Holographic Spacetimes
From Entanglement Renormalization

Brian Swingle
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1. Entanglement
2. Motivations
3. Related results
4. Preview

MOTIVATION AND BACKGROUND
Entanglement

Renyi entropy:

\[ S_n(A) = S_n(B) = \ln 2 \]

\[ S_n = \frac{1}{1-n} \ln \left( \text{tr}(\rho_A^n) \right) \]

\[ S(A) = \lim_{n \to 1} S_n(A) \]
## Why do we care?

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My dream: entanglement renormalization everywhere!

Intermediate value theorem: if it works for Ising and it works for SYM then it works for everything!
Renormalization

An example:

\[ H = -w \sum_{r \neq \sigma} c_r^\dagger c_r\sigma + h.c. + \sum_r U n_{r\uparrow} n_{r\downarrow} - \sum_r \mu (n_{r\uparrow} + n_{r\downarrow}) \]

\[ S = \sum \int d\hat{n} \int \frac{dk}{2\pi} \psi_{\hat{n}\sigma}^\dagger (i\partial_t - v_F k) \psi_{\hat{n}\sigma} \]

\[ \langle \psi_{\hat{n}\uparrow} \psi_{\hat{n}\downarrow} \rangle \neq 0 \]

... but rarely so clear-cut in experiment
Recent results: Fermi surfaces

- **Widom formula:** \((Klich-Gioev \ '06, BGS \ '09)\)
  \[
  S_n(A) = \left(1 + \frac{1}{n}\right) \frac{1}{24} \frac{1}{(2\pi)^d} \int_{\partial A} \int_{FS} |n_x \cdot n_k| \ln (L)
  \]

- **Fermi liquids:** \(U(1)^\infty\) symmetry
  \[
  H_{\text{Landau}} = \sum_k \epsilon_k n_k + \frac{1}{2} \sum_{kk'} f_{kk'} n_k n_{k'} \quad (n_k = c_k^\dagger c_k)
  \]
  \((BGS \ '10, Ding-Seidel-Yang \ '12)\)

- **Exactly solvable models!**
  1. Variable quasiparticle residue
  2. Quantum phase transition
  3. Non-Fermi liquid metal \((BGS \ '12)\)
Recent results: Fermi surfaces

(BGS-McMinis-Tubman coming soon)

- Sub-leading oscillating terms

\[ \delta S_n = \int \frac{f_n}{4} \int_{\partial A} \int_{FS} |n_x \cdot n_k| \frac{\cos (2k_F L_{\text{eff}}(x,k))}{(2k_F L_{\text{eff}}(x,k))^2/n} \]
Recent results: topological phases

- Topological entanglement entropy:
  \[ S(A) = \alpha L - \beta \]  
  (Levin-Wen, Kitaev-Preskill ’05)

- Entanglement spectrum: 
  \( \rho_A = e^{-K_A}, \text{spec}(K_A) \) 
  (Li-Haldane ’09, ...)

- Bulk-edge correspondence: \( K_A \) same universal features as physical edge Hamiltonian! 
  (Senthil-BGS ’11)

\[ K_{\text{half space}} \rightarrow \text{Rindler space} \]
\[ K_{\text{ball}} \rightarrow \text{hyperbolic space} \]
  (Unruh-Davies-Fulling)
  (Casini-Myers-Huerta ’11)
Thermal-entanglement crossover

\[ \rho_A(T) = \text{tr}_B \left( e^{-\frac{H}{T}} / Z \right) \]

\[ S(A, T) = \text{universal cross-over function} \]
\[ + \text{non-universal area law terms} \]

\[ S(A, T) = \frac{1}{12} \frac{1}{(2\pi)^d} \int_{\partial A} \int_{FS} |n_x \cdot n_k| \ln \left( \frac{v_F}{\pi T \epsilon} \sinh \left( \frac{\pi T L_{\text{eff}}(x,k)}{v_F} \right) \right) \]

\[ (BGS '10) \]

\[ S(A, T) \sim T^\phi f(T L^{1/z}) \quad S \sim L \ln L \text{ for NFL} \]

\[ (\text{Senthil-BGS '11}) \]
RG monotonicity?

- Wonderful results for CFTs!
  
  \((Zamolodchikov, \ Schwimmer-Komagordski '11, \ Casini-Huerta '12)\)

- Nothing(?) for anything more general ...

1. Fermi points, 1d \(\epsilon_k = \nu k + \nu' k^3 / k_0^2\)

2. Limit cycles (other talks!)

3. Lifshitz points

4. Disorder \((Refael-Moore '04)\)

5. CFT -> FS -> “CFT” e.g. d-wave SC

Is there a general framework?
Preview

Large N
- Operator dimensions
- Entropy/area
- Mutual information

cMERA
- Scaling transformations
- Entanglement per scale

Fermi surfaces
- Emergent dimensions/branching MERA
- Field theory route to hyperscaling spacetimes?
1. Entanglement and locality
2. Disentanglers and coarse-grainers
3. Operators and correlation functions
Spin chains

- Hilbert space: \( \bullet = \downarrow \) or \( \uparrow \)
  \[ \dim(\mathcal{H}(\bullet)) = 2 \]

- Hamiltonian:
  \[ H = \sum_{r} h_r^{(1)} + \sum_{r,r'} h_{rr'}^{(2)} + \sum_{r,r',r''} h_{rr',rr''}^{(3)} + \ldots \]
  **Locality:** \( h_{rr'}^{(2)} \to 0 \) as \( |r - r'| \to \infty \), etc.

- Later, large \( N \) limit:
  \[ \dim(\mathcal{H}(\bullet)) = e^N \]
Short Range Entanglement

• Example 1:

\[ A \quad S(A) = 0 \]

• Example 2:

\[ A \quad S(A) = 2 \ln 2 \]
Building up entanglement

= disentangler
Longer range entanglement

= coarse-grainer
Entanglement renormalization

Entanglement

Locality

Coarse-graining

multi-scale entanglement renormalization ansatz (MERA)
$k = 3, \ 3 \rightarrow 1 \ \text{scheme}$
Unitaries and isometries

• Disentanglers are **unitaries**

• Coarse-grainers are **isometries**
Contracting the network
Operators

Contractible region

operator insertion
Renormalized operators

\[ O_{ren} = \mathcal{C}(O) = V^\dagger OV \]

\[ V = (\otimes U)(\otimes W) \]

(Montaner et al. '08, Pfeifer et al. '09)
Scaling operators

\[ \mathcal{C}(O(x)) = k^{-\Delta} O(x/k) = k^{-\Delta} \]

\[ \mathcal{C}(O_1(0)O_2(x)) = k^{-\Delta_1} O_1(0) k^{-\Delta_2} O_2(x/k) \]

Collision after \( \log_k (x/\epsilon) \) steps, \( k^{-2\Delta} \log_k (x/\epsilon) = \left( \frac{\epsilon}{x} \right)^{2\Delta} \)
Entanglement renormalization and holography

- Laplacian: $\text{spec}(\triangle_G) \sim \text{spec}(\triangle_{\text{AdS}})$
- Geodesics and correlations
- Entropy bounds

(Ryu-Takayanagi ’06)
(BGS ’09)
(see also van Raamsdonk ’09)
1. Large N limit
2. Operator dimensions
3. Geometry from entanglement
4. Mutual information
5. Comparing to holography

LARGE N AND STRONG COUPLING

(BGS '12 Monday night)
\[ \dim(\mathcal{H}(\bullet)) = e^N \]

\[ M = \begin{pmatrix} M_{11} \ldots M_{1N_c} \\ \vdots \\ M_{N_c1} \ldots M_{N_cN_c} \end{pmatrix} \]

\[ N = N_c^2 \]

\[ S = \frac{1}{g^2} \int dt d^d x \text{ tr} \left( \frac{1}{2} F^2 + (D M_1)^2 + (D M_2)^2 + [M_1, M_2]^2 \ldots \right) \]
Operator dimensions

- and are strongly mixing at large N and strong coupling
  \[ O_{\text{ren}} = \mathcal{C}(O) = V^\dagger O V \]

- Operators reduced to identity
  \[ U \text{ is } \chi^2 \times \chi^2 \text{ generic unitary,} \]
  \[ \int dU_{\text{Haar}} U^\dagger (O \otimes 1) U = \frac{tr(O)}{\chi} (1 \otimes 1) \]

- Corrections ~ \(1/\chi^p\) typical dimension:
  \[ 1/\chi^p \sim k^{-\Delta} \rightarrow \Delta \sim N \text{ or greater} \]
Low dimension operators

• Symmetry protects existence of a few low dimension operators, e.g.
  – Translation invariance $\mathcal{T} V = V \mathcal{T}$
  $$\partial O = \frac{O(x) - O(0)}{x}$$

  $$\mathcal{C}(\partial O) = k^{-\Delta} \frac{O(x/k) - O(0)}{x} = k^{-\Delta - 1} \frac{O(x/k) - O(0)}{x/k}$$

  – Hamiltonian density $O = h$, $\Delta_h = d + \epsilon$

  – Global symmetries, SUSY, ...
Additive entanglement entropy

and are strongly mixing:

every bond cut gives a definite entropy \( \delta S \sim N \)

(c.f. quantum expanders – ask me later)

(Hastings ‘07, Ben-Aroya Ta-Shma ‘07)
Geometry and entropy

Number of bonds cut = \(2 \log_k \left( \frac{R}{\epsilon} \right)\)

Entropy = \(S = \delta S \left(2 \log_k \left( \frac{R}{\epsilon} \right)\right)\)
Mutual information

\[ I(A, B) = S(A) + S(B) - S(AB) \]

\[ I = \delta S \left( 4 \log_k \left( \frac{R}{\epsilon} \right) - 2 \log_k \left( \frac{x}{\epsilon} \right) - 2 \log_k \left( \frac{(2R + x)/\epsilon}{\epsilon} \right) \right) \]

\[ I = 2\delta S \log_k \left( \frac{R^2}{(2R+x)x} \right), \quad x \ll R \]
Mutual information

\[ x \gg R \]

\[ \mathcal{I}(A, B) = S(A) + S(B) - S(AB) \]
\[ \rightarrow \mathcal{I}(A, B) \approx 0 \]

Corrections from low dim. operators:

\[ \rho_{AB} = \rho_A \otimes \rho_B + \delta \rho, \quad \mathcal{I}(A, B) \sim \delta \rho^2 \]
\[ \delta \rho \sim \langle O_A O_B \rangle_c, \quad \mathcal{I} \sim \langle O_A O_B \rangle^2_c \quad \text{(consistent with Wolf et al. '07)} \]

(information locking, data hiding states?)

(Hayden et al.)
Holography proper

e.g. Einstein gravity \((Maldacena, Witten, Gubser-Polyakov-Klebanov)\)

- Most operators have large dimension
  \[ \Delta \sim L_{\text{AdS}}/\ell_s \sim \lambda^{1/4} \quad (\mathcal{N} = 4, 3+1d), \text{D-branes, etc.} \]

- Entropy and area, RT formula: \((Ryu-Takayanagi '06)\)
  \[ S(A) = \frac{\vert\Sigma(A)\vert}{4G_N}, \Sigma(A) = \text{bulk surface anchored at } \partial A \]

- Phase transitions in MI \((Headrick ...\)

**CONJECTURE:** basic MERA network \(\sim\) coarse holography; strongly coupled disentanglers and coarse-grainiers \(\sim\) sharp holography

\((Heemskerk-Penedones-Polchinski-Sully '09)\)
1. Continuous MERA
2. Entanglement per scale
3. RG causality
4. Fermi surfaces and branching MERA

ADDITIONAL TOPICS

(BGS '12 monday night)
Continuous MERA

- Replace discrete lattice with a continuous theory with cutoff
- UV state $|UV\rangle$, IR state $|IR\rangle$ (unentangled)

$$
|UV\rangle = V |IR\rangle
$$

$$
V = e^{-iu_IR D} \quad \mathcal{R} = \epsilon e^u
$$

$D =$ dilatation generator (regulated)

(Haegeman-Osborne-Verschelde-Verstraete '11)
Correlations

\[ e^{iuD} O(x) e^{-iuD} = e^{-\Delta u} O(xe^{-u}) \]

Compare: \( \mathcal{C}(O) = V^\dagger OV = k^{-\Delta} O \)

\[
\langle UV | O(x) O(0) | UV \rangle \\
e \langle IR | e^{i(u_{IR}-u)D} \left[ e^{-u\Delta} O(xe^{-u}) e^{-u\Delta} O(0) \right] e^{i(u_{IR}-u)D} | IR \rangle \\
e^{-2\Delta \ln(x/\epsilon)} \langle O(\epsilon) O(0) \rangle
\]

\[ (\text{Haegeman et al. '11}) \]
Geometry

- State at scale $u$: $|u\rangle = e^{-iuD}|IR\rangle$
- DEFINE bulk “area element” =
  $$dS(A) = S(A, u + du) - S(A, u)$$
- CHALLENGE: show that this satisfies the axioms of an area function
  $$A \rightarrow |\Sigma(A)| \propto S(A)$$
  (BGS ‘09, see also nice work in Nozaki-Ryu-Takayanagi ‘12)
- Use twist fields (focus on $d=1$): $\Phi_n$
  $$\langle \Phi_n(L)\Phi_n(0) \rangle_n = \text{tr}(\rho_{[0,L]}^n)$$
  (Cardy-Calabrese, Caraglio-Gliozzi ‘08)
RG quantum quench

- Twist fields are non-local ($|IR\rangle$ is unentangled)
  \[ \langle IR | \Phi_n(L) \Phi_n(0) | IR \rangle_n = 1 \]
- Twist fields are scaling fields $\Delta_n = \frac{c}{12} \left( n - \frac{1}{n} \right)$

- State $|u\rangle = e^{-iuD} |IR\rangle$ like a quantum quench! Compute entropy added ...
  \[ \langle u | \Phi_n(L) \Phi_n(0) | u \rangle_n = e^{-2u\Delta_n} \langle IR | \Phi_n(Le^{-u}) \Phi_n(0) | IR \rangle_n \text{ Provided } u < \ln(L) \]

\[ \frac{dS(L)}{du} = \frac{c}{3}, \text{ saturates at } u = \ln \left( \frac{L}{\epsilon} \right) \]
• Exponential “RG light cone”:
  \[ e^{i u D} O(x) e^{-i u D} = e^{-\Delta u} O(x e^{-u}) \]
• UV perturbations don’t affect much of the RG circuit

(see also Evenbly and Vidal)
Fermi surface

- How should we understand the Fermi surface? Emergent dimensions!?

- UV-IR duality $\delta k \sim 1/r$
- Growing number of low energy modes: (1+1d CFTs)
  \[
dS(A) \sim \left( \frac{R}{r} \right)^{d-1} \left( k_F r \right)^{d-1} \frac{dr}{r}
\]

- Locality? Fermi-liquid – no! FS+gauge field – maybe?

(Evenbly-Vidal `12)
Hyperscaling violation geometry?

• Different kind of dilatation generator

\[ D \sim \int_{FS} D_1 d \]

\[ V = e^{-iu_{IR}D} \]

\[ |UV\rangle = V |IR\rangle \]

• Entanglement per scale

\[ \frac{dS(A)}{du} \sim (k_F R)^{d-1} du \]

• Consistent with a geometrical calculation!

(Ogawa-Takayanagi-Ugajin ’11, Huijse-Sachdev-BGS ’12)
Summary

• Geometrical picture of entropy in cMERA
• Fermi surfaces as emergent dimensions
• RG causality
• Sparse spectrum of scaling dimensions emerges with definite entropy at large N
• Phase transitions in mutual information
• General (holographic?) framework for combining entanglement and renormalization?
Future

Dynamics

• More covariant form of ER/holo relationship
• Entanglement in time dependent states

Holography and numerics

• Look for CFTs by imposing entanglement + symmetry
• Suggestions from holography to improve tensor network algorithms

Many other directions, so get involved! Thanks!