Renormalized entanglement entropy and the number of degrees of freedom

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Based on **arXiv:1202.2070** with Mark Mezei

Goal

For any renormalizable quantum field theory, construct an observable which could:

- probe and characterize quantum entanglement at a given scale.

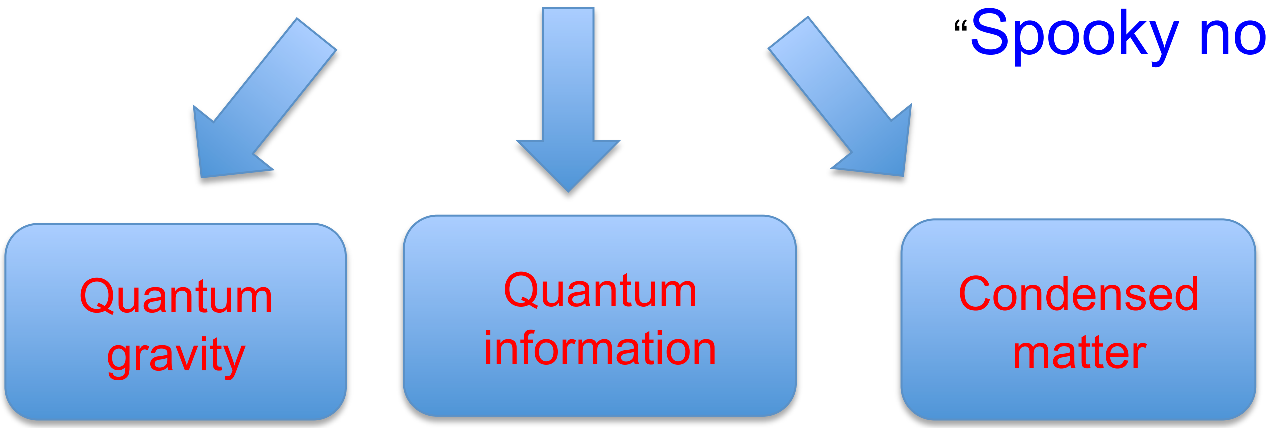
- track the number of degrees of freedom of the system at a given scale.

Quantum entanglement

Quantum
entanglement

$$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

"Spooky non-locality"

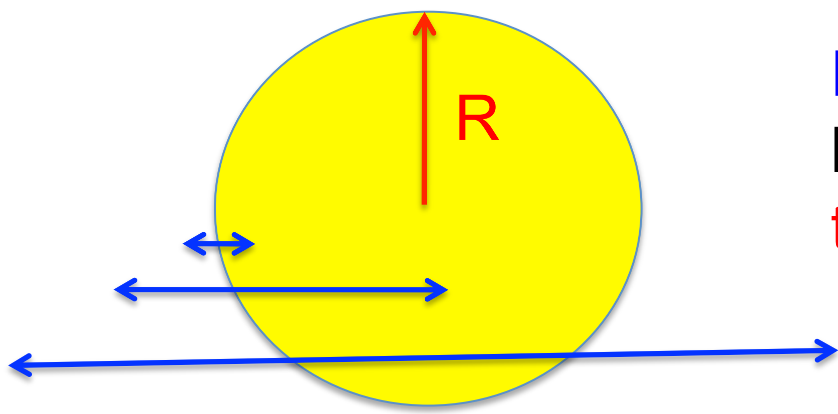


Bi-partite entanglement: entanglement entropy

$$S^{(\Sigma)} = -\mathrm{Tr}\,\rho_A \log \rho_A$$

(will focus on vacuum)

Entanglement entropy



dominated by short-distance physics

Expect it to depend on physics at length scales ranging from size R all the way to short-distance cutoff δ .

$$S^{(\Sigma)} \propto \frac{A_\Sigma}{\delta^{d-2}} + \cdots$$

δ : Short-distance cutoff

(Bombelli et al, Srednicki)

"unpleasant" features:

- ill-defined in the continuum limit: Divergent for a renormalizable QFT
- Long range correlations hard to extract.

Even in the large R limit, still sensitive to all the shorter-distance d.o.f., not clear it will reduce to the behavior of the IR fixed point.

Common practice:

subtract the UV divergent parts by hand, often ambiguous (e,g. typically not invariant under reparametrizations of the cutoff)

Even after the subtraction, could still depend on physics at scales much smaller than the size of the entangled region.

Free massive fields

For a free massive scalar field for a spherical region in the regime mR >> 1 in d=3:

Herzberg and Wilczek, Heurta

$$S_{\text{scalar}}(mR) = \#\cancel{\frac{R}{\delta}} - \frac{\pi}{6}\, mR - \frac{\pi}{240}\,\frac{1}{mR} + \cdots$$

The finite part diverges linearly in R and does not have a well defined limit in the large R limit.

At long distances, the system contains nothing.

Ideally, we would have liked to have the EE to go to zero.

Entanglement entropy contains too much information (junk):

e.g. in the infinite R limit, it does not reduce to physics at the IR fixed point, and still depends on physics at much shorter length scales.

Would like to be able to directly probe entanglement relations at a given scale.

Here we make a simple proposal.

"Renormalized entanglement entropy"

For any entangling (smooth) surface Σ with a scalable size R:

$$\mathcal{S}_d^{(\Sigma)}(R) = \begin{cases} \frac{1}{(d-2)!!}\left(R\frac{d}{dR}-1\right)\left(R\frac{d}{dR}-3\right)\cdots\left(R\frac{d}{dR}-(d-2)\right)S^{(\Sigma)}(R) & \text{d odd} \\ \frac{1}{(d-2)!!}R\frac{d}{dR}\left(R\frac{d}{dR}-2\right)\cdots\left(R\frac{d}{dR}-(d-2)\right)S^{(\Sigma)}(R) & \text{d even} \end{cases}.$$

d=2: $\mathcal{S}_2(R) = R\frac{dS}{dR}$

d=3: $\mathcal{S}_3^{(\Sigma)}(R) = R\frac{\partial S^{(\Sigma)}}{\partial R} - S^{(\Sigma)}$

d=4:

$$\mathcal{S}_4^{(\Sigma)}(R) = \frac{1}{2}R\partial_R(R\partial_R S^{(\Sigma)} - 2S^{(\Sigma)}) = \frac{1}{2}\left(R^2\frac{\partial^2 S^{(\Sigma)}}{\partial R^2} - R\frac{\partial S^{(\Sigma)}}{\partial R}\right)$$

Renormalized entanglement entropy

Will show:

- UV finite, well-defined in the continuum limit
- R-independent for a scale invariant system $\mathcal{S}_d^{(\Sigma)}(R) = s_d^{(\Sigma)}$
- For a general quantum field theory
$$\mathcal{S}_d^{(\Sigma)}(R) \to \begin{cases} s_d^{(\Sigma,\mathrm{UV})} & R \to 0 \\ s_d^{(\Sigma,\mathrm{IR})} & R \to \infty \end{cases} .$$

➢ It is most sensitive to degrees of freedom at scale R.

For a sphere: $s_d^{(\mathrm{sphere})} = \text{central charge}$ (CFT)

Monotonicity of $\mathcal{S}_d^{\mathrm{sphere}}(R)$ would then lead to a c-theorem.

UV finiteness

The divergent part of EE should only depend on local physics at the cutoff scale near the entangling surface,

Grover, Turner, Vishwanath

$$S_{\rm div}^{(\Sigma)} = \int_\Sigma d^{d-2}\sigma\,\sqrt{h}F(K_{ab},h_{ab})$$

h: induced metric,
K: extrinsic curvature

F: sum of all possible geometric invariants

$S_{\rm div}^{(\Sigma)} = S_{\rm div}^{(\bar\Sigma)}$ ⇒ Function F must be even in K.

For a scalable smooth surface $\quad h_{ab} \sim R^2, \qquad K_{ab} \sim R, \qquad D_a \sim R^0$

$$F \sim 1 + K^2 + K^4 + \cdots \sim 1 + \frac{1}{R^2} + \frac{1}{R^4} + \cdots$$

⇒ $$S_{\rm div}^{(\Sigma)} = a_1 R^{d-2} + a_2 R^{d-4} + \cdots$$

UV finiteness (II)

$$S_{\rm div}^{(\Sigma)} = a_1 R^{d-2} + a_2 R^{d-4} + \cdots$$

CFT: $a_1 \sim \frac{1}{\delta_0^{d-2}}, \quad a_2 \sim \frac{1}{\delta_0^{d-4}}, \cdots$ δ_0 : bare UV cutoff

$$S_{\rm div}^{(\Sigma)} = \begin{cases} \frac{R^{d-2}}{\delta_0^{d-2}} + \cdots + \frac{R}{\delta_0} & \text{odd } d \\ \frac{R^{d-2}}{\delta_0^{d-2}} + \cdots + \frac{R^2}{\delta_0^2} + \log\frac{R}{\delta_0} & \text{even } d \end{cases}$$

Terms with negative powers of R are all UV finite.

previously known from holographic calculations (Ryu, Takayanagi)

General QFT: $a_1 = \frac{1}{\delta_0^{d-2}} h_1(\mu\delta_0)$ μ : some mass scale $\mu\delta_0 \ll 1$

$$h_1(\mu\delta_0) = c_0 + c_2(\mu\delta_0)^{2\alpha} + c_3(\mu\delta_0)^{3\alpha} + \cdots$$

In a renormalizable theory: cannot contain any singular dependence on μ in the $\mu \to 0$ limit.

UV finiteness (III)

Given: $$S_{\text{div}}^{(\Sigma)} = a_1 R^{d-2} + a_2 R^{d-4} + \cdots$$

$$\mathcal{S}_d^{(\Sigma)}(R) = \begin{cases} \frac{1}{(d-2)!!}\left(R\frac{d}{dR}-1\right)\left(R\frac{d}{dR}-3\right)\cdots\left(R\frac{d}{dR}-(d-2)\right)S^{(\Sigma)}(R) & \text{d odd} \\ \frac{1}{(d-2)!!}R\frac{d}{dR}\left(R\frac{d}{dR}-2\right)\cdots\left(R\frac{d}{dR}-(d-2)\right)S^{(\Sigma)}(R) & \text{d even} \end{cases}.$$

will then get rid of all UV divergent terms for any QFT.

The differential operator also gets rid of finite terms of the same R-dependence.

Such terms can be modified by redefining the cutoff, thus not well defined in the continuum limit (“contaminated”).

$\mathcal{S}_d^{(\Sigma)}(R)$ is thus UV finite, and unambiguous (independent of reparametrizations of the cutoff).

CFT

For a scale invariant system, we must have:

$$\mathcal{S}_d^{(\Sigma)}(R) = s_d^{(\Sigma)}$$

Converting it back to the EE itself, we then have

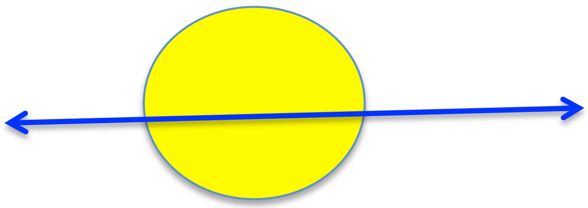
$$S^{(\Sigma)} = \begin{cases} \frac{R^{d-2}}{\delta_0^{d-2}} + \cdots + \frac{R}{\delta_0} + (-1)^{\frac{d-1}{2}} s_d^{(\Sigma)} & \text{odd d} \\ \frac{R^{d-2}}{\delta_0^{d-2}} + \cdots + \frac{R^2}{\delta_0^2} + (-1)^{\frac{d-2}{2}} s_d^{(\Sigma)} \log \frac{R}{\delta_0} + \text{const} & \text{even d} \end{cases}$$

This agrees with what was previously found from holographic calculations. (Ryu, Takayanagi)

$s_d^{(\Sigma)}$ is the "universal" part of the entanglement entropy.

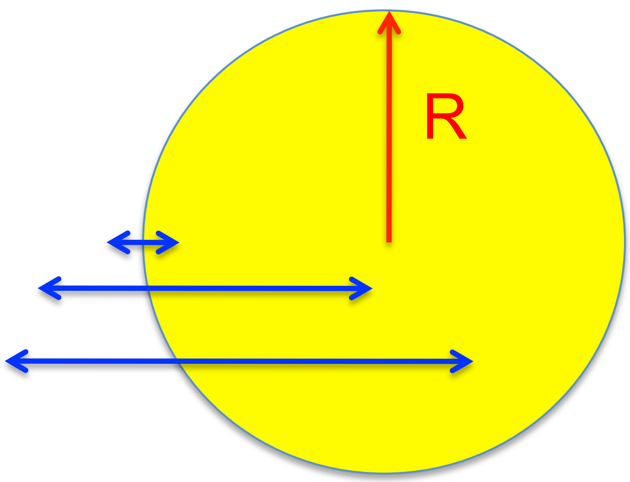
General QFTs

In the small R limit: $$S^{(\Sigma)}(R) \to S^{(\Sigma,\mathrm{UV})}, \quad R \to 0$$



$$\mathcal{S}_d^{(\Sigma)}(R) \to s_d^{(\Sigma,\mathrm{UV})}, \quad R \to 0$$

In the large R limit: $S^{(\Sigma)}(R)$ depends on scales from δ_0 to R including all intermediate scales of the system, $\mu_1, \mu_2, \cdots$



Nevertheless

$$\mathcal{S}_d^{(\Sigma)}(R) \to s_d^{(\Sigma,\mathrm{IR})}, \quad R \to \infty$$

General QFTs (II)

Introducing a floating cutoff δ :

$$\frac{1}{\mu_1}, \frac{1}{\mu_2}, \cdots \ll \delta \ll R$$

So that between δ and R there is no other physical length scales and the system can be well approximated by the IR fixed point,

$$\Rightarrow \quad S^{(\Sigma)}(\delta_0, R, \mu_1, \cdots) = S^{(\Sigma)}(\delta, R) \approx S^{(\Sigma,\mathrm{IR})}(\delta, R)$$

$$\Rightarrow \quad \mathcal{S}_d^{(\Sigma)}(R) \to s_d^{(\Sigma,\mathrm{IR})}, \qquad R \to \infty$$

In other words, in the large R limit, the contributions to EE from various mass parameters must have the form:

$$c_1 R^{d-2} + c_2 R^{d-4} + \cdots + O(R) + O((\mu R)^{-\#}) \qquad \text{(odd d)}$$

General QFTs (III)

Similarly, for any length L << R,

$$L \ll \delta \ll R$$

$\mathcal{S}_d^{(\Sigma)}(R)$ should not be sensitive to contributions from d.o.f. at scale L.

(Their contributions should be suppressed by positive powers of L/R.)

$\Rightarrow$ $\mathcal{S}_d^{(\Sigma)}(R)$ can be considered to directly probe and characterize entanglement at scale R.

The R-dependence can be interpreted as describing the “RG” flow of entanglement entropy with distance scale.

Summary

$$\mathcal{S}_d^{(\Sigma)}(R) = \begin{cases} \frac{1}{(d-2)!!}\left(R\frac{d}{dR}-1\right)\left(R\frac{d}{dR}-3\right)\cdots\left(R\frac{d}{dR}-(d-2)\right)S^{(\Sigma)}(R) & \text{d odd} \\ \frac{1}{(d-2)!!}R\frac{d}{dR}\left(R\frac{d}{dR}-2\right)\cdots\left(R\frac{d}{dR}-(d-2)\right)S^{(\Sigma)}(R) & \text{d even} \end{cases} .$$

- UV finite, well-defined in the continuum limit

- R-independent for a scale invariant system $\mathcal{S}_d^{(\Sigma)}(R) = s_d^{(\Sigma)}$

- For a general quantum field theory $\mathcal{S}_d^{(\Sigma)}(R) \to \begin{cases} s_d^{(\Sigma,\mathrm{UV})} & R \to 0 \\ s_d^{(\Sigma,\mathrm{IR})} & R \to \infty \end{cases} .$

- most sensitive to degrees of freedom at scale R.

can be considered as describing the RG flow of entanglement entropy

Note: definition not unique, simplest

Gapped systems

For a free massive scalar field for a spherical region
in the regime mR >> 1 in d=3:

$$S_{\text{scalar}}(mR) = \# \frac{R}{\delta} - \frac{\pi}{6} mR - \frac{\pi}{240} \frac{1}{mR} + \cdots$$

$$\mathcal{S}_{\text{scalar}}(mR) = + \frac{\pi}{120} \frac{1}{mR} + \cdots \qquad \to 0$$

In odd d, for generic gapped systems, we expect: (e.g. d=3)

$$\mathcal{S}_3^{(\Sigma)}(R) \to \gamma, \qquad R \to \infty \qquad \gamma :$$

Topological entanglement entropy

(Kitaev, Preskill; Levin, Wen)

In even d: $\mathcal{S}_{2n}^{(\Sigma)}(R) \to 0, \quad R \to \infty, \qquad n = 1, 2, \cdots$

An application: non-Fermi liquids

For a system with a Fermi surface, expect at large R:

$$\mathcal{S}_d^{(\Sigma)}(R) \propto A_{FS} \propto k_F^{d-2}$$

$$\mathcal{S}_d^{(\Sigma)}(R) \propto k_F^{d-2} R^{d-2} \propto A_{FS} A_\Sigma, \qquad R \to \infty$$

$$S^\Sigma(R) \sim k_F^{d-2} R^{d-2} \log(k_F R) \sim A_{FS} A_\Sigma \log(A_{FS} A_\Sigma)$$

Similarly for higher co-dimensional Fermi surfaces:

Wolf; Gioev, Klich
Swingle,
Swingle, Senthil

$$\mathcal{S}_d^{(\Sigma)}(R) \propto (k_F R)^{d-n}$$

(independent whether the system has quasiparticles or not)

$$S^\Sigma(R) \propto \begin{cases} (k_F R)^{d-n} \log(k_F R) & \text{n even} \\ (k_F R)^{d-n} & \text{n odd} \end{cases}$$

Renyi entropy

$$R_n(A) = \frac{1}{1-n} \log \mathrm{Tr} \rho_A^n$$

One can similarly define “renormalized Renyi entropies,” and exactly parallel discussion shows that for all n, Renyi entropies have the same structure as the entanglement entropy for a CFT.

The results for the (non)-Fermi liquids also apply to Renyi entropies.

EE and the number of d.o.f.

$\mathcal{S}_d^{(\Sigma)}(R)$ characterizes entanglement at scale R.

describes the RG flow of entanglement entropy

Could $\mathcal{S}_d^{(\Sigma)}(R)$ track the flow of the number of d.o.f. as we vary R?

Since RG flow leads to a loss of short-distance d.o.f.

$$R\frac{d\mathcal{S}_d^{(\Sigma)}(R)}{dR} < 0 \ ?$$

(much stronger condition)

$$\Rightarrow \quad s_d^{(\Sigma,\mathrm{UV})} > s_d^{(\Sigma,\mathrm{IR})}$$

i.e. a c-theorem.

d=2

$$\mathcal{S}_2(R) = R\frac{dS}{dR}$$

For a CFT $\qquad \mathcal{S}_2 = \frac{c}{3}$ $\qquad$ Holzhey, Larsen, Wilczek

For Lorentz-invariant, unitary QFTs $\qquad$ Casini and Huerta

$\mathcal{S}_2(R)$ monotonic $\qquad$ alternative proof of Zamolodchikov's c-theorem

Proof uses Lorentz symmetry and strong sub-additivity condition

$$S(A) + S(B) \geq S(A \cap B) + S(A \cup B)$$

Higher dimensions

$\mathcal{S}_d^{(\Sigma)}(R)$ now depends on the shape of Σ.

Will all shapes work?

d=4: for a CFT

Solodukhin

$$s_4^{(\Sigma)} = 2a_4 \int_\Sigma d^2\sigma \sqrt{h}\, E_2 + c_4 \int_\Sigma d^2\sigma \sqrt{h}\, I_2$$

a_4,c_4 : coefficients of trace anomaly

I_2 vanishes for sphere, $s_4^{(\mathrm{sphere})} = 4a_4$

For a general shape, will be a combination of a and c.

Thus only for a sphere, do we always have

$$s_4^{(\Sigma,\mathrm{UV})} > s_4^{(\Sigma,\mathrm{IR})}$$

Higher dimensions (II)

For all even spacetime dimensions:

Myers, Sinha
Casini, Myers, Heurta

$$s_{2n}^{(\mathrm{sphere})} = 4a_{2n}$$

Casini, Myers, Heurta

For all odd dimension: $s_d^{(\mathrm{sphere})} = (\log Z)_{\mathrm{finite}}$

$(\log Z)_{\mathrm{finite}}$: finite part of the Euclidean partition for the CFT on S^d

There are supports that these quantities could satisfy

$$s_d^{(\mathrm{sphere,UV})} > s_d^{(\mathrm{sphere,IR})}$$

Cardy,
Myers, Sinha
Jefferis, Klebanov, Pufu and Safdi

even dimension: coefficient of divergent term, local expression

Thus now focus on a sphere

$\mathcal{S}_d(R)$ if monotonic

- lead to the conjectured c-theorem in all dimensions

- give a scale-dependent measure of the number of d.o.f. for a general QFT.

d=3

$$\mathcal{S}_3(R) = R\frac{dS}{dR} - S$$

Free massive scalar and various holographic examples:

Conjecture: $\mathcal{S}_3(R)$ monotonically decreasing with R and non-negative

for all Lorentz invariant, unitary QFTs

Monotonicity $\Longleftrightarrow$ $S''(R) < 0$

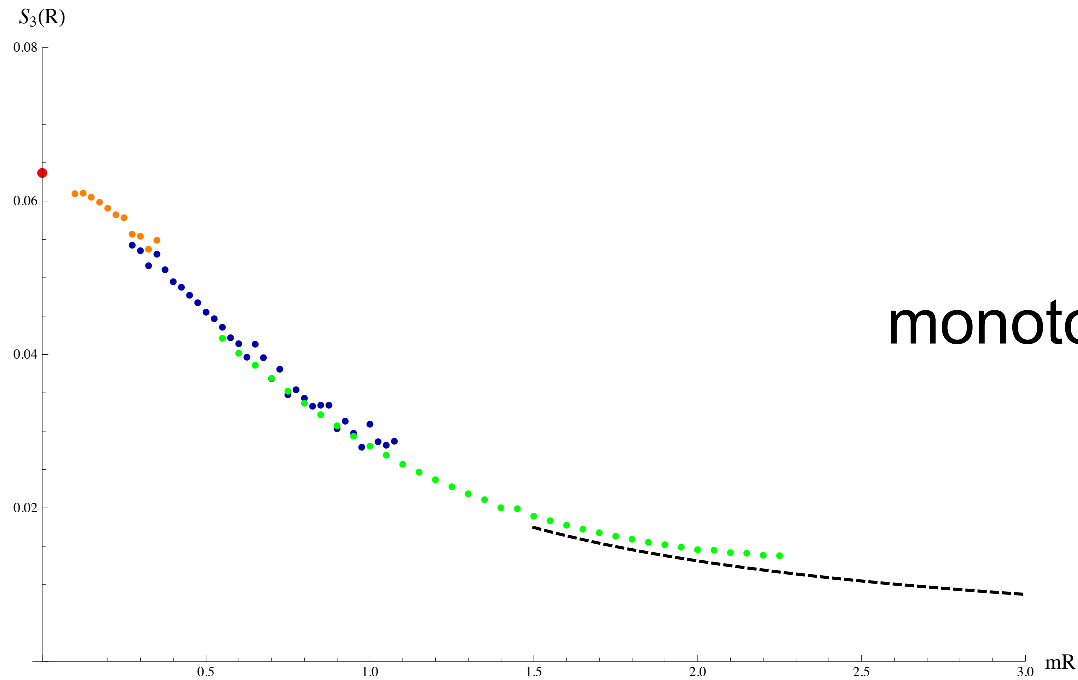
Casini and Huerta have given a proof shortly after (1202.5650).

But their proof does not appear to give non-negativeness.

Free massive scalar field

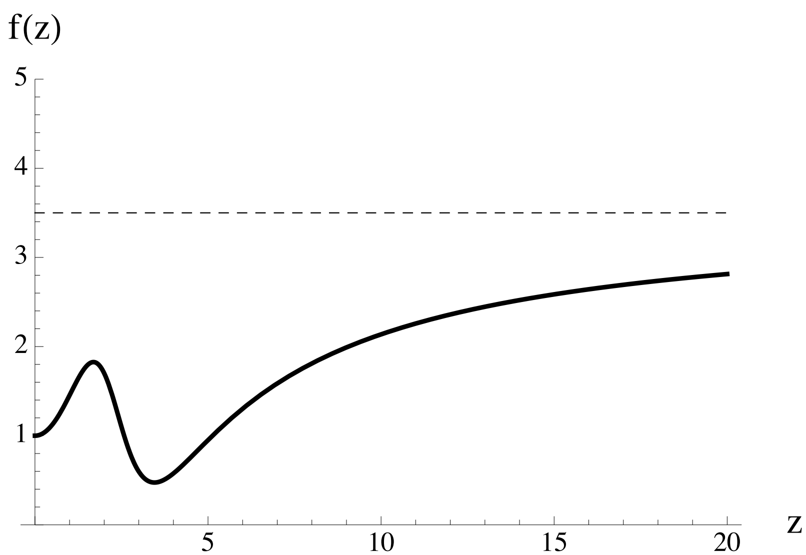
For a free massive scalar field for a spherical region in the regime mR >> 1 in d=3:

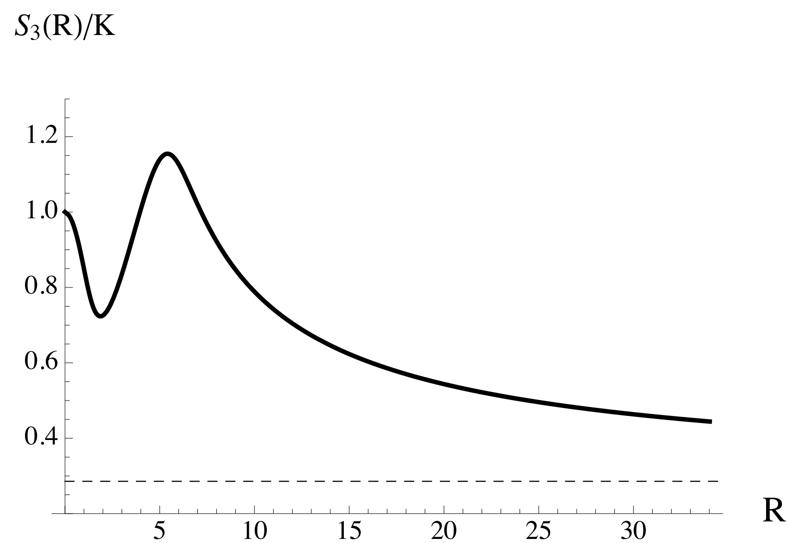
$$\mathcal{S}_{\text{scalar}}(mR) = +\frac{\pi}{120}\,\frac{1}{mR} + \cdots$$



monotonic for all R

Importance of unitarity





Null energy condition

⇒ monotonicity of f(z)

⇒ monotonicity of $\mathcal{S}_3(R)$

d=4

$$\mathcal{S}_4(R) = \frac{1}{2}\left(R^2\frac{d^2S}{dR^2} - R\frac{dS}{dR}\right)$$

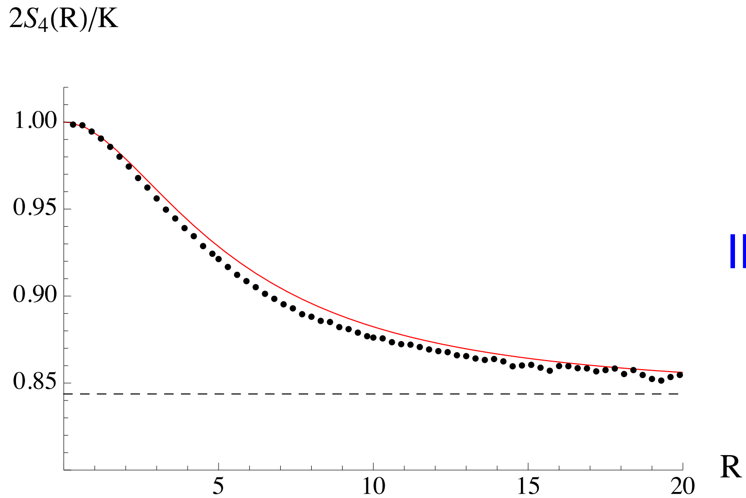
Various holographic examples:

$\mathcal{S}_4(R)$ neither monotonic nor non-negative

Nevertheless $\mathcal{S}_4(R \to 0) > \mathcal{S}_4(R \to \infty)$

from a-theorem

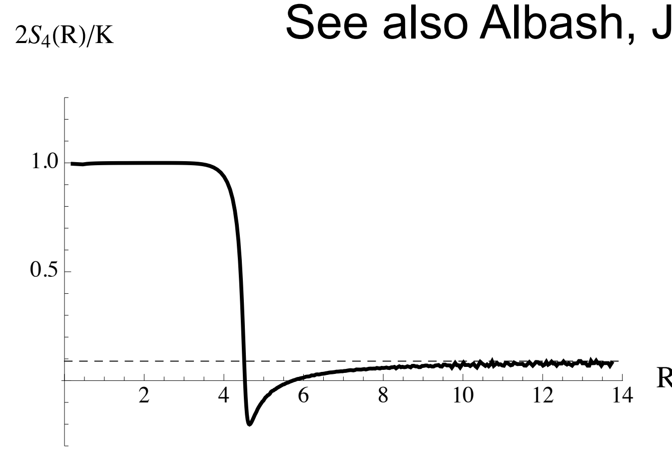
Some examples



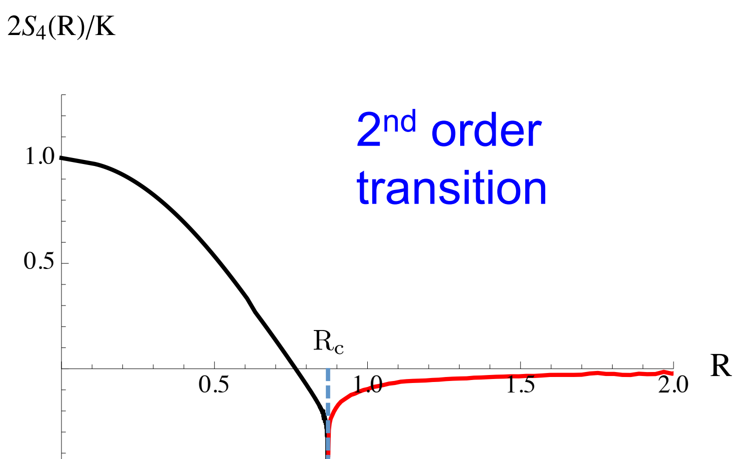
d=4: Leigh-Strassler flow

IR: CFT

See also Albash, Johnson



A toy flow

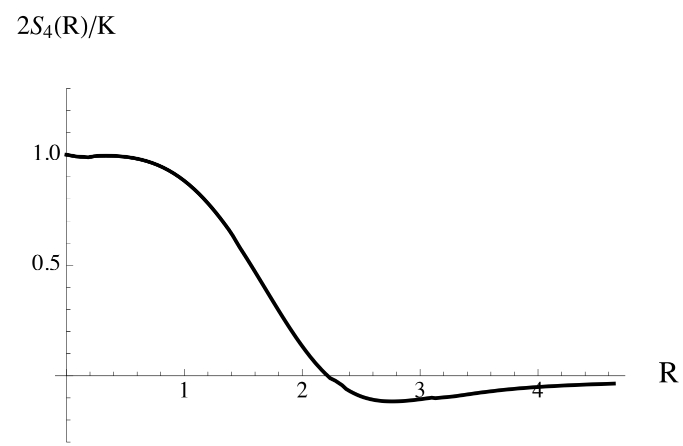


2nd order transition

IR: gapped phases

GPPZ flow

(Girardello, Petrini, Porrati,Zaffaroni)



Coulomb branch flow

(Freedman, Gubser, Pilch, Warner)

d=4

$\mathcal{S}_4(R)$ neither positive-definite nor always monotonic

- the function form should be modified

- Monotonicity of $\mathcal{S}_4$ or its improvement would imply an inequality for S with least three derivatives.

$$R^3\partial_R^3 S + R^2\partial_R^2 S < R\partial_R S$$

Not clear it could arise from the strong subadditivity condition.

Despite this: $\mathcal{S}_4$ characterizes "scale-dependent entanglement"

Its non-monotonicity and non-positive-definiteness likely reflects some interesting underlying physics

d>4

When two fixed points are close, i.e .

$$\frac{s_d^{\rm UV} - s_d^{\rm IR}}{s_d^{\rm UV}} \ll 1$$

$\mathcal{S}_d(R)$ is always monotonically decreasing in holographic systems

Behavior near a UV fixed point

In all holographic theories:

For small R $\mathcal{S}_d(R) = s_d^{(\mathrm{UV})} - A(\alpha)(\mu R)^{2\alpha} + \cdots \; A(\alpha) > 0$

$\alpha = d - \Delta$ (source flow) $\qquad \alpha = \Delta$ (vev flow)

$$\mathcal{S}_d = s_d^{(\mathrm{UV})} - O(g^2)$$ g: least relevant coupling

See also Klebanov, Nishioka, Pufu, Safdi

Free massive field in 2+1 dimension:

$$\partial_{m^2 R}\mathcal{S}_3\Big|_{m^2R^2=0} \neq 0$$

Klebanov, Nishioka, Pufu, Safdi

Behavior near an IR fixed point

HL, Mezei, to appear

$$\tilde{\alpha} = \Delta - d < \begin{cases} \frac{1}{2} & \text{odd d} \\ 1 & \text{even d} \end{cases}$$

Δ Dimension of leading irrelevant operator

For large R $\quad \mathcal{S}_d(R) = s_d^{(\mathrm{IR})} + \frac{B(\tilde{\alpha})}{(\tilde{\mu}R)^{2\tilde{\alpha}}} + \cdots \quad B(\tilde{\alpha}) > 0$

$$\sim s_d^{(IR)} + O(g^2)$$

For $\tilde{\alpha}$ outside the above range:

$$\mathcal{S}_d(R) = s_d^{(\mathrm{IR})} + \begin{cases} \frac{\#}{R} + \cdots & \text{odd d} \\ \frac{\#}{R^2} + \cdots & \text{even d} \end{cases}$$

$$\mathcal{S}_d = s_d^{(\mathrm{IR})} + C g^{\gamma} \quad \gamma = \begin{cases} \tilde{\alpha}^{-1} & \text{odd d} \\ 2\tilde{\alpha}^{-1} & \text{even d} \end{cases} < 2$$

C: “nonlocal”
Sign of C not definite in d=4

“Phase transitions”

We also observed that in holographic systems, the entanglement entropy has “phase transitions” in the Lorentz-invariant vacuum as a function of size:

can be first order or second order

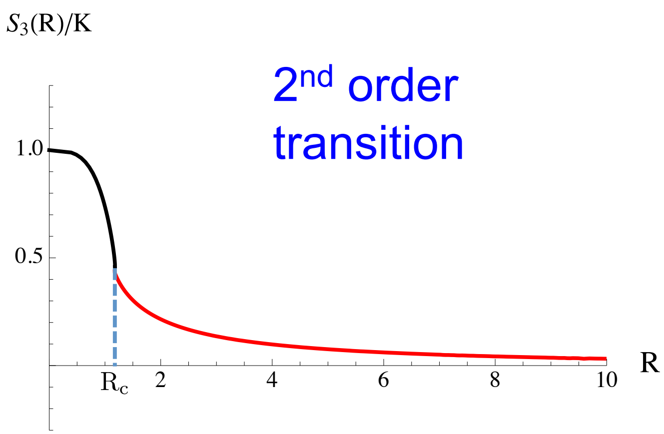
Involving topology change or no topology change

See also Klebanov, Nishioka, Pufu,Safdi

Not yet clear what these phase transitions tell us.

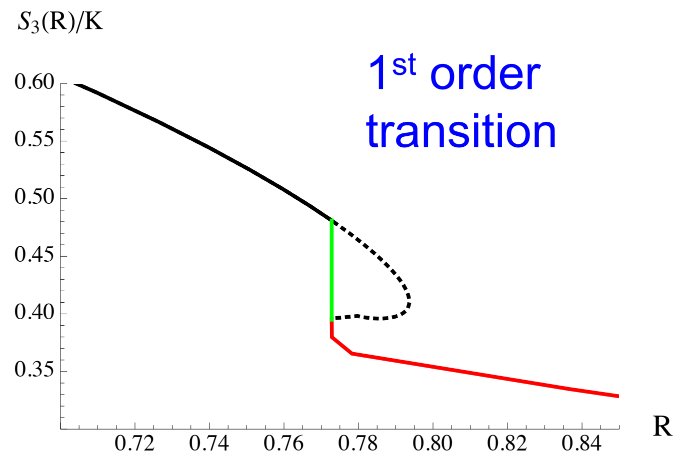
Nishioka and Takayanagi
Klebanov, Kutasov, Murugan
Pakman, Parnachev
Headrick
Albash and Johnson….

Some examples

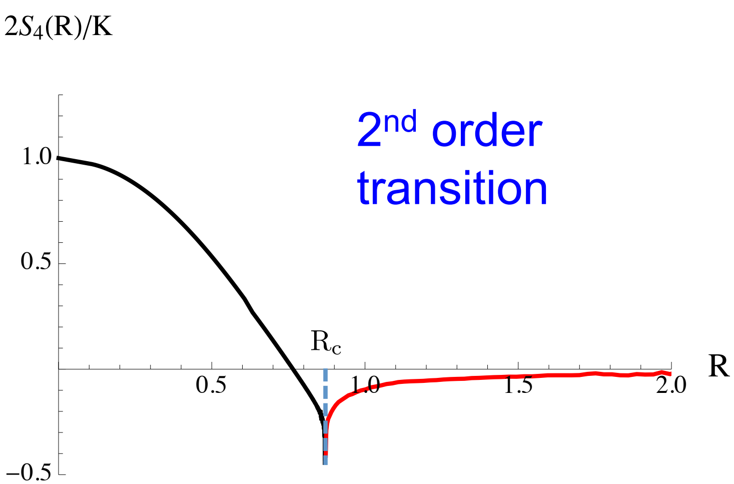


2nd order transition

d=3

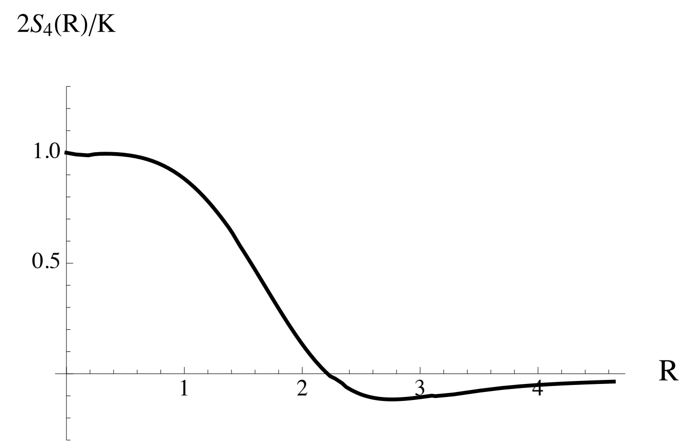


1st order transition



2nd order transition

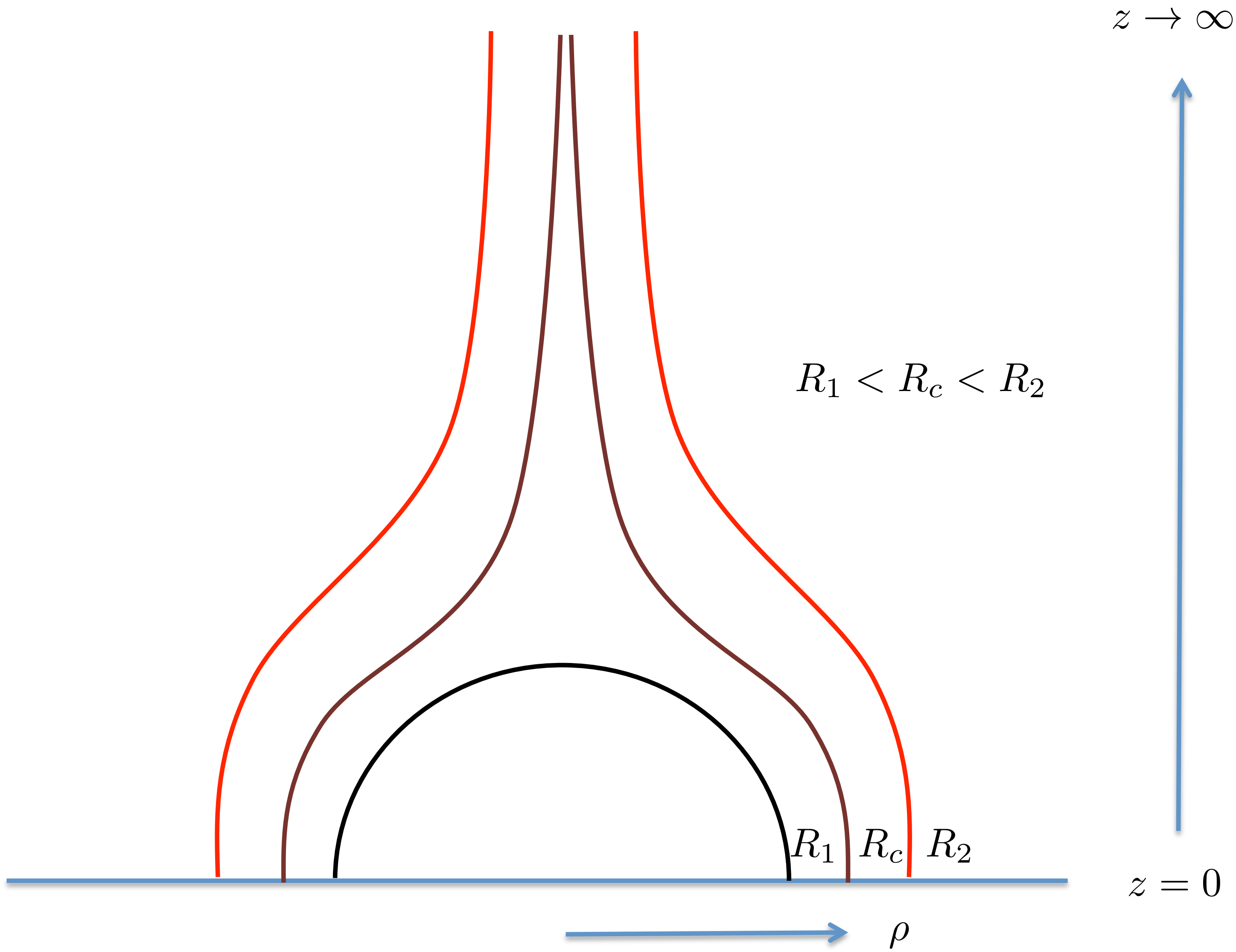
d=4



GPPZ flow (Girardello, Petrini, Porrati, Zaffaroni)

Coulomb branch flow (Freedman, Gubser, Pilch, Warner)

2nd order phase transitions involving topology change



Summary

For any renormalizable quantum field theory (not necessarily Lorentz-invariant), "renormalized entanglement entropy:"

- probe and characterize quantum entanglement at a given scale.
- for d=2,3, C-function, candidate for a measure of the number of degrees of freedom of the system at a given scale (with Lorentz symmetry)
- Non-relativistic ?
- an intrinsically finite definition (mutual information) ?