

The Harmony of Superstring Amplitudes: Implications from/to Field-Theory

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Harmony of scattering amplitudes from/in String Theory

I will report on my work with:

Carlos Mafra, DAMTP Cambridge

Oliver Schlotterer, AEI Potsdam

- Complete N-Point Superstring Disk Amplitude I.
Pure Spinor Computation, [arXiv:1106.2645](#)
- Complete N-Point Superstring Disk Amplitude II.
Amplitude and Hypergeometric Function Structure, [arXiv:1106.2646](#)
- Explicit BCJ Numerators from Pure Spinors, [arXiv:1104.5224](#)

Harmony of scattering amplitudes from/in String Theory

Scattering amplitudes in gauge and gravity theory
have a **remarkably rich yet simple structure**



Cabrillo Highway

Detour via string theory to arrive at harmony !

Harmony from/in String Theory

Various **relations within or between** gravity and gauge theory scattering amplitudes suggest a **unification** within or between these theories of the sort **inherent to string theory** !

Tree-level:

- Color decomposition of gauge theory amplitudes into sum over partial subamplitudes

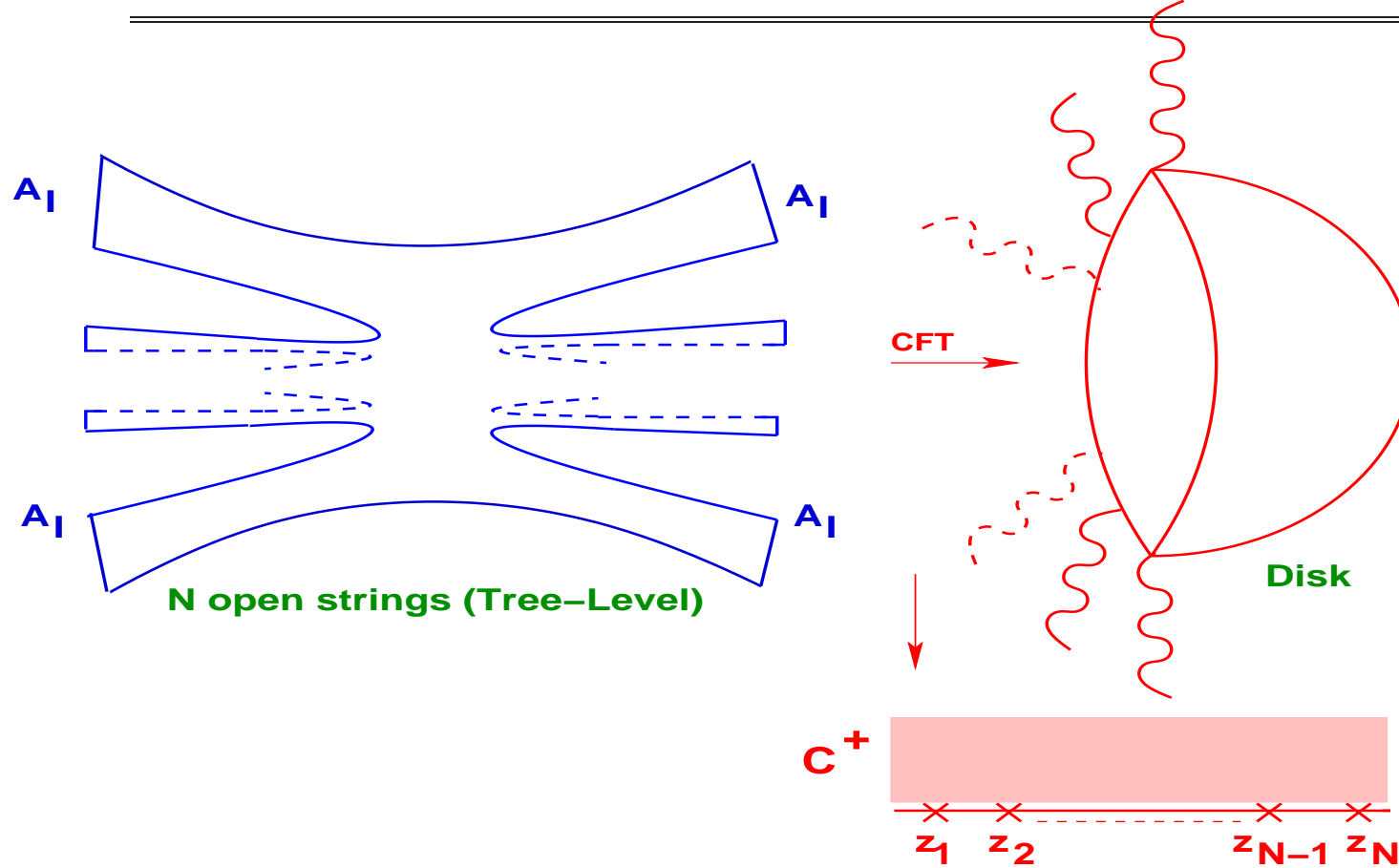
- BCJ: relations within gauge theory between color and kinematics

- KLT: relation between gravity and gauge theory

⋮

↔ (String) world-sheet derivation of amplitude relations
(By applying world-sheet string techniques \implies new algebraic identities)

Disk scattering of open strings



$$A(1, 2, \dots, N) = g_{YM}^{N-2} \sum_{\sigma \in S_N} \text{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \dots T^{a_{\sigma(N)}}) A(\sigma(1), \sigma(2), \dots, \sigma(N))$$

$A(1, 2, \dots, N)$ tree-level color-ordered N -leg partial amplitude (helicity subamplitude)

Disk scattering of open strings

N open string amplitude given by CFT computation with gluon vertex operators $V_g(z)$:

$$A(1, \dots, N; \alpha') = V_{\text{CKG}}^{-1} \int_{z_1 < \dots < z_N} \left(\prod_{j=1}^N dz_j \right) \langle V_g(z_1) \dots V_g(z_N) \rangle$$

$$A(1, \dots, N; \alpha') = V_{\text{CKG}}^{-1} \int_{z_1 < \dots < z_N} \left(\prod_{j=1}^N dz_j \right) \sum_{\mathcal{K}_I} \mathcal{K}_I \prod_{i < j}^N |z_i - z_j|^{s_{ij}} (z_i - z_j)^{n_{ij}^I}$$
$$s_{ij} = \alpha' (k_i + k_j)^2$$

↪ It would be desirable to obtain **simple** and **compact** result !

Full N -point open superstring amplitude

Compact and short expression in terms of a *minimal basis*
of $(N - 3)!$ *building blocks*

$$A(1, 2, \dots, N; \alpha') = \sum_{\sigma \in S_{N-3}} A_{YM}(1, 2_\sigma, \dots, (N-2)_\sigma, N-1, N) F_{(1, \dots, N)}^\sigma(\alpha')$$

Mafra, Schlotterer, St.St., arXiv:1106.2645 and arXiv:1106.2646

A_{YM} Yang–Mills subamplitudes

$F^\sigma(\alpha')$ generalized Euler integral
multiple Gaussian hypergeometric functions

Pure spinor formalism allows for remarkable **simplifications**
to package kinematics and α' -dependence

Structure of N -point superstring amplitude

Remarks:

$$\bullet N = 4 : \left\{ \begin{array}{l} A_{YM}(1, 2, 3, 4) \rightarrow 2g_{YM}^2 \frac{1}{su} t_8(\xi_1, k_1, \xi_2, k_2, \xi_3, k_3, \xi_4, k_4) \\ F_{1234} \rightarrow \frac{\Gamma(1-s) \Gamma(1-u)}{\Gamma(1-s-u)} \end{array} \right. \text{Green, Schwarz, 1982}$$

- $D = 4$, MHV

$$A_{YM}(1^-, 2^-, 3^+, \dots, N^+) = g_{YM}^{N-2} \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle N1 \rangle}, \quad \begin{array}{l} \text{Parke, Taylor, 1986} \\ \text{Berends, Giele, 1988} \end{array}$$

- $(N - 3)!$ dimensional minimal basis of hypergeometric functions F^σ

Structure of N -point superstring amplitude

- For any external state of SYM VM (FT = STTH Ward identities)
- Consider (adjacent) Eikonal Regge limit:
 - ▷ Keep $2(N - 3)$ kinematic invariants $s_{1i}, s_{2j}, s_{i,i+1} \neq 0$
 - ▷ For the remaining $\frac{1}{2}(N - 3)(N - 4)$ kinematic invariants $s_{ij} \rightarrow 0$

Cheung, O'Connell, Wecht, [arXiv:1002.4674](https://arxiv.org/abs/1002.4674)

$$F_{(1,\dots,N)}^\sigma \longrightarrow 0 \quad , \quad \sigma \neq (1,\dots,N)$$
$$\implies A(1, 2, \dots, N; \alpha') \longrightarrow A_{YM}(1, 2, \dots, N) F(\alpha')$$

Set of $(N - 3)!$ basis functions F^σ

$$F_{(1,\dots,N)}^\sigma(\alpha') = \int_{z_i < z_{i+1}} \prod_{j=2}^{N-2} dz_j \left(\prod_{i < l} |z_{il}|^{s_{il}} \right) \left\{ \prod_{k=2}^{N-2} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \right\}, \quad \sigma \in S_{N-3}$$

$$F_{(1,\dots,N)}^\sigma(\alpha') = \int_{z_i < z_{i+1}} \prod_{j=2}^{N-2} dz_j \left(\prod_{i < l} |z_{il}|^{s_{il}} \right) \times \left(\prod_{k=2}^{[N/2]} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \right) \left(\prod_{k=[N/2]+1}^{N-2} \sum_{n=k+1}^{N-1} \frac{s_{kn}}{z_{kn}} \right), \quad \sigma \in S_{N-3}$$

E.g. : $N = 5$

$$\left\{ \begin{array}{l} F^{(23)} = s_{12} s_{34} \int_0^1 dx \int_0^1 dy x^{s_{45}} y^{s_{12}-1} (1-x)^{s_{34}-1} (1-y)^{s_{23}} (1-xy)^{s_{24}} \\ = 1 + \zeta(2) (s_{13}s_3 - s_{34}s_4 - s_{15}s_5) \\ - \zeta(3) (s_1^2 s_3 + 2s_1 s_2 s_3 + s_1 s_3^2 - s_3^2 s_4 - s_3 s_4^2 - s_1^2 s_5 - s_1 s_5^2) + \mathcal{O}(\alpha'^4), \\ \\ F^{(32)} = s_{13} s_{24} \int_0^1 dx \int_0^1 dy x^{s_{45}} y^{s_{12}} (1-x)^{s_{34}} (1-y)^{s_{23}} (1-xy)^{s_{24}-1} \\ = \zeta(2) s_{13} s_{24} - \zeta(3) s_{13} s_{24} (s_1 + s_2 + s_3 + s_4 + s_5) + \mathcal{O}(\alpha'^4). \end{array} \right.$$

with: $s_i \equiv s_{i,i+1} = \alpha' (k_i + k_{i+1})^2$, $i + 5 \equiv i$

$$\begin{aligned}
B_N[n] &= \int_0^1 dx_1 \dots \int_0^1 dx_{N-3} \prod_{a=1}^{N-3} x_a^{1+a-N+n_a} \prod_{b=a}^{N-3} x_a^{2\alpha' k_{b+3}} \binom{b+2}{k_1 + \sum_{j=a+3}^{b+2} k_j} \\
&\times \left(1 - \prod_{j=a}^b x_j \right)^{2\alpha' k_{2+a} k_{3+b} + n_{ab}}, \quad b \geq a = 1, 2, \dots, N-3, \\
&\quad n_a, n_{ab} = 0, \pm 1
\end{aligned}$$

$\frac{1}{2}N(N-3)$ Laurent polynomials = number of kinematic invariants

N	dimension	function F^σ	reference
4	1	${}_2F_1$	Green, Schwarz, et al., 1982
5	2	${}_3F_2$	Medina, et al., hep-th/0208121
6	6	triple hypergeometric function $F^{(3)}$	Oprisa, St.St., hep-th/0509042
7	24	multiple hypergeometric function	St.St., Taylor, arXiv:0708.0574
N	$(N-3)!$	multiple hypergeometric function	Mafra, Schlotterer, St.St.

Structure of integrals is analyzed:

- ▷ *multiple pole* structure (dual channels)
- ▷ *transcendentality* properties
- ▷ *Gröbner basis* analysis to account for various integral relations

α' -expansion \iff multiple Euler–Zagier sums

E.g. $N=7$:

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \frac{x^{s_2} (1-x)^{s_3} y^{t_2} (1-y)^{s_4} z^{t_6} (1-z)^{s_5} w^{s_7} (1-w)^{s_6}}{(1-xy)(1-wz)(1-yz)} (1-wxyz)^{s_1-t_1+t_4-t_7}$$

$$\times (1-xy)^{-s_3-s_4+t_3} (1-wz)^{-s_5-s_6+t_5} (1-yz)^{-s_4-s_5+t_4} (1-wyz)^{s_5+t_1-t_4-t_5} (1-xyz)^{s_4-t_3-t_4+t_7}$$

$$= \mathcal{I}_0 + \mathcal{I}_{1a} (s_1 + s_2 + s_3 + s_4 + s_5 + s_6 + s_7) + \mathcal{I}_{1b} (t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7) + \mathcal{O}(\alpha'^2)$$

with Multiple Euler–Zagier sums $\mathcal{I}_0, \mathcal{I}_{1a}, \mathcal{I}_{1b}$:

$$\mathcal{I}_0 = \int \frac{1}{(1-xy)(1-wz)(1-yz)} = \sum_{\substack{n_1, n_2=0 \\ n_3=1}}^{\infty} \frac{1}{n_3 (1+n_1) (n_1+n_2+1) (n_2+n_3)} = \frac{27}{4} \zeta(4)$$

$$\mathcal{I}_{1a} = \int \frac{\ln w}{(1-xy)(1-wz)(1-yz)} = \sum_{\substack{n_1, n_3=1 \\ n_2=0}}^{\infty} \frac{1}{n_1 n_3^2 (n_1+n_2) (n_2+n_3)} = \frac{7}{2} \zeta(5) - 4\zeta(2)\zeta(3)$$

$$\mathcal{I}_{1b} = \int \frac{\ln y}{(1-xy)(1-wz)(1-yz)} = \sum_{\substack{n_1, n_2=1 \\ n_3=0}}^{\infty} \frac{1}{n_1 n_2 (n_2+n_3) (n_1+n_3)^2} = -\frac{9}{2} \zeta(5) + \zeta(2)\zeta(3)$$

I. Color vs. kinematics in string theory

String amplitude at fixed color ordering $(1, \dots, N)$
combines two sectors or moduli

$$A(1, \dots, N) \in \underbrace{A_{YM, \pi}}_{(N-3)! \text{ basis}} \otimes \underbrace{F^\pi(\alpha')}_{\text{dual basis ?}}$$

Can the integrals F^π be reduced to an $(N - 3)!$ basis ?

Or more generally:

Do we have an analog of
Kleiss–Kuijf and BCJ relations
in kinematic space ?

Recall: $(N - 3)!$ (independent) building blocks A_{YM} , i.e. for any Π :

$$A_{YM}(1_{\Pi}, \dots, N_{\Pi}) = \sum_{\sigma \in S_{N-3}} K_{\Pi}^{\sigma} A_{YM, \sigma}$$

• K_{Π}^{σ} can be derived from string theory monodromy relations $\alpha' \rightarrow 0$:

$$A(1, 2, \dots, N) + e^{i\pi s_{12}} A(2, 1, 3, \dots, N - 1, N) + e^{i\pi(s_{12} + s_{13})} A(2, 3, 1, \dots, N - 1, N) \\ + \dots + e^{i\pi(s_{12} + s_{13} + \dots + s_{1N-1})} A(2, 3, \dots, N - 1, 1, N) = 0$$

(imaginary part) field-theory relations (BCJ relations):

$$s_{12} A_{YM}(2, 1, 3, \dots, N - 1, N) + \dots + (s_{12} + s_{13} + \dots + s_{1N-1}) A_{YM}(2, 3, \dots, N - 1, 1, N) = 0$$

(real part) field-theory relations (Kleiss-Kuijf relations):

$$A_{YM}(1, 2, \dots, N) + A_{YM}(2, 1, 3, \dots, N - 1, N) + \dots + A_{YM}(2, 3, \dots, N - 1, 1, N) = 0$$

St.St., arXiv:0907.2211 & Bjerrum-Bohr, Damgaard, Vanhove, arXiv:0907.1425

• K_{Π}^{σ} can also be derived *directly* from string theory:

$$K_{\Pi}^{\sigma} = F_{\Pi}^{\sigma}(\alpha') \Big|_{\alpha'=0}$$

Color vs. kinematics in string theory

Further insights from looking at different representations (basis) for same amplitude:

$$A(1, \dots, N) = \sum_{\pi \in S_{N-3}} A_{YM, \pi} F_{(1, \dots, N)}^{\pi}(\alpha')$$

with some basis of $(N - 3)!$ independent basis amplitudes $A_{YM, \pi}$ yields:

$$F_{(1, \dots, N)}^{\sigma} = \sum_{\pi \in S_{N-3}} (K^{-1})_{\pi}^{\sigma} F_{(1, \dots, N)}^{\pi}$$

\implies for a given **fixed color ordering** $(1, \dots, N)$ any function F^{σ} (referring to the **kinematics** σ) may be expressed in terms of a basis of $(N - 3)!$ functions F^{π} referring to the **kinematics** π .

Color vs. kinematics in string theory

⇒ We have found a **dual system of equations** for the functions F
 (**complementary** to BCJ and KK relations)

reducing the set of functions to a **minimal basis** of dimension $(N-3)!$

reduction to	$A_{YM,\pi}$	$F^\pi(\alpha')$
$(N-2)!$	Kleiss–Kuijf	partial fraction $\frac{1}{z_{12}z_{23}} - \frac{1}{z_{13}z_{23}} = \frac{1}{z_{12}z_{13}}$ ⇒ Gröbner basis analysis
$(N-3)!$	BCJ	tot. derivatives $0 = \int \partial_k \prod_{i<j} z_{ij} ^{s_{ij}} \dots$

II. Effective D -brane action (α' -expansion)

Series of higher derivative terms (α' -corrections to SYM):

$$\mathcal{L}_{\text{effective}}^{Dp} = \text{Tr} \sum_{n \geq 4}^{\infty} \sum_{m=0}^{\infty} \alpha'^{n-2+m} \sum_{\substack{i_r \in \mathbb{N}, i_1 > 1 \\ i_1 + \dots + i_d = n-2+m}} \zeta(i_1, \dots, i_d) d_{m,n,\vec{i}} (t_{m,n}^{\vec{i}} D^{2m} F^n)$$

transcendentality degree

$$\sum_{r=1}^d i_r = n - 2 + m$$

and depth d

$$\zeta(i_1, \dots, i_d) = \sum_{n_1 > \dots > n_d > 0} \prod_{r=1}^d n_r^{-i_r}, \quad i_r \in \mathbb{N}, \quad i_1 > 1$$

α'^0 1	F^2			
α' 0	F^3	$D^2 F^2$		
α'^2 $\zeta(2)$	F^4	$D^2 F^3$	$D^4 F^2$	
α'^3 $\zeta(3)$	F^5	$D^2 F^4$	$D^6 F^2$	
α'^4 $\zeta(4)$	F^6	$D^4 F^4$	$D^2 F^5$	
α'^5 $\zeta(2)\zeta(3), \zeta(5)$	F^7	$D^6 F^4$	$D^4 F^5$	$D^2 F^6$
\vdots	\dots	\dots	\dots	\dots

Degree of transcendentality \iff order in α' -expansion

Arrange higher derivative terms

String contact terms: Non-trivial relations to field-theory

$$A(1, 2, \dots, N; \alpha') = \sum_{\sigma \in S_{N-3}} \underbrace{A_{YM}(1, 2_\sigma, \dots, (N-2)_\sigma, N-1, N)}_{\text{all kinematics}} \underbrace{F_{(1, \dots, N)}^\sigma(\alpha')}_{\text{expansion coefficients}}$$

E.g.: $N = 4$:

$$A(1, 2, 3, 4) = A_{YM}(1, 2, 3, 4) F$$

$$\implies \boxed{A_{F^4}(1, 2, 3, 4) = s u A_{YM}(1, 2, 3, 4)}$$

Arrange higher derivative terms

E.g.: $N = 5$: $\mathcal{A}(1, 2, 3, 4, 5) = A_{YM}(1, 2, 3, 4, 5) F_1 + A_{YM}(1, 3, 2, 4, 5) F_2$

$$\mathcal{A}(1, 2, 3, 4, 5) = A_{YM}(1, 2, 3, 4, 5) C_1 + A_{F^4}(1, 2, 3, 4, 5) C_2$$

$$\Rightarrow \boxed{A_{F^4}(1, 2, 3, 4, 5) = (s_{12}s_{34} - s_{34}s_{45} - s_{12}s_{51}) A_{YM}(1, 2, 3, 4, 5) + s_{13} s_{24} A_{YM}(1, 3, 2, 4, 5)}$$

Higher derivative terms

- *organized* according to the *YM amplitudes* A_{YM}
- *constructed* by YM amplitudes A_{YM}
(serve as *building blocks* in the effective action) with the expansion coefficients encoded in the functions F .
- *entirely described* in terms of the fundamental *YM three-vertices* (as a consequence of BCJ).

III. Degree of transcendentality in the α' -expansion

Transcendental power series expansion:

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{I_6(x, y, z)}{xyz} = \frac{1}{s_1 s_5 t_1} - \zeta(2) \left(\frac{s_3}{s_1 s_5} + \frac{s_4}{s_1 t_1} + \frac{s_2}{s_5 t_1} \right) + \zeta(3) \left(\frac{s_3 + s_4 - t_3}{s_1} + \frac{s_2 + s_3 - t_2}{s_5} + \frac{s_3^2 + s_3 t_1}{s_1 s_5} + \frac{s_4^2 + s_4 s_5}{s_1 t_1} + \frac{s_2^2 + s_1 s_2}{s_5 t_1} \right) + \mathcal{O}(\alpha').$$

To each power α'^n in α' a Riemann zeta constant of fixed weight $n + 3$ (with $n \geq -1$) appears

with: $I_6(x, y, z) = x^{s_5} y^{t_1} z^{s_1} (1-x)^{s_4} (1-y)^{s_3} (1-z)^{s_2} (1-xy)^{s_3 s_5} (1-yz)^{s_2 t_1} (1-xyz)^{s_2 s_5}$

Degree of transcendentality in the α' -expansion

On the other hand:

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{I_6(x, y, z)}{(1 - xy)(1 - yz)} = 2 \zeta(2) + [2 \zeta(2) - 4 \zeta(3)] (t_1 + t_2 + t_3) \\ - [2 \zeta(2) - \zeta(3)] (s_1 + s_2 + s_3 + s_4 + s_5 + s_6) + \mathcal{O}(\alpha^2) ,$$

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{I_6(x, y, z)}{(1 - xyz)^2} = \zeta(2) + \zeta(2) (s_3 + s_6 - t_2 - t_3) \\ - \zeta(3) (s_1 + s_2 + 2s_3 + s_4 + s_5 + 2s_6 + t_1 - t_2 - t_3) + \mathcal{O}(\alpha^2) ,$$

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{I_6(x, y, z)}{(1 - x)(1 - y)(1 - z)(1 - xyz)} = \alpha'^{-2} (\dots) + \alpha'^{-1} (\dots) \\ + \alpha'^0 [(\dots) + \zeta(2) (\dots)] + \alpha' [(\dots) + \zeta(2) (\dots) + \zeta(3) (\dots)] + \mathcal{O}(\alpha^2)$$

Transcendentality criterion from gluon amplitudes

↪ Regard the rational functions as originating from a CFT computation of a six-gluon amplitude:

$$A(1, \dots, N) = \langle c(z_1) c(z_{N-1}) c(z_N) \rangle \\ \times \left(\prod_{l=2}^{N-2} \int_{z_{l-1}}^1 dz_l \right) \langle V_g^{(-1)}(z_i) V_g^{(-1)}(z_j) \prod_{l \neq i, j}^N V_g^{(0)}(z_l) \rangle$$

with vertex operators:

$$V_g^{(-1)}(z) = g_A T^a e^{-\phi} \xi_\mu \psi^\mu e^{ik_\rho X^\rho} \\ V_g^{(0)}(z) = T^a \frac{g_A}{(2\alpha')^{1/2}} \xi_\mu \left[i\partial X^\mu + 2\alpha' (k_\lambda \psi^\lambda) \psi^\mu \right] e^{ik_\rho X^\rho},$$

and correlators:

$$\langle \partial X^\mu(z_1) X^\nu(z_2) \rangle = -\frac{2\alpha' \eta^{\mu\nu}}{z_{12}}, \quad \langle \psi^\mu(z_1) \psi^\nu(z_2) \rangle = \frac{\eta^{\mu\nu}}{z_{12}}$$

Degree of transcendentality in the α' -expansion

More details and structure in z -space:

$$\left[\begin{array}{l} z_1=0, z_2=xyz \\ z_3=xy, z_4=x, z_5=1 \end{array} \right]$$

$$\frac{1}{(1-xyz)^2} \simeq \frac{1}{z_{13}z_{14}z_{25}^2z_{36}z_{46}} ,$$

$$\frac{1}{(1-x)(1-y)(1-z)(1-xyz)} \simeq \frac{1}{z_{16}^2z_{23}z_{25}z_{34}z_{45}}$$

$$\int z_{ij}^{s_{ij}-2} r(z_{kl}) = \frac{1}{s_{ij}-1} \int r(z_{kl}) \partial_{z_i} z_{ij}^{s_{ij}-1} = -\frac{1}{s_{ij}-1} \int z_{ij}^{s_{ij}-1} \partial_{z_i} r(z_{kl})$$

$$\frac{1}{xyz} \simeq \frac{z_{16}^2}{z_{12}z_{13}z_{14}z_{15}z_{26}z_{36}z_{46}z_{56}} \rightarrow \frac{1}{z_{12}z_{13}z_{14}z_{15}} ,$$

$$\frac{1}{(1-xy)(1-yz)} \simeq \frac{1}{z_{13}z_{15}z_{24}z_{26}z_{35}z_{46}} \rightarrow \frac{1}{z_{13}z_{15}z_{24}z_{35}} ,$$

taking into account: $z_6 = z_\infty = \infty$
 and c -ghost correlator $\langle c(z_1)c(z_5)c(z_6) \rangle = z_{15}z_\infty^2$

Transcendentality criterion from gluon amplitudes

$$\frac{1}{xyz} \simeq \frac{1}{z_{12}z_{13}z_{14}z_{15}} \simeq (\xi_1\xi_6)(\xi_2k_1)(\xi_3k_1)(\xi_4k_1)(\xi_5k_1)$$

$$\xi_1^{\mu_1}\xi_2^{\mu_2}\xi_3^{\mu_3}\xi_4^{\mu_4}\xi_5^{\mu_5}\xi_6^{\mu_6}k_1^\lambda k_1^\sigma k_1^\rho k_1^\tau \langle \psi_1^{\mu_1}\psi_6^{\mu_6} \rangle \langle \partial X_2^{\mu_2} X_1^\lambda \rangle \langle \partial X_3^{\mu_3} X_1^\sigma \rangle \langle \partial X_4^{\mu_4} X_1^\rho \rangle \langle \partial X_5^{\mu_5} X_1^\tau \rangle$$

$$\frac{1}{(1-xy)(1-yz)} \simeq \frac{1}{z_{13}z_{15}z_{24}z_{35}} \simeq (\xi_2\xi_6)(\xi_1k_3)(\xi_3k_5)(\xi_4k_2)(\xi_5k_1)$$

$$\xi_1^{\mu_1}\xi_2^{\mu_2}\xi_3^{\mu_3}\xi_4^{\mu_4}\xi_5^{\mu_5}\xi_6^{\mu_6}k_1^{\lambda_1}k_2^{\lambda_2}k_3^{\lambda_3}k_5^{\lambda_5} \langle \psi_2^{\mu_2}\psi_6^{\mu_6} \rangle \langle \partial X_1^{\mu_1} X_3^{\lambda_3} \rangle \langle \partial X_3^{\mu_3} X_5^{\lambda_5} \rangle \langle \partial X_4^{\mu_4} X_2^{\lambda_2} \rangle \langle \partial X_5^{\mu_5} X_1^{\lambda_1} \rangle$$

$$\xi_1^{\mu_1}\xi_2^{\mu_2}\xi_3^{\mu_3}\xi_4^{\mu_4}\xi_5^{\mu_5}\xi_6^{\mu_6}k_1^{\lambda_1}k_2^{\lambda_2}k_3^{\lambda_3}k_5^{\lambda_5} \langle \psi_2^{\mu_2}\psi_6^{\mu_6} \rangle \langle \psi_1^{\mu_1}\psi_3^{\lambda_3} \rangle \langle \psi_3^{\mu_3}\psi_5^{\lambda_5} \rangle \langle \partial X_4^{\mu_4} X_2^{\lambda_2} \rangle \langle \psi_5^{\mu_5}\psi_1^{\lambda_1} \rangle$$

However:

$$\langle \partial X_1 X_3 \rangle \langle \partial X_3 X_5 \rangle \langle \partial X_5 X_1 \rangle - \langle \psi_1 \psi_3 \rangle \langle \psi_3 \psi_5 \rangle \langle \psi_5 \psi_1 \rangle = 0$$

IV. Higher-point closed superstring amplitudes

(Color ordered) gluon amplitudes give rise to graviton amplitudes in type I or Type II superstring theory (field-theory for $\alpha' \rightarrow 0$)

At tree-level:

gravity = gauge theory \otimes gauge theory

Amplitudes (on-shell S -matrix):

KLT relations

$$\begin{aligned}
 M_4(1, 2, 3, 4)_{S^2} &= (2\alpha'\pi)^{-1} \sin(\pi s_{12}) \bar{A}_4(1, 2, 3, 4)_{D_2} A_4(1, 2, 4, 3)_{D_2} \\
 M_5(1, 2, 3, 4, 5)_{S^2} &= (2\alpha'\pi)^{-2} \left\{ \sin(\pi s_{12}) \sin(\pi s_{34}) \bar{A}_5(1, 2, 3, 4, 5)_{D_2} A_5(2, 1, 4, 3, 5)_{D_2} \right. \\
 &\quad \left. + \sin(\pi s_{13}) \sin(\pi s_{24}) \bar{A}_5(1, 3, 2, 4, 5)_{D_2} A_5(3, 1, 4, 2, 5)_{D_2} \right\} \\
 &\quad \vdots
 \end{aligned}$$

E.g.: $M(1^-, 2^-, 3^+, 4^+) = \left(\frac{\kappa}{2}\right)^2 \frac{\langle 12 \rangle^8 [12]}{N(4) \langle 34 \rangle} \frac{B(s_{12}, s_{14})}{B(-s_{12}, -s_{14})} \rightarrow \left(\frac{\kappa}{2}\right)^2 \frac{\langle 12 \rangle^8 [12]}{N(4) \langle 34 \rangle}$

with: and: $\langle ij \rangle [ij] = s_{ij} = \alpha' k_i k_j$, $N(n) = \prod_{i=1}^{n-1} \prod_{j=i+1}^n \langle ij \rangle$

Tree-level higher order gravitational couplings

For $N = 4$, use: $\frac{B(s,u)}{B(-s,-u)} = -e^{-2 \sum_{n=1}^{\infty} \frac{\zeta(2n+1)}{2n+1} (s^{2n+1} + t^{2n+1} + u^{2n+1})}$

	$N = 4$	$N = 5$	$N = 6$	$N = 7$	$N = 8$
$\alpha'^3 \zeta(3)$	R^4				
$\alpha'^4 \zeta(4)$	$D^2 R^4$	R^5			
$\alpha'^5 \zeta(5)$	$D^4 R^4$	$D^2 R^5$	R^6		
$\alpha'^5 \zeta(2)\zeta(3)$	$D^4 R^4$	$D^2 R^5$	R^6		
$\alpha'^6 \zeta(3)^2$	$D^6 R^4$	$D^4 R^5$	$D^2 R^6$	$R^7 ?$	
$\alpha'^6 \zeta(6)$	$D^6 R^4$	$D^4 R^5$	$D^2 R^6$	$R^7 ?$	
$\alpha'^7 \zeta(7)$	$D^8 R^4$	$D^6 R^5$	$D^4 R^6$	$D^2 R^7 ?$	$R^8 ?$
$\alpha'^7 \zeta(3)\zeta(4)$	$D^8 R^4$	$D^6 R^5$	$D^4 R^6$	$D^2 R^7 ?$	$R^8 ?$
$\alpha'^7 \zeta(2)\zeta(5)$	$D^8 R^4$	$D^6 R^5$	$D^4 R^6$	$D^2 R^7 ?$	$R^8 ?$
$\alpha'^8 \zeta(3)\zeta(5)$	$D^{10} R^4$	$D^8 R^5$	$D^6 R^6$	$D^4 R^7 ?$	$D^2 R^8 ?$
$\alpha'^8 \zeta(8)$	$D^{10} R^4$	$D^8 R^5$	$D^6 R^6$	$D^4 R^7 ?$	$D^2 R^8 ?$
$\alpha'^8 \zeta(2)\zeta(3)^2$	$D^{10} R^4$	$D^8 R^5$	$D^6 R^6$	$D^4 R^7 ?$	$D^2 R^8 ?$
$\alpha'^8 \zeta(5, 3)$	$D^{10} R^4$	$D^8 R^5$	$D^6 R^6$	$D^4 R^7 ?$	$D^2 R^8 ?$

Relevance and implications to field–theory

\implies Constraints on higher order gravitational couplings:
very restricted sets of MZVs appear

Results constrain candidate counter terms in $N=8$ SUGRA:
serve as unique candidate counter terms for $D = 4, N = 8$
SUGRA for loop levels $L \leq 6$

[Terms $D^4 R^4$ and $D^6 R^4$ represent the only local
supersymmetric and $SU(8)$ –symmetric operators
invariant under $D = 4, N = 8$ SUGRA
(matrix elements fulfill SUSY Ward identities)]

$D^4 R^4, D^6 R^4$ have non–vanishing single–soft scalar limits
 \implies operators violate continuous $E_{7(7)}$ –symmetry
 \implies no counter terms at 5– and 6–loop

Beisert, Elvang, Freedman, Kiermaier, Morales, St. St. [arXiv:1009.1643](https://arxiv.org/abs/1009.1643)

Multi zeta values (MZVs)

Many relations over \mathbb{Q} , e.g.:

$$\begin{aligned} \zeta(2, 1) &= 2 \zeta(3) \quad , \quad \zeta(4, 1) = 2 \zeta(5) - \zeta(2) \zeta(3) \\ \zeta(5, 3) &= -\frac{5}{2} \zeta(6, 2) - \frac{21}{25} \zeta(2)^4 + 5 \zeta(3) \zeta(5) \\ &\vdots \end{aligned}$$

The set of integral linear combinations of MZVs is a ring

e.g.: $\zeta(m) \zeta(n) = \zeta(m, n) + \zeta(n, m) + \zeta(m + n)$

Zagier: For a given weight $w \in \mathbb{N}$ the dimension d_w of the space spanned by MZVs: $d_w = d_{w-2} + d_{w-3}$, $d_0, d_1 = 0$,

w	d_w	basis
2	1	$\zeta(2)$
3	1	$\zeta(3)$
4	1	$\zeta(2)^2$
5	2	$\zeta(5), \zeta(2)\zeta(3)$
6	2	$\zeta(2)^3, \zeta(3)^2$
7	3	$\zeta(7), \zeta(2)\zeta(5), \zeta(3)\zeta(2)^2$
8	4	$\zeta(2)^4, \zeta(2)\zeta(3)^2, \zeta(3)\zeta(5), \zeta(5, 3)$
9	5	$\zeta(2)^3\zeta(3), \zeta(3)^3, \zeta(2)^2\zeta(5), \zeta(2)\zeta(7), \zeta(9)$
10	7	$\zeta(2)^5, \zeta(2)^2\zeta(3)^2, \zeta(2)\zeta(3)\zeta(5), \zeta(5)^2, \zeta(3)\zeta(7), \zeta(2)\zeta(5, 3), \zeta(7, 3)$

Gravitational amplitudes in superstring theory

Question: Can we also cast the gravity amplitude M into compact form with simple building blocks ?

KLT:
$$M(1, \dots, N)_{S^2} \sim \sum_{\sigma, \rho} e^{i\pi\phi(\sigma, \rho)} \bar{A}_N(\rho)_{D_2} A_N(\rho)_{D_2}$$

sum over $\frac{1}{2}(N-1)! \times \frac{1}{2}(N-1)!$ open string amplitudes

↔ With our gluon results we can cast $M(1, 2, \dots, N)$ in terms of a basis of $(N-3)! \times (N-3)!$ building blocks $\bar{A}_{YM, \sigma} \times A_{YM, \pi}$:

$$M(1, \dots, N)_{S^2} = \sum_{\sigma, \rho \in S_{N-3}} \mathcal{M}(\rho, \sigma) \bar{A}_{YM}(\rho) A_{YM}(\sigma)$$

Gravitational amplitudes in superstring theory

For $N = 4$:

$$M(1, 2, 3, 4) = \frac{B(s_{12}, s_{14})}{B(-s_{12}, -s_{14})} \frac{s_{12} s_{23}}{s_{13}} |A_{YM}(1, 2, 3, 4)|^2$$

For $N = 5$:

$$M(1, 2, 3, 4, 5)_{FT} = (s_1 s_2 s_4 + s_1 s_3 s_4 + s_1 s_3 s_5 + s_2 s_3 s_5 + s_2 s_4 s_5)^{-1} \\ \times \left\{ s_{12} s_{23} s_{34} s_{45} s_{51} |A_{YM}(1, 2, 3, 4, 5)|^2 \right. \\ \left. - s_{13} s_{14} s_{24} s_{25} s_{35} |A_{YM}(1, 3, 5, 2, 4)|^2 \right\}$$

↪ more investigation on world-sheet symmetries

C. Mafra, O. Schlotterer, S. Stieberger, work in progress.