# Limit cycles and scale invariance II 

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with<br>Jeff Fortin and Andy Stergiou

In the previous talk...

and even...


In this talk..

or even...


That is, we study flows between cycles
or from fixed points (FP) to cycles
or from cycles to fixed points

For this talk "cycles" mean "recursive flows"
that is, either limit cycles or ergodic flows

Two approaches:
1.Perturbative
i. Valid when both ends and the complete flow between them lie in the perturbative regeme
ii. Stronger: can study whole flow, not just compare end points
iii.Well established
2.Non-perturbative
i. More generally valid
ii. Weaker: compare quantities, like $a$, at ends
iii. Relies on (few and reasonable) unproven assumptions

## Outline

- Intro
- Non-perturbative:
- KS and LPR setup for cycles
- Need of function $S$ for extension to cycles
- Perturbative:
- review of JO's perturbative proof of a $c$ theorem
- the virial current and definition of $S$
- computation of $S$
- establish properties of $S: S=0$ at $\mathrm{FP}, S=Q$ on cycles
- Non-perturbative:
- Complete the KS proof for generic ends

Here and below:
JO: Jack and Osborn, Analogs for the $c$ theorem for four-dimensional renormalizable field theories.
Nucl.Phys. B343 (1990) 647-688
KS: Komargodski and Schwimmer, On renormalization group flows in four dimensions, JHEP 1112 (2011) 099
LPR: Luty, Polchinski and Rattazzi, The $a$-theorem and the asymptotics of 4D quantum field theory. e-Print: arXiv:1204.5221 [hep-th]

Talk based on our paper, A generalized $c$-theorem and the consistency of scale without conformal invariance. e-Print: arXiv: 1208.3674 [hep-th]

## The KS proof of the $a$ theorem

The presentation follows more closely LPR, with modifications.


Compute

$$
\left\langle T_{\mu}^{\mu}\left(-p_{1}\right) T_{\mu}^{\mu}\left(-p_{2}\right) T_{\mu}^{\mu}\left(p_{1}\right) T_{\mu}^{\mu}\left(p_{2}\right)\right\rangle
$$

along the RG flow, with $p_{1}^{2}=p_{2}^{2}=0$ as if on mass-shell massless particles, and

$$
\left(p_{1}+p_{2}\right)^{2}=s \quad\left(p_{1}-p_{1}\right)^{2}=t=0 \quad\left(p_{1}-p_{2}\right)^{2}=u=-s
$$

$$
\left\langle T_{\mu}^{\mu}\left(-p_{1}\right) T_{\mu}^{\mu}\left(-p_{2}\right) T_{\mu}^{\mu}\left(p_{1}\right) T_{\mu}^{\mu}\left(p_{2}\right)\right\rangle=\mathcal{A}_{\mathrm{fwd}}(s)
$$

The amplitude has a cut along the whole real $s$-axis, but is analytic in the upper half-plane:

$$
\oint_{C} d s \frac{\mathcal{A}_{\mathrm{fwd}}(s)}{s^{3}}=0
$$



$$
\frac{1}{2 \pi i} \int_{I_{1}} d s \frac{\mathcal{A}_{\mathrm{fwd}}(s)}{s^{3}}+\frac{1}{2 \pi i} \int_{I_{3}} d s \frac{\mathcal{A}_{\mathrm{fwd}}(s)}{s^{3}}=-\frac{1}{2 \pi i} \int_{I_{2}} d s \frac{\mathcal{A}_{\mathrm{fwd}}(s)}{s^{3}}=-\frac{1}{\pi} \int_{I_{2}, s \geq 0} d s \frac{\operatorname{Im} \mathcal{A}_{\mathrm{fwd}}(s)}{s^{3}} \leq 0
$$

Consider next IR and UV limit of 4-pt function

Trace anomaly in curved background:

$$
T_{\mu}^{\mu}=\beta_{i} \mathcal{O}_{i}-\frac{a}{16 \pi^{2}} G+\frac{c}{16 \pi^{2}} F-\frac{b}{16 \pi^{2}} R^{2}
$$

where

$$
\begin{gather*}
F=R^{\mu \nu \rho \sigma} R_{\mu \nu \rho \sigma}-\frac{4}{d-2} R^{\mu \nu} R_{\mu \nu}+\frac{2}{(d-2)(d-1)} R^{2},  \tag{squareWeyl}\\
G=\frac{2}{(d-3)(d-2)}\left(R^{\mu \nu \rho \sigma} R_{\mu \nu \rho \sigma}-4 R^{\mu \nu} R_{\mu \nu}+R^{2}\right),
\end{gather*}
$$

(Euler density)

Since $\quad T_{\mu}^{\mu}=2 g^{\mu \nu} \frac{\delta W}{\delta g^{\mu \nu}} \quad$ and $\quad \beta_{i}=0 \quad$ at fixed points
we can compute the $4-\mathrm{pt}$ function by thrice differentiating this and setting $g_{\mu \nu}=\eta_{\mu \nu}$
Alternatively, set $\quad g_{\mu \nu}=e^{2 \tau(x)} \eta_{\mu \nu}$ and thrice differentiate w.r.t $\tau(x)$

$$
\begin{aligned}
e^{-4 \tau} G & =8\left(\partial^{2} \tau\right)^{2}-8 \tau_{, \mu \nu} \tau^{, \mu \nu}-16 \tau_{, \mu} \tau_{,, \nu} \tau^{, \mu \nu}-8(d-3) \tau_{, \mu} \tau^{\mu} \partial^{2} \tau+2(d-1)(d-4)\left(\tau_{, \mu} \tau^{, \mu}\right)^{2} \\
e^{-4 \tau} R^{2} & =4\left(\partial^{2} \tau\right)^{2}-4(d-2) \tau_{, \mu} \tau^{, \mu} \partial^{2} \tau+(d-2)^{2}\left(\tau_{, \mu} \tau^{, \mu}\right)^{2} .
\end{aligned}
$$

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\end{aligned}
$$

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F=R^{\mu \nu \rho \sigma} R_{\mu \nu \rho \sigma}-\frac{4}{d-2} R^{\mu \nu} R_{\mu \nu}+\frac{2}{(d-2)(d-1)} R^{2},  \tag{squareWeyl}\\
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e^{-4 \tau} R^{2} & =4\left(\partial^{2} \tau\right)^{2}-4(d-2) \tau_{, \mu \tau} \tau^{\mu} \partial^{2} \tau+(d-2)^{2}\left(\tau_{, \mu} \tau_{, \mu} \tau^{, \mu}\right)^{2} .
\end{aligned}
$$

Hence

$$
\mathcal{A}_{\mathrm{fwd}}=-32 \frac{a}{16 \pi^{2}} s^{2}
$$

Using this in $\quad \frac{1}{2 \pi i} \int_{I_{1}} d s \frac{\mathcal{A}_{\mathrm{fwd}}(s)}{s^{3}}+\frac{1}{2 \pi i} \int_{I_{3}} d s \frac{\mathcal{A}_{\mathrm{fwd}}(s)}{s^{3}} \leq 0 \quad \mathrm{KS}($ and LPR) obtain

$$
a_{\mathrm{UV}} \geq a_{\mathrm{IR}}
$$



Comments:

- Not quite at FPs on semicircles $I_{1}$ and $I_{3}$ : need to establish corrections vanish in the limit. To this end, LPR use conformal perturbation theory. Marginal deformations are most dangerous.
- No use made of Wess-Zumino action: instead of integrating anomaly and differentiating four times, we differentiate three times. This will be useful below.
- Global result (or "weak" version of a theorem): no local monotonicity along the Rg flow information.

Towards generalization to cycles:

- Rather than the FP relation $T_{\mu}^{\mu}=-\frac{a}{16 \pi^{2}} G+\frac{c}{16 \pi^{2}} F-\frac{b}{16 \pi^{2}} R^{2}$
use the cycle relation $\partial_{\mu} \mathcal{D}^{\mu}=T_{\mu}^{\mu}-\partial_{\mu} V^{\mu}=-\frac{a}{16 \pi^{2}} G+\frac{c}{16 \pi^{2}} F-\frac{b}{16 \pi^{2}} R^{2}$
- Need interpolating function to compute amplitude (of what?) on $I_{2}$

More on interpolation: FP


The virial current is defined only on cycle: $V_{\mu}=\left.D_{\mu} \phi^{T} Q \phi \quad \beta(g)\right|_{\text {on cycle }}=\left.(Q g)\right|_{\text {on cycle }}$
We want a 4-pt function of $X$, where $X=T_{\mu}^{\mu}$ at FPs, and $X=T_{\mu}^{\mu}-\partial_{\mu} V^{\mu} \quad$ on cycles

We take $\quad X=T_{\mu}^{\mu}-\partial_{\mu} J^{\mu}$
where the new current $\quad J_{\mu}=D_{\mu} \phi^{T} S \phi$
is given in terms of a function $S(g)$ that has the properties


Fortunately, JOs quantity $S$ has these properties!

## Perturbative approach

(Note: most of the results below are valid more generally, not just in perturbation theory.
It is only some positivity properties of "metrics" that require perturbation theory).

JO consider trace anomaly in curved background with spacetime dependent coupling 'constants'

Why is this necessary?
KS/LPR use $W\left[e^{-2 \tau(x)} \eta_{\mu \nu}, g^{i}(\mu)\right]=W\left[\eta_{\mu \nu}, g^{i}\left(e^{\tau(x)} \mu\right)\right] . \quad$ ( $W=$ generating function)
(or even the same with general metric),
obtained by making a $\tau$-dependent field redefinition in the quantum action.

The anomaly has now all terms of dimension 4 constructed out of the mertic and the couplings.
So in addition to

$$
\beta_{i} \mathcal{O}_{i}-\frac{a}{16 \pi^{2}} G+\frac{c}{16 \pi^{2}} F-\frac{b}{16 \pi^{2}} R^{2}
$$

one has terms like

$$
\frac{1}{2} \chi_{i j}^{g} \partial_{\mu} g^{i} \partial_{\nu} g^{j} G^{\mu \nu} \quad \text { and } \quad \frac{1}{2} \chi_{i j}^{a} \nabla^{2} g^{i} \nabla^{2} g^{j}
$$

See JO for the complete (intimidating) expression!

Bonus: taking $\frac{\delta \tilde{S}_{0}}{\delta g^{i}(x)}$ where $\quad \tilde{S}_{0} \quad$ is the renormalized quantum action gives finite (renormalized) operators:

$$
\left[O_{i}(x)\right] \equiv \frac{\delta \tilde{S}_{0}}{\delta g^{i}(x)}
$$

Similarly,

$$
T_{\mu \nu}=\frac{2}{\sqrt{g}} \frac{\delta \tilde{S}_{0}}{\delta g^{\mu \nu}(x)}
$$

defines a finite stress-energy tensor.

Note: This does not mean that all finite operators must be of this form!

## Consistency conditions

Now, since

$$
W\left[e^{-2 \tau(x)} \eta_{\mu \nu}, g^{i}(\mu)\right]=W\left[\eta_{\mu \nu}, g^{i}\left(e^{\tau(x)} \mu\right)\right] .
$$

terms in the anomaly satisfy relations. Among them, of particular interest

$$
8 \partial_{i} a=\chi_{i j}^{g} \beta^{j}-\beta^{j} \partial_{j} w_{i}-\partial_{i} \beta^{j} w_{j} \quad \quad \beta^{i}=\mu \frac{d g^{i}}{d \mu}, \partial_{i}=\frac{\partial}{\partial g^{i}}
$$

(Notes: 1. I have rescaled JO quantities by $16 \pi^{2} ; 2$. Not a gradient flow)

$$
\Rightarrow 8 \mu \frac{d a}{d \mu}=\chi_{i j}^{g} \beta^{i} \beta^{j}-\partial_{j} w_{i} \beta^{i} \beta^{j}-\partial_{i} \beta^{j} w_{j} \beta^{i}
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$$

perturbatively
positive definite

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$$

perturbatively positive definite
indefinite sign
and
lower order in loop expansion

## Consistency conditions

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$$
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$$

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$$
\Rightarrow 8 \mu \frac{d a}{d \mu}=\chi_{i j}^{g} \beta^{i} \beta^{j}-\partial_{j} w_{i} \beta^{i} \beta^{j}-\partial_{i} \beta^{j} w_{j} \beta^{i}
$$

perturbatively positive definite

```
indefinite sign
and
    lower order in loop expansion
```

In fact, can make $a$ increase then decrease as it moves away from UV-FP! No "local" or "strong" $a$-theorem

However, JO note that $\quad 8 \partial_{i} a=\chi_{i j}^{g} \beta^{j}-\beta^{j} \partial_{j} w_{i}-\partial_{i} \beta^{j} w_{j}$

$$
\Rightarrow \quad 8 \partial_{i}\left(a+\frac{1}{8} w_{j} \beta^{j}\right)=\chi_{i j}^{g} \beta^{j}-\beta^{j} \partial_{[j} w_{i]}
$$

So there is a perturbative $c$-theorem for $\quad \tilde{a} \equiv a+\frac{1}{8} w_{j} \beta^{j}$

$$
\Rightarrow 8 \mu \frac{d \tilde{a}}{d \mu}=\chi_{i j}^{g} \beta^{i} \beta^{j} \geq 0
$$

Nota bene: In Yang mills (with matter) with $\beta^{g}=-\beta_{0} g^{3} / 16 \pi^{2}-\beta_{1} g^{5} /\left(16 \pi^{2}\right)^{2}-\cdots$

$$
a=a_{0}+\frac{n_{V} \beta_{1}}{8\left(16 \pi^{2}\right)^{2}} g^{4}+\mathcal{O}\left(g^{6}\right)
$$

may increase away from the trivial UV-FP.
But

$$
\tilde{a}=a_{0}-\frac{n_{V} \beta_{0}}{4\left(16 \pi^{2}\right)} g^{2}+\mathcal{O}\left(g^{4}\right)
$$

always decreases away form the trivial UV-FP.

## Dimension 3 operators

The new dimension-4 terms in the anomaly arise from the need to add local counterterms to the quantum action

$$
\tilde{S}_{0} \rightarrow \tilde{S}_{0}+\tilde{S}_{\text {c.t. }}
$$

Then, for example, anomaly terms like

$$
\frac{1}{2} \chi_{i j}^{g} \partial_{\mu} g^{i} \partial_{\nu} g^{j} G^{\mu \nu} \quad \text { and } \quad \frac{1}{2} \chi_{i j}^{a} \nabla^{2} g^{i} \nabla^{2} g^{j}
$$

arise as beta-functions of the counterterms

$$
\frac{1}{2} \mathscr{G}_{i j} \partial_{\mu} g^{i} \partial_{\nu} g^{j} G^{\mu \nu} \quad \text { and } \quad \frac{1}{2} \mathscr{A}_{i j} \nabla^{2} g^{i} \nabla^{2} g^{j}
$$

But in theories with scalars or spinors we need additional counterterms for finiteness of the quantum action.

The need arises because there are dimension- 3 operators (currents) and one can form dimension-1 "currents" from derivatives on coupling "constants."

Very relevant example:

$$
\mathscr{L}_{K}=\frac{1}{2} \partial^{\mu} \phi_{a} \partial_{\mu} \phi_{a} \quad a=1, \ldots, n_{s}
$$

Then we must add for finiteness a counterterm

$$
\mathcal{L}_{\text {c.t. }}=\left(\partial^{\mu} g_{I}\right)\left(N_{I}\right)_{a b} \phi_{b} \partial_{\mu} \phi_{a}
$$

with $\quad\left(N_{I}\right)_{a b}=-\left(N_{I}\right)_{b a}$
and the index $I$ runs over all the (dimensionless) coupling constants, eg, $I=a b c d$ in

$$
V=\frac{1}{4!} g_{a b c d} \phi_{a} \phi_{b} \phi_{c} \phi_{d}=g_{I} \mathcal{O}_{I}
$$

Example: $n_{s}$ real scalars, $n_{f}$ Weyl spinors with

$$
V=\frac{1}{4!} \lambda_{a b c d} \phi_{a} \phi_{b} \phi_{c} \phi_{d}+\left(\frac{1}{2} y_{a \mid i j} \phi_{a} \psi_{i} \psi_{j}+\text { h.c. }\right)
$$



A simple 1-loop computation gives:

$$
\left(N_{c \mid i j}\right)_{a b}=-\frac{1}{16 \pi^{2} \epsilon} \frac{1}{2}\left(y_{a \mid i j}^{*} \delta_{b c}-y_{b \mid i j}^{*} \delta_{a c}\right)
$$

So this is real stuff, can't be ignored.

Does it go away in the $x$-independent-coupling limit?

Look both at trace anomaly and consistency conditions.

Preliminary: finite operators

$$
\mathcal{L}=\frac{1}{2} D^{\mu} \phi_{a} D_{\mu} \phi_{a}+\frac{1}{4!} g_{a b c d} \phi_{a} \phi_{b} \phi_{c} \phi_{d}=\frac{1}{2} D^{\mu} \phi^{T} D_{\mu} \phi+g_{I} \mathcal{O}_{I}
$$

where the new counterterm is in $D_{\mu} \phi_{a}=\left(\partial_{\mu} \delta_{a b}+\left(\partial_{\mu} g_{I}\right)\left(N_{I}\right)_{a b}\right) \phi_{b}$

Then

$$
\left[\mathcal{O}_{I}\right]=\frac{\delta}{\delta g_{I}} \int \mathcal{L}=\mathcal{O}_{I}-\partial_{\mu} J_{I}^{\mu}
$$

where $\quad J_{I}^{\mu}=\partial^{\mu} \phi^{T} N_{I} \phi$

Trace anomaly (take limit of flat space, $x$-independent couplings):

$$
\begin{aligned}
T_{\mu}^{\mu} & =\beta_{I} \mathcal{O}_{I} \\
& =\beta_{I}\left(\mathcal{O}_{I}-\partial_{\mu} J_{I}^{\mu}\right)+\beta_{I} \partial_{\mu} J_{I}^{\mu} \\
& =\beta_{I}\left[\mathcal{O}_{I}\right]+\partial_{\mu} J^{\mu}
\end{aligned}
$$

Hence $\quad J^{\mu}=\beta_{I} J_{I}^{\mu}=\partial^{\mu} \phi^{T} \beta_{I} N_{I} \phi \quad$ is finite

Now $\quad N_{I}=\frac{N_{I}^{1}}{\epsilon}+\frac{N_{I}^{2}}{\epsilon^{2}}+\cdots \quad \quad \beta_{I}=-\epsilon g_{I}+\cdots$

So there is a finite current given in terms of

$$
S=-g_{I} N_{I}^{1}
$$

As we shall see, on cycles $S=Q$

May also re-write:

$$
\begin{aligned}
T_{\mu}^{\mu}-\partial_{\mu} J^{\mu} & =\beta_{I}\left[\mathcal{O}_{I}\right] \\
& =\beta_{I}\left(\mathcal{O}_{I}-\partial_{\mu} J_{I}^{\mu}\right) \\
& =B_{I} \mathcal{O}_{I}
\end{aligned}
$$

where

$$
B_{I}=\beta_{I}-(S g)_{I} \quad(S g)_{a b c d}=S_{a e} g_{e b c d}+\text { perms }
$$

Condition for cycles or FPs: $\quad B_{I}=0$

## Consistency Conditions

Preliminary: symmetry considerations

$$
\mathcal{L}=\frac{1}{2} D^{\mu} \phi_{a} D_{\mu} \phi_{a}+\frac{1}{4!} g_{a b c d} \phi_{a} \phi_{b} \phi_{c} \phi_{d}=\frac{1}{2} D^{\mu} \phi^{T} D_{\mu} \phi+g_{I} \mathcal{O}_{I}
$$

Introduce background gauge field

$$
D_{\mu} \phi_{a}=\left(\partial_{\mu} \delta_{a b}+\left(\partial_{\mu} g_{I}\right)\left(N_{I}\right)_{a b}\right) \phi_{b} \rightarrow D_{\mu} \phi_{a}=\left(\partial_{\mu} \delta_{a b}+\left(A_{\mu}\right)_{a b}+\left(D_{\mu} g_{I}\right)\left(N_{I}\right)_{a b}\right) \phi_{b}
$$

$\mathrm{SO}\left(n_{s}\right)$ local invariance:

$$
\delta \phi_{a}=-\omega_{a b} \phi_{b} \quad \delta g_{I}=-(\omega g)_{I} \quad \delta A_{\mu}=D_{\mu} \omega
$$

The generating functional is a locally $\operatorname{SO}\left(n_{s}\right)$ invariant functional

$$
W\left[g^{\mu \nu}, g_{I}-(\omega g)_{I}, A_{\mu}+D_{\mu} \omega\right]=W\left[g^{\mu \nu}, g_{I}, A_{\mu}\right]
$$

(for this one needs to extend the set of counterterms and replace $D_{\mu}=\partial_{\mu}+A_{\mu}$ for $\nabla_{\mu}$ as needed)
Automatically finite currents are obtained from

$$
-\left[\phi_{a} \overleftrightarrow{D}^{\mu} \phi_{b}\right]=\frac{\delta}{\delta\left(A_{\mu}\right)_{b a}} \int \mathcal{L}
$$

$\square$

$$
\left[D^{\mu} \phi^{T} w \phi\right]=D_{\mu} \phi^{T}\left(w+N_{I}(w g)_{I}\right) \phi \quad \text { with } w^{T}=-w
$$

Among JO consistency conditions:

$$
\begin{aligned}
8 \partial_{I} a & =\chi_{I J}^{g} B_{J}-B_{J} \partial_{J} w_{I}-\left(\partial_{I} B_{J}\right) w_{J}-\left(P_{I} g\right)_{J} w_{J} \\
& =\chi_{I J}^{g} B_{J}-\beta_{J} \partial_{J} w_{I}-\left(\partial_{I} \beta_{J}\right) w_{J}-\left(\rho_{I} g\right)_{J} w_{J}
\end{aligned}
$$

and

$$
B_{I} P_{I}=0
$$

Compare with previous: $\quad 8 \partial_{i} a=\chi_{i j}^{g} \beta^{j}-\beta^{j} \partial_{j} w_{i}-\partial_{i} \beta^{j} w_{j}$ obtained in absence of dim-3 operators (and hence of $N_{I}$ counterterms)

Recall, for $\quad \tilde{a} \equiv a+\frac{1}{8} w_{j} \beta^{j} \quad$ get $c$-theorem: $\quad 8 \mu \frac{d \tilde{a}}{d \mu}=\beta^{i} \partial_{i} \tilde{a}=\chi_{i j}^{g} \beta^{i} \beta^{j}$

Now additional term spoils positivity in general
with

$$
\begin{array}{lll}
\tilde{a} \equiv a+\frac{1}{8} w_{I} \beta_{I} & \Rightarrow & -8 \frac{d \tilde{a}}{d t}=8 \mu \frac{d \tilde{a}}{d \mu}=\beta_{I} \partial_{I} \tilde{a}=\chi_{I J}^{g} \beta_{I} B_{J}-\beta_{I}\left(\rho_{I} g\right)_{J} w_{J} \\
\tilde{A} \equiv a+\frac{1}{8} w_{I} B_{I} & \Rightarrow & -8 \frac{d \tilde{A}}{d \eta}=B_{I} \partial_{I} \tilde{A}=\chi_{I J}^{g} B_{I} B_{J} \geq 0
\end{array}
$$

$\tilde{A}$ decreases (is non-increasing) along $\eta$-flows, and equals $a$ on FPs and cycles (where $B_{I}=0$ )

Relation between flows:
RG-flow: $\quad-\frac{d g_{I}}{d t}=\beta_{I}(g(t)), \quad \eta$-flow: $\quad-\frac{d \bar{g}_{I}}{d \eta}=B_{I}(\bar{g}(\eta))$.

$$
\bar{g}(\eta)=F(\eta) g(\eta) \quad F(\eta)=T\left(\exp \left[-\int_{-\infty}^{\eta} d \eta^{\prime} S\left(\eta^{\prime}\right)\right]\right) \in S O\left(n_{s}\right)
$$

$\mathrm{SO}\left(n_{s}\right)$ invariance: $\quad \tilde{A}(\bar{g}(\eta))=\tilde{A}(g(\eta)) \quad \Rightarrow \quad-\frac{d \tilde{A}}{d t} \geq 0$
that is $\tilde{A}$ is non-increasing along RG flows, a perturbative $c$-function
$\mathrm{SO}\left(n_{s}\right)$ invariance gives $\tilde{A}=$ constant on cycles: $\quad g(t)=F^{-1}(t) \bar{g}_{*} \quad \Rightarrow \quad \tilde{A}(g(t))=\tilde{A}\left(\bar{g}_{*}\right)$

Alternatively: Covariance under $\mathrm{SO}\left(n_{s}\right)$ gives constraints,

$$
a(g+\delta g)-a(g)=(\omega g)_{I} \partial_{I} a=0
$$

and similarly for any $\operatorname{SO}\left(n_{s}\right)$ invariant, like $\tilde{a}, \tilde{A}, \chi_{I J}^{g} \beta_{I} B_{J}, \ldots$
On cycle, $\beta_{I}=(Q g)_{I}$

$$
\frac{d a}{d t}=-\beta_{I} \partial_{I} a=-(Q g)_{I} \partial_{I} a=0
$$

## Compute $S$

Recall $S=-g_{I} N_{I}^{1}$
For theory of $n_{s}$ real scalars, $n_{f}$ Weyl spinors (symmetry group $G_{F}=\mathrm{SO}\left(n_{s}\right) \times \operatorname{SU}\left(n_{f}\right)$ ) with

$$
V=\frac{1}{4!} \lambda_{a b c d} \phi_{a} \phi_{b} \phi_{c} \phi_{d}+\left(\frac{1}{2} y_{a \mid i j} \phi_{a} \psi_{i} \psi_{j}+\text { h.c. }\right)
$$

found, at 1-loop: $\quad\left(N_{c \mid i j}\right)_{a b}=-\frac{1}{16 \pi^{2} \epsilon} \frac{1}{2}\left(y_{a \mid i j}^{*} \delta_{b c}-y_{b \mid i j}^{*} \delta_{a c}\right)$
More conveniently, rewrite

$$
16 \pi^{2}\left(N_{I}^{1}\right)_{a b} \partial^{\mu} g_{I}=-\frac{1}{2}\left[\operatorname{tr}\left(y_{a} \partial^{\mu} y_{b}^{*}\right)+\text { h.c. }-\{a \leftrightarrow b\}\right],
$$

Hence $S=0$ (1-loop)

Why? Individual topology symmetric under $a \leftrightarrow b$


Contributions to $N_{I}$ still symmetric at 2-loops


JO: $S=0$ to 2-loops in fermion-scalar model, and to 3-loops in pure scalar model

Contributions to $N_{I}$ not symmetric at 3-loops;


Be mesmerized:

$$
\begin{aligned}
\left(16 \pi^{2}\right)^{3}\left(N_{I}^{1}\right)_{a b} \partial^{\mu} g_{I} \supset & -\frac{1}{2} \operatorname{tr}\left(y_{a} \partial^{\mu} y_{c}^{*} y_{d} y_{e}^{*}\right) \lambda_{b c d e}-\frac{1}{3} \operatorname{tr}\left(y_{a} y_{c}^{*} \partial^{\mu} y_{d} y_{e}^{*}\right) \lambda_{b c d e}-\frac{1}{2} \operatorname{tr}\left(y_{a} y_{c}^{*} y_{d} \partial^{\mu} y_{e}^{*}\right) \lambda_{b c d e} \\
& -\frac{5}{24} \operatorname{tr}\left(y_{a} y_{c}^{*} y_{d} y_{e}^{*}\right) \partial^{\mu} \lambda_{b c d e}-\frac{1}{24} \operatorname{tr}\left(y_{b} \partial^{\mu} y_{c}^{*} y_{d} y_{e}^{*}\right) \lambda_{a c d e}-\frac{5}{24} \operatorname{tr}\left(y_{b} y_{c}^{*} \partial^{\mu} y_{d} y_{e}^{*}\right) \lambda_{a c d e} \\
& -\frac{1}{24} \operatorname{tr}\left(y_{b} y_{c}^{*} y_{d} \partial^{\mu} y_{e}^{*}\right)_{a c c e}-\frac{5}{24} \operatorname{tr}\left(\partial^{\mu} y_{b} y_{c}^{*} y_{d} y_{e}^{*}\right) \lambda_{a c d e}-\frac{7}{32} \operatorname{tr}\left(y_{a} \partial^{\mu} y_{c}^{*} y_{d} y_{d}^{*} y_{b} y_{c}^{*}\right) \\
& -\frac{7}{96} \operatorname{tr}\left(y_{a} y_{c}^{*} \partial^{\mu} y_{d} y_{d}^{*} y_{b} y_{c}^{*}\right)-\frac{23}{96} \operatorname{tr}\left(y_{a} y_{c}^{*} y_{d} \partial^{\mu} y_{d}^{*} y_{b} y_{c}^{*}\right)-\frac{7}{96} \operatorname{tr}\left(y_{a} y_{c}^{*} y_{d} y_{d}^{*} \partial^{\mu} y_{b} y_{c}^{*}\right) \\
& -\frac{7}{32} \operatorname{tr}\left(y_{a} y_{c}^{*} y_{d} y_{d}^{*} y_{b} \partial^{\mu} y_{c}^{*}\right)+\frac{1}{16} \operatorname{tr}\left(y_{a} \partial^{\mu} y_{c}^{*} y_{c} y_{d}^{*} y_{b} y_{d}^{*}\right)-\frac{5}{48} \operatorname{tr}\left(y_{a} y_{c}^{*} \partial^{\mu} y_{c} y_{d}^{*} y_{b} y_{d}^{*}\right) \\
& -\frac{1}{48} \operatorname{tr}\left(y_{a} y_{c}^{*} y_{c} \partial^{\mu} y_{d}^{*} y_{b} y_{d}^{*}\right)-\frac{7}{96} \operatorname{tr}\left(y_{a} y_{c}^{*} y_{c} y_{d}^{*} \partial^{\mu} y_{b} y_{d}^{*}\right)+\frac{1}{16} \operatorname{tr}\left(y_{a} y_{c}^{*} y_{c} y_{d}^{*} y_{b} \partial^{\mu} y_{d}^{*}\right) \\
& + \text { h.c. }-\{a \leftrightarrow b\},
\end{aligned}
$$

First ever computation of non-vanishing $S$ :
$\left(16 \pi^{2}\right)^{3} S_{a b}=\frac{5}{8} \operatorname{tr}\left(y_{a} y_{c}^{*} y_{d} y_{e}^{*}\right) \lambda_{b c d e}+\frac{3}{8} \operatorname{tr}\left(y_{a} y_{c}^{*} y_{d} y_{d}^{*} y_{b} y_{c}^{*}\right)+$ h.c. $-\{a \leftrightarrow b\}$

Checked that:

- $S=0$ at non-trivial FPs of models discussed in Stergiou's talk
- $S=Q$ at all cycles in those models


## $S=Q$ at cycles

$$
-8 \frac{d \tilde{A}}{d \eta}=B_{I} \partial_{I} \tilde{A}=\chi_{I J}^{g} B_{I} B_{J} \geq 0
$$

$$
G_{F} \text { symmetry: } \quad \frac{d \tilde{A}}{d \eta}=0 \quad \text { at cycle }
$$

$$
\text { Positivity (perturbatively) of } \chi_{I J}^{g} \quad \Rightarrow \quad B_{I}=0 \quad \text { at cycle }
$$

$$
B_{I}=\beta_{I}-(S g)_{I} \quad \Rightarrow \quad B_{I}=(Q g)_{I}-(S g)_{I} \quad \text { at cycle }
$$

$$
\Rightarrow \quad((Q-S) g)_{I}=0 \quad \Rightarrow S=Q+\Delta Q \quad \text { where } \quad(\Delta Q g)_{I}=0 \quad \text { simply defines new }
$$ conserved current, ambiguity in $Q$

So, redefining $\quad Q^{\prime}=Q+\Delta Q$
we have shown that on cycles $S=Q$

## $S=0$ at Fixed Points

Use again

$$
-8 \frac{d \tilde{A}}{d \eta}=B_{I} \partial_{I} \tilde{A}=\chi_{I J}^{g} B_{I} B_{J} \geq 0
$$

but now with $\quad B_{I}=-(S g)_{I} \quad$ and $\quad \frac{d \tilde{A}}{d \eta}=0 \quad$ at Fixed Points

$$
\Rightarrow(S g)_{I}=0 \quad \text { at Fixed Points }
$$

Either $S=0$ or there is an emergent symmetry at the FP and $J^{\mu}$ is the associated conserved current

## Back to

## Non-perturbative Approach

Start by showing again last slide of that section

More on interpolation: FP


The virial current is defined only on cycle: $V_{\mu}=\left.D_{\mu} \phi^{T} Q \phi \quad \beta(g)\right|_{\text {on cycle }}=\left.(Q g)\right|_{\text {on cycle }}$
We want a 4-pt function of $X$, where $X=T_{\mu}^{\mu}$ at FPs, and $X=T_{\mu}^{\mu}-\partial_{\mu} V^{\mu} \quad$ on cycles

We take $\quad X=T_{\mu}^{\mu}-\partial_{\mu} J^{\mu}$
where the new current $\quad J_{\mu}=D_{\mu} \phi^{T} S \phi$
is given in terms of a function $S(g)$ that has the properties


Fortunately, JOs quantity $S$ has these properties!

Only left to do: tie loose ends.

Interpretation: in $X=T_{\mu}^{\mu}-\partial_{\mu} J^{\mu}, X$ is just $X=\partial_{\mu} \mathcal{D}^{\mu} \quad$ which vanishes at scale invariant cycles and on a curved background (but with $x$-independent couplings and vanishing $A_{\mu}$ )

$$
\partial_{\mu} \mathcal{D}^{\mu}=B_{I} \mathcal{O}_{I}-\frac{a}{16 \pi^{2}} G+\frac{c}{16 \pi^{2}} F-\frac{b}{16 \pi^{2}} R^{2}
$$

with $\quad B_{I}=\beta_{I}-(S g)_{I} \quad$ defined everywhere in theory space, and vanishing on FPs and cycles

Why ignore terms involving $D_{\mu} g_{I}$ ?
The $x$-dependence comes in through $g_{I}\left(\mu e^{\tau(x)}\right)$
However, for fixed points $g_{I}\left(\mu e^{\tau(x)}\right)=g_{I *}$ is $\tau(x)$-independent

At cycles couplings are covariantly constant. Hence additional terms are also absent. That is, dependence comes as rotation in $G_{F}$ and anomaly terms are $G_{F}$ invariant.

Last, recall: Not quite on semicircles $I_{1}$ and $I_{3}$. LPR use conformal perturbation theory to establish corrections vanish in the limit.


But they only use scaling properties, which are also valid in scale but not conformal theory. Their proof goes through.

## Afterthoughts

1.Simplification in search for models:

Previously: solve $\beta_{I}(g)-(Q g)_{I}=0$
both for $g$ and $Q$ using 3-loop beta-function

Now: solve $\quad \beta_{I}(g)=0$
for $g$ only, using 1-loop beta-function; plug into $S$; if $S=0$ then FP, else cycle with $Q=S$
2. Are there flows

or


Numerical study in progress.
3.Wess-Zumino Action with $x$-dependent couplings?

Quick review:
Consider infinitesimal variations $\tau \rightarrow \tau+\sigma$ in $W\left[e^{-2 \tau(x)} g_{\mu \nu}(x), g^{i}\left(e^{-\tau(x)} \mu\right)\right]$
The $a$-term in the anomalous variation is

$$
16 \pi^{2} \Delta W=-\int \sqrt{g} \sigma a G+\cdots
$$

and KS show this is produced by the variation of the Wess-Zumino action

$$
16 \pi^{2} \Delta W_{\mathrm{WZ}}=-\int \sqrt{g}\left\{\tau a G-4 a\left[G^{\mu \nu} \tau_{, \mu} \tau_{, \nu}+\tau_{, \mu} \tau^{, \mu} \nabla^{2} \tau+\frac{1}{2}\left(\tau_{, \mu} \tau^{, \mu}\right)^{2}\right]\right\}
$$

We can extend this result to include a particular JO term

$$
16 \pi^{2} \Delta W=-\int \sqrt{g}\left[\sigma a G+\partial_{\mu} \sigma w_{i} \partial_{\nu} g^{i} G^{\mu \nu}\right]+\cdots
$$

Now the WZ action is

$$
16 \pi^{2} \Delta W_{\mathrm{WZ}}=-\int \sqrt{g}\left\{\tau a G-4 \tilde{a}\left[G^{\mu \nu} \tau_{, \mu} \tau_{, \nu}+\tau_{, \mu} \tau^{, \mu} \nabla^{2} \tau+\frac{1}{2}\left(\tau_{, \mu} \tau^{, \mu}\right)^{2}\right]\right\}
$$

where $\tilde{a} \equiv a+\frac{1}{8} w_{j} \beta^{j} \quad$ is our old friend for which there is a perturbative $c$-theorem (in the absence of dimension 3-operators)

Note the first terms is still plain $a$ : cannot simply redefine $a$ everywhere.

## However

we are unable to construct a complete WZ action (that is, one that accounts for the complete set of operators in the anomaly as listed in JO)

However
as we have seen, there is no need to have a WZ action to complete the KS argument

Thanks for your attention

The End

