$
Correlation function/Amplitude duality

\[
\lim_{x_{i,i+1}^2 \to 0} \ln \left( \frac{G_4(x_i)}{G_4^{(0)}(x_i)} \right) = 2 \ln \left( \frac{A_4(p_i)}{A_4^{(0)}(p_i)} \right), \quad p_i = x_i - x_{i+1}
\]

Is understood at the level of *integrands* (and not in terms of divergent *integrals*)

\[
\lim_{x_{i,i+1}^2 \to 0} G^{(3)} = M^{(3)} + M^{(1)}M^{(2)}
\]

3-loop correlator:

\[
\lim_{x_{i,i+1}^2 \to 0} G^{(3)} = T(1, 3; 2, 4) + T(1, 3; 4, 2) + T(2, 4; 1, 3) + T(2, 4; 3, 1)
\]
\[
+ L(1, 3; 2, 4) + L(2, 4; 1, 3) + (g \times h)(1, 3; 2, 4) + (g \times h)(2, 4; 1, 3)
\]

Compare with 3-loop 4-gluon amplitude

\[
M^{(1)} = g(1, 2, 3, 4),
\]
\[
M^{(2)} = h(1, 3; 2, 4) + h(2, 4; 1, 3),
\]
\[
M^{(3)} = T(1, 3; 2, 4) + T(1, 3; 4, 2) + T(2, 4; 1, 3) + T(2, 4; 3, 1) + L(1, 3; 2, 4) + L(2, 4; 1, 3)
\]

Precise agreement with the amplitude/correlator duality!
Four loops

4-loop 4-point integrand *in the planar limit*

\[ f^{(4)}(1, \ldots, 8) = \frac{\sum_{\sigma \in S_8} [c_A P_A(x_{\sigma(i)}) + c_B P_B(x_{\sigma(i)}) - c_C P_C(x_{\sigma(i)})]}{\prod_{1 \leq i < j \leq 8} x_{ij}^2} , \]

Multi-graphs with 8 vertices of degree 3

Conformal polynomials

\[ P_A(x_1, \ldots, x_8) = \frac{1}{24} x_1^2 x_2 x_1 x_3 x_2^2 x_1 x_6 x_2 x_5 x_2 x_3 x_3 x_4 x_2 x_4 x_5 x_4 x_6 x_2 x_5 x_6 x_6 x_7 x_8 , \]

\[ P_B(x_1, \ldots, x_8) = \frac{1}{8} x_1^2 x_2 x_1 x_3 x_2 x_4 x_2 x_5 x_2 x_6 x_3 x_4 x_2 x_5 x_4 x_5 x_5 x_6 x_7 x_8 , \]

\[ P_C(x_1, \ldots, x_8) = \frac{1}{16} x_1^2 x_2 x_1 x_3 x_2 x_4 x_3 x_3 x_4 x_3 x_5 x_4 x_5 x_5 x_6 x_6 x_7 x_8 . \]

The coefficients \( c_A = c_B = c_C = 1 \) follow from the OPE constraint for \( \ln G_4 \) at 4 loops

We constructed integrand of 4-point correlator up to 6 loops!
Back to the amplitudes: 4-loop 4-gluons

Amplitude/correlator duality

$$M_4^{(4)} = \lim_{x^2_{i,i+1} \to 0} \left[ \frac{1}{2} G_4^{(4)} - \frac{1}{4} G_4^{(3)} G_4^{(1)} - \frac{1}{8} (G_4^{(2)})^2 + \frac{3}{16} G_4^{(2)} (G_4^{(1)})^2 - \frac{5}{128} (G_4^{(1)})^4 \right]$$

All pseudo-conformal integrals that contribute to four-loop four-point amplitude

Perfect agreement with the known 4-loop result [Bern,Czakon,Dixon,Kosower,Smirnov'06]

$$M_4^{(4)} = I^{(a)}(s, t) + I^{(a)}(t, s) + 2I^{(b)}(s, t) + 2I^{(b)}(t, s) + 2I^{(c)}(s, t) + 2I^{(c)}(t, s) + I^{(d)}_4(s, t)$$

$$+ I^{(d)}(t, s) + 4I^{(e)}(s, t) + 4I^{(e)}(t, s) + 2I^{(f)}(s, t) + 2I^{(f)}(t, s) - 2I^{(d2)}_4(s, t) - 2I^{(d2)}(t, s) - I^{(f2)}(s, t)$$

All 15 relative signs/coefficients follow from $c_A = c_B = c_C = 1$!

Agreement between correlators and amplitudes verified up to 6 loops
4-point correlator predicts integrand for 4-gluon amplitude in light-cone limit \( x_{i,i+1}^2 \to 0 \)

... and it contains a lot of information about anomalous dimension at short distances \( x_1 \to x_2 \) through the OPE

\[
\mathcal{O}(x_1) \mathcal{O}(x_2) = \frac{c_I}{x_{12}^4} \mathcal{I} + \frac{c_K(a)}{(x_{12}^2)^{1-\frac{1}{2} \gamma_K}} \mathcal{K}(x_2) + \ldots, \quad \mathcal{K} = \text{tr}[\Phi I \Phi I] - \text{Konishi operator}
\]

Asymptotics of 4-point correlator in the short-distance limit \( x_1 \to x_2, x_3 \to x_4 \)

\[
\ln G_4 \sim \frac{1}{2} \gamma_K(a) \ln(x_{12}^2 x_{34}^2) + \ldots
\]

Evaluation of \( \gamma_K(a) \) is reduced to extracting single-log part of a 2-point propagator-type integral for \( \ln G_4 \) – enormous simplification

Using state-of-the-art technology, we have been able to compute \( \gamma_K(a) \) up to five loops!

\[
\gamma_K(a) = 3a - 3a^2 + \frac{21}{4}a^3 + \left( -\frac{39}{4} + \frac{9}{4} \zeta_3 - \frac{45}{8} \zeta_5 \right)a^4
\]

\[
+ \left( \frac{27}{4} \zeta(3) - \frac{81}{16} \zeta(3)^2 - \frac{135}{16} \zeta(5) + \frac{945}{32} \zeta(7) + \frac{237}{16} \right)a^5 + O(a^6)
\]

The result exactly matches the AdS/CFT prediction [Bajnok,Hegedus,Janik,Lukowski'09]

Six loops does not seem impossible . . .
Conclusions

✔ The all-loop integrand of 4-point correlator possesses a complete symmetry under the exchange of the four external and all internal (integration) points

✔ This symmetry alone + OPE constraints allow us to construct 6-loop integrand of the correlation function in the planar limit (without doing Feynman diagram calculation!)

✔ In the light-cone limit, the scattering amplitude/correlator duality predicts the integrand for 4-gluon amplitude

✔ In the short-distance limit, the OPE leads to analytical result for the Konishi anomalous dimension at 5 loops

✔ Straightforward generalization to higher loops

✔ What are hidden symmetries of $\mathcal{N} = 4$ SYM:
  ✗ Dual (super)conformal symmetry of the amplitudes
  ✗ permutation symmetry of the correlator
  ✗ ???
5-loop 4-gluon amplitude

All pseudo-conformal integrals that contribute to five-loop four-point amplitude

The relative signs are determined from unitarity cuts

The complete five-loop four-point MSYM planar amplitude [Bern, Carrasco, Johansson, Kosower, Smirnov'07]

\[
M_4^{(5)} (1, 2, 3, 4) = -\frac{1}{32} \left[ (I_1 + 2I_2 + 2I_3 + 2I_4 + I_5 + I_6 + 2I_7 + 4I_8 + 2I_9 + 4I_{10} + 2I_{11} + 4I_{12} \\
+ 4I_{13} + 4I_{14} + 4I_{15} + 2I_{16} + 4I_{17} + 4I_{18} + 4I_{19} + 4I_{20} + 2I_{21} + 2I_{23} + 4I_{24} + 4I_{25} \\
+ 4I_{26} + 2I_{27} + 4I_{28} + 4I_{29} + 4I_{30} + 2I_{31} + I_{32} + 4I_{33} + 2I_{34} + \{s \leftrightarrow t\} + I_{22} \right]
\]