Entanglement, holography, and RG flows in 3d and 5d

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Based on work with C. Herzog, D. Jafferis, I. Klebanov, T. Nishioka, S. Sachdev, B. Safdi, and T. Tesileanu.

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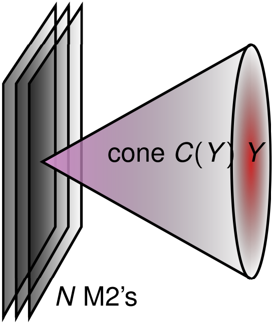
Introduction

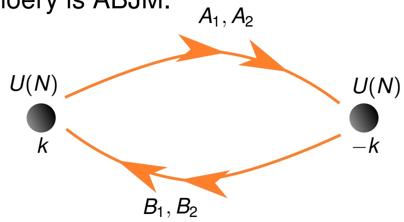
4 facts about holography, entanglement, and RG flows:

1. There are many conjectured dualities between gravity in AdS and SCFTs. In this talk AdS_4 and AdS_6.

Example: $AdS_4 \times Y$ geometry close to N M2-branes.



When $Y = S^7/\mathbb{Z}_k$, dual gauge thoery is ABJM.



For many other Sasaki-Einstein spaces Y, the duals are quiver gauge theories.

Also: warped solutions and holographic RG flows.

BRANE

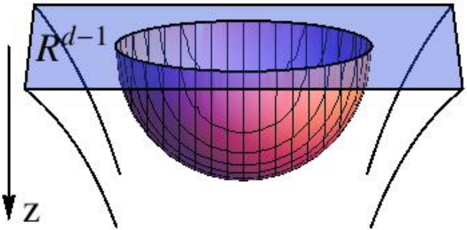
The brane is flat, not stuffed.
It is 3 fabric layers thick.
Made of acrylic felt, fleece, cord.

From the word "membrane," a **BRANE** is a "membrane-like object in higher-dimensional space that can carry energy and confine particles and forces." This plush brane contains two closed and three open strings. Theorists believe these "strings" of string theory correspond to different particles. A graviton, for example, is a mode of a closed string. Open strings, on the other hand, would be gauge bosons and fermions.

$16.99 PLUS SHIPPING

GLUON PHOTON NEUTRINO TACHYON ELECTRON COSMIC MICROWAVE BACKGROUND RADIATION DOW
NEUTRINO MUON UP QUARK **BRANE** NEUTRON DOWN QUARK NEUTRINO TACHYON ELECTRON UP QU

2. The entanglement entropy can be computed holographically from the area of a minimal surface in the bulk.



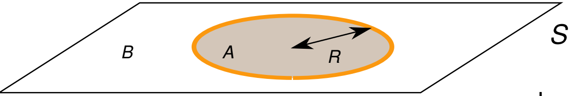
- In AdS_{d+1}, minimize [Ryu, Takayanagi '06]

$$S = \frac{1}{4G_{d+1}} \int d^{d-1}x \sqrt{g_{\text{induced}}}$$

- In 10d string theory, minimize [Ryu, Takayanagi; Klebanov, Kutasov, Murugan '08]

$$S = \frac{1}{32\pi^6 \ell_s^8} \int d^8x \, e^{-2\phi} \sqrt{g_{\text{induced}}} \,.$$

3. For a CFT_3: Universal part of disk EE = -the free energy F on S^3 [Casini, Huerta, Myers '11] .



Disk of radius R, UV cutoff ϵ:

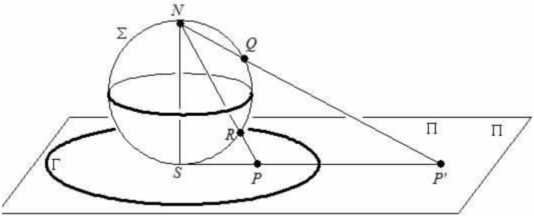
$$S(R) = \alpha \frac{R}{\epsilon} - F + O(\epsilon/R)\,,$$

where $S = -\operatorname{tr}(\rho_A \log \rho_A)$ with $\rho_A \equiv \operatorname{tr}_B |0\rangle\langle 0|$.

- Map from $\mathbb{R}^3$ to S^3 using stereographic projection.

$$\log |Z_{S^3}| = ar^3 + br + F\,.$$

- Remove cubic and linear divergence.



4. *F*-theorem for all 3d Lorentz-invariant theories: $F_{\rm UV} > F_{\rm IR}$ whenever there is an RG flow connecting $\text{CFT}_{\rm UV}$ to $\text{CFT}_{\rm IR}$.
   - First established in holographic theories [Myers, Sinha '10] .
   - [Liu and Mezei '12] propose that the Renormalized Entanglement Entropy

     $$\mathcal{F}(R) = RS'(R) - S(R)$$

     interpolates monotonically between $F_{\rm UV} = \mathcal{F}(0)$ and $F_{\rm IR} = \mathcal{F}(\infty)$.
   - Proof that $\mathcal{F}'(R) = RS''(R) < 0$ in [Casini, Huerta '12] using subadditivity of entanglement entropy.

Motivational questions:

- Is there an ordering on the set of 3d / odd dimensional CFTs?
  - In even-dimensional CFT, such an ordering is given by the conformal anomaly coefficients c (in 2d) and a (in 4d) [Zamolodchikov '86; Cardy '88; Komargodski, Schwimmer '11]

$$\langle T^\mu_\mu \rangle_{2\mathrm{d}} = -\frac{c}{12} R \,, \quad \langle T^\mu_\mu \rangle_{4\mathrm{d}} = -\frac{a}{16\pi^2} \text{Euler density} + \frac{c}{16\pi^2} \text{Weyl}^2$$

- Are all instances of AdS/CFT correct?
  - For $AdS_4 \times Y$ backgrounds in M-theory, $F \propto N^{3/2}/\sqrt{\mathrm{Vol}(Y)}$.
  - For warped AdS_4 backgrounds in massive type IIA, $F \propto N^{5/3}$.
  - For warped AdS_6 backgrounds in massive type IIA, $F \propto N^{5/2}$.
- Is there an equivalent of a-maximization in $\mathcal{N} = 2$ 3d SCFT?

Is F calculable?

So far, F was computed in:

- Free theories: free scalar, free fermion, CS theory, free Maxwell field (through a direct evaluation of the path integral on S^3).
- Slightly relevant perturbations of a CFT (through conformal perturbation theory).
- Large N expansions:
    - Gauge theories with many flavors (through perturbation theory).
    - Gauge theories with many colors (through holography).
- Supersymmetric CFTs (through supersymmetric localization). Overlap with all of the above.

Outline

- Free field results.
- Large N_f expansion.
- Supersymmetry and holography in 3d.
- Supersymmetry and holography in 5d.

Simplest example: Free scalar field

- For a free *massless* scalar field, we calculate free energy on S^3.
- Action for a conformally coupled scalar

$$S = \frac{1}{2}\int d^3x\,\sqrt{g}\left[(\partial_\mu\phi)^2 + \frac{1}{8}R\phi^2\right].$$

- Evaluate $e^{-F} = \int D\phi\, e^{-S}$ by doing explicitly the Gaussian integral:

$$F = \frac{1}{2}\operatorname{tr}\log\frac{\mathcal{O}_S}{2\pi}\,, \qquad \mathcal{O}_S \equiv -\nabla^2 + \frac{R}{8}\,.$$

- Eigenvalues of $\mathcal{O}_S$ are

$$\lambda_n = \left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right), \qquad n \geq 0$$

with multiplicity $m_n = (n+1)^2$.

- Evaluating $F = \frac{1}{2}\sum_{n=0}^{\infty} m_n \log(\lambda_n/(2\pi))$ through zeta-function regularization, one obtains

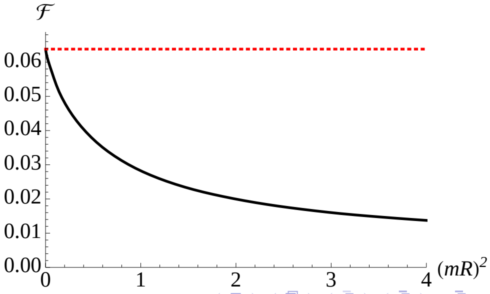
$$F_S = \frac{1}{16}\left(2\log 2 - \frac{3\zeta(3)}{\pi^2}\right) \approx 0.0638\,.$$

- Note $F_S > 0$, which is consistent with the RG flow triggered by $m^2\phi^2$ between the free scalar CFT and the trivial CFT.

One can compute $S(R)$ numerically and evaluate

$$\mathcal{F}(R) = RS'(R) - S(R)\,.$$

The derivative w.r.t. the coupling $g = m^2$ is nonzero at the UV fixed point!



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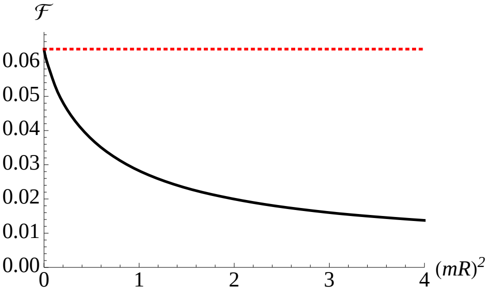

More Gaussian integrals

Other theories where we can also do a Gaussian integral:

- Free massless Dirac fermion:

$$S = \int d^3x \sqrt{g} \psi^\dagger (i \not{D}) \psi \,, \qquad F_D = \frac{\log 2}{4} + \frac{3\zeta(3)}{8\pi^2} \approx 0.219 \,.$$

  - Note: $F_D > 0$, in agreement with the F-theorem. This time $\mathcal{F}(mR)$ for a massive scalar is monotonic and stationary in the UV.

- $U(1)$ CS theory at level k

$$S = \frac{ik}{4\pi} \int A \wedge dA \,, \qquad F_{\text{CS}} = \frac{1}{2} \log k \,.$$

  - Careful about gauge fixing!

- Maxwell field on S^3 (non-conformal theory!):

$$S = \frac{1}{4e^2} \int d^3x \sqrt{g} F_{\mu\nu} F^{\mu\nu} \,, \qquad F_{\text{Maxwell}} = -\frac{1}{2} \log(e^2 r)$$

  - We should really assign $F_{\text{Maxwell}} = \infty$ in the UV.

Perturbation theory around free field results

- Consider CS theory at level k with N_f fermions and N_b bosons with electric charges q_f and q_b. In flat space:

$$S = \int d^3x \left[\sum_{\alpha=1}^{N_f} \psi_\alpha^\dagger (i\not{D}) \psi_\alpha + \sum_{a=1}^{N_b} |D_\mu z_a|^2 \right] + \frac{ik}{4\pi} \int A \wedge dA$$

- After integrating out ψ_α and z_a, we get an effective action for A:

$$S_{\text{eff}} = \frac{ik}{4\pi} \int A \wedge dA + \frac{1}{2} \int dx\, dy\, A_\mu(x) A_\nu(y) \langle J^\mu(x) J^\nu(y) \rangle + \dots\,,$$

where in flat space

$$\langle J^\mu(x) J^\nu(y) \rangle = \frac{N_f q_f^2 + N_b q_b^2}{8\pi^2} \frac{\delta^{\mu\nu} |x-y|^2 - 2(x^\mu - y^\mu)(x^\nu - y^\nu)}{|x-y|^6}\,.$$

- If we take k, N_f and N_b to be very large, the A integral is dominated by the quadratic terms in the effective action.

Perturbation theory around free field results

- The answer is

$$F = \frac{\log 2}{4}(N_f + N_b) + \frac{3\zeta(3)}{8\pi^2}(N_f - N_b) + \frac{1}{2}\log\left[\pi\sqrt{\left(\frac{N_f q_f^2 + N_b q_b^2}{8}\right)^2 + \left(\frac{k}{\pi}\right)^2}\right] + \cdots .$$

- As suggested later, this may be a good approximation down to small values of k, N_f, and N_b.

Supersymmetric theories: Intro to 3d SUSY

- $\mathcal{N} = 2$ SUSY multiplets in 3d can be obtained by dimensional reduction of $\mathcal{N} = 1$ SUSY multiplets in 4d.
- $\mathcal{N} = 2$ vector multiplet: gauge field A_μ, real scalar σ, complex spinor λ transforming in the adjoint representation of the gauge group.
- $\mathcal{N} = 2$ chiral multiplet: complex scalar ϕ and complex spinor ψ.
- $\mathcal{N} = 4$ vector multiplet $=$ one $\mathcal{N} = 2$ vector + one adjoint chiral.
- $\mathcal{N} = 4$ hypermultiplet $=$ two $\mathcal{N} = 2$ chirals transforming in conjugate representations.

Supersymmetric theories: Localization on S^3

- Due to [Kapustin, Willett, Yaakov '10; Jafferis '10] .
- Exact evaluation of $Z_{S^3} = \int [DX] e^{-S}$ for $\mathcal{N} \geq 2$ SUSY.
- Clever trick: change the theory by considering $S_t = S + t\{Q, \mathcal{V}\}$ where Q is a supercharge and $\mathcal{V}$ is such that $\{Q, \mathcal{V}\}$ is positive definite.
  - Example: For a vector multiplet take $\mathcal{V} = \operatorname{tr}\left((Q\lambda)^\dagger \lambda\right)$ and for a chiral multiplet take $\mathcal{V} = \operatorname{tr}\left[(Q\psi)^\dagger \psi + \psi^\dagger (Q\psi^\dagger)^\dagger\right]$.
- One can show that $Z = \int [DX] e^{-S_t}$ is independent of t.
- Then take t to be large. The integral localizes where $\{Q, \mathcal{V}\} = 0$:

$$Z = \sum_{\{Q,\mathcal{V}\}=0} e^{-S_t\big|_{\{Q,\mathcal{V}\}=0}} \int D(\delta X) e^{-\frac{1}{2}\int d^3x \sqrt{g} \frac{\delta^2 S_t}{\delta X^2}\big|_{\{Q,\mathcal{V}\}=0} (\delta X)^2} .$$

  This is an exact result!!!
- $\{Q, \mathcal{V}\} = 0$ implies $A_\mu = \phi = 0$ and $\sigma = \text{const}$ (Coulomb branch).

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- $\{Q, \mathcal{V}\} = 0$ implies $A_\mu = \phi = 0$ and $\sigma = \text{const}$ (Coulomb branch).

- Catch: for some $\mathcal{N}=2$ theories we may only have a UV description, and we may not know the superconformal algebra in the IR.
- If r_i is a reference R-charge, and Q_i^a are $U(1)$ flavor charges,

$$R[X_i] = r_i + \sum_a t_a Q_i^a \,.$$

- If at the IR fixed point we hit a non-trivial SCFT, we require $\frac{\partial F}{\partial t_a} \propto \int \langle J_a \rangle = 0$ [Jafferis '10] .
- F is actually maximized because $\frac{\partial^2 F}{\partial t_a \partial t_b} \propto -\int \langle J_a J_b \rangle$, which is negative definite [Closset *et al.* '12] .
- Answer:

$$Z = \int_{\text{Cartan}} d\sigma \, e^{i\pi k \sigma^2} \det{}_{\text{Ad}} \left(\sinh(\pi\sigma)\right) \prod_{\text{chirals } X_i \text{ in } \mathcal{R}_i} e^{\mathrm{tr}_{\mathcal{R}_i} \ell(1-R[X_i]+i\sigma)} \,,$$

where

$$\ell(z) = -z \ln\left(1 - e^{2\pi i z}\right) + \frac{i}{2}\left(\pi z^2 + \frac{1}{\pi}\mathrm{Li}_2\left(e^{2\pi i z}\right)\right) - \frac{i\pi}{12} \,.$$

$1/N$ expansion

- Localization and F-maximization allow us to compute F in all $\mathcal{N} \geq 2$ supersymmetric theories.

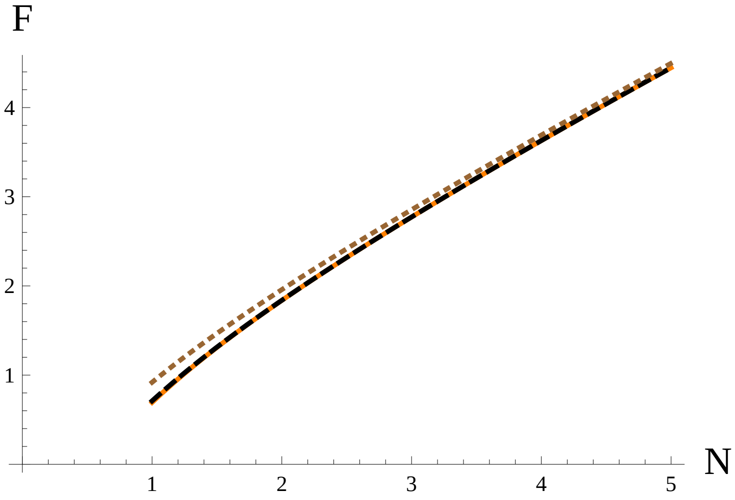
- In $\mathcal{N} = 4$ $U(1)$ gauge theory with N hypers, the partition function is

$$Z = e^{-F} = \frac{1}{2^N} \int_{-\infty}^{\infty} \frac{d\lambda}{\cosh^N(\pi\lambda)} = \frac{2^{-N}\Gamma\left(\frac{N}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{N+1}{2}\right)} .$$

- $1/N$ expansion is quite powerful:

$$F = -\log Z = N \log 2 + \frac{1}{2} \log\left(\frac{N\pi}{2}\right) - \frac{1}{4N} + \frac{1}{24N^3} + \dots .$$

Exact result in orange, large N approximation in black and brown.



- Consider $\mathcal{N} = 2$ CS theory with level k and N pairs of fundamental and anti-fundamental flavors with no superpotential.
- Assume the flavor fields have R-charge

$$\Delta = \frac{1}{2} + \frac{\Delta_1}{N} + \dots$$

- Denoting $\kappa = 2k/(N\pi)$, the free energy is

$$F(\Delta) = N \log 2 + \frac{1}{2} \log\left(\frac{N\pi}{2}\sqrt{1+\kappa^2}\right) - \left(\frac{\pi^2\Delta_1^2}{2} + \frac{2\Delta_1}{1+\kappa^2} + \frac{1-\kappa^2}{4(1+\kappa^2)^2}\right)\frac{1}{N} + \dots .$$

- Extremizing over Δ_1 one finds

$$\Delta_1 = -\frac{2}{\pi^2(1+\kappa^2)} .$$

- Add superpotential $\sum (Q\tilde{Q})^2 \Longrightarrow$ flow to IR where $\Delta_1 = 0$. $F_{\text{UV}} > F_{\text{IR}}$ because of F-maximization.

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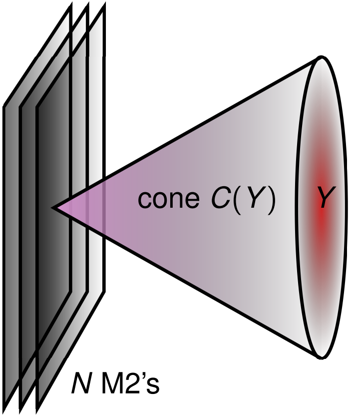
Holographic computations

M-theory background $AdS_4 \times Y$, where Y is threaded by N units of 7-form flux.

- If L is the radius of AdS_4, the evaluation of the on-shell action gives

$$F = \frac{\pi L^2}{2G_4} = N^{3/2}\sqrt{\frac{2\pi^6}{27\mathrm{Vol}(Y)}}\,,$$

  where G_4 is the effective Newton constant in 4d.
- Same result can be obtained from computing the holographic entanglement entropy.



- Simplest case: $Y = S^7$.
- A presentation of the dual field theory is: $\mathcal{N} = 4$ $U(N)$ vector multiplet, adjoint hyper, and fundamental hyper.

$$Z = \int d\lambda_i e^{-F}\,, \qquad F = \sum_{i\neq j}\left[F_V(\lambda_i - \lambda_j) + F_H(\lambda_i - \lambda_j)\right] + \sum_i F_H(\lambda_i)\,,$$

$$F_V(\lambda) = -\log(2\sinh(\pi\lambda)) = -\pi|\lambda| + O(e^{-2\pi|\lambda|})\,,$$

$$F_H(\lambda) = \log(2\cosh(\pi\lambda)) = \pi|\lambda| + O(e^{-2\pi|\lambda|})\,.$$

- Integral is dominated by the minimum of F.
- Take $\lambda_i = N^\alpha x_i$ and introduce $\rho(x) = \frac{1}{N}\sum_{i=1}^N \delta(x - x_i)$.
- Long-range forces cancel:

$$F = \frac{\pi}{4}N^{2-\alpha}\int dx\, \rho(x)^2 + \pi N^{1+\alpha}\int dx\, \rho(x)|x|\,.$$

- Balancing out these terms implies $\alpha = 1/2$, so $F \propto N^{3/2}$.

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- Minimizing $F[\rho]$ under the constraints $\int dx\,\rho(x) = 1$ and $\rho(x) \geq 0$ almost everywhere,

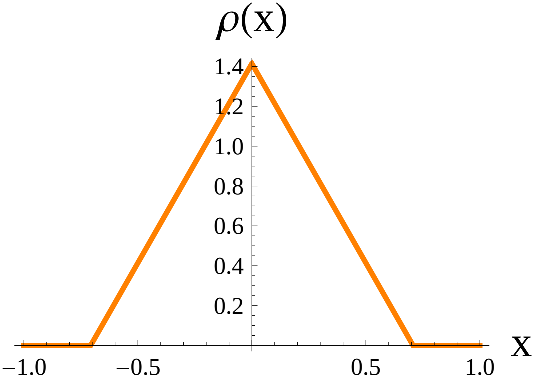
$$\rho(x) = \sqrt{2} - 2|x|\,, \qquad x \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]\,.$$

- The minimum is

$$F = \frac{\pi\sqrt{2}}{3} N^{3/2}$$

in agreement with the holographic computation.

- Perfect agreement for many other 3d field theories with AdS_4 duals.

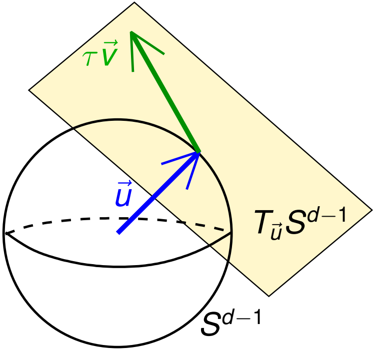


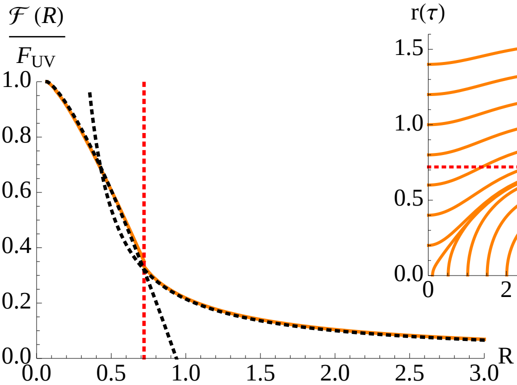
Holographic RG flow in 11d

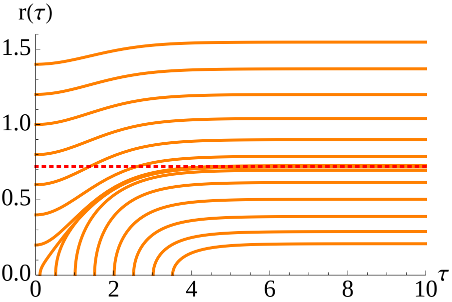
M-theory background corresponding to a holographic RG flow:

$$ds^2 = H(\tau)^{-2/3}(-dt^2 + dr^2 + r^2 d\theta^2) + H(\tau)^{1/3} ds^2_{TS^4}$$

- τ is the radial coordinate in each tangent plane.
- At large τ spacetime approaches $AdS_4 \times V_{5,2}$.
- At $\tau = 0$ we have $\mathbb{R}^{2,1} \times S^4$.
- There is a mass gap.
- Look for the minimal surface $r(\tau)$ for which $r(\infty) = R$, and then vary R.







- Change of topology from $B_2 \times V_{5,2}$ to $S^1 \times TS^4$. See also [Liu and Mezei '12] .
- One can prove that in holographic theories $\mathcal{F}(R)$ is always stationary in the UV.

AdS_6/CFT_5

- The only known AdS_6 solutions are in massive type IIA supergravity ($N_f < 8$ D8 branes, O8 planes, a stack of N D4-branes).
- The metric is a warped product of AdS_6 and half of an S^4. Pretty singular:

$$ds^2 = \frac{1}{(\sin\alpha)^{1/3}}\left[L^2\frac{-dt^2 + d\vec{x}^2 + dz^2}{z^2} + \frac{4L^2}{9}\left(d\alpha^2 + \cos^2\alpha\, ds^2_{S^3}\right)\right]$$

where

$$\frac{L^4}{\ell_s^4} = \frac{18\pi^2 N}{8 - N_f}\,.$$

There is also a nontrivial dilaton

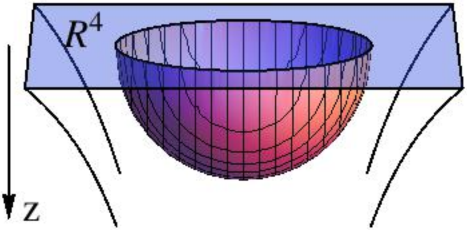
$$e^{-2\phi} = \frac{3(8 - N_f)^{3/2}\sqrt{N}}{2\sqrt{2}\pi}(\sin\alpha)^{5/3}\,.$$

- The on-shell Lagrangian is not integrable!
- The three-ball entanglement entropy is finite:

$$S = \frac{1}{32\pi^6 \ell_s^8} \int d^8x\, e^{-2\phi} \sqrt{g} \propto \int d\alpha\, (\sin\alpha)^{1/3} (\cos\alpha)^3 \,.$$

- Finding the minimal surface that ends on a sphere of radius R at the boundary,

$$F = -\frac{9\sqrt{2}\pi N^{5/2}}{5\sqrt{8 - N_f}} \,.$$



Dual field theory

- Dual field theory is known due to [Seiberg '96] .
  - No known Lagrangian description!
  - Under a relevant deformation it flows to $\mathcal{N} = 1$ $USp(2N)$ super-Yang-Mills with N_f fundamental hyper and one hyper in the antisymmetric representation of $USp(2N)$.
- SYM theory localizes on Coulomb branch, like in 3d [Hosomichi *et al.* '12; Kallen *et al.* '12] :

$$Z = \int_{\text{Cartan}} d\sigma e^{-\left(\frac{4\pi^3 r}{g_{\text{YM}}^2} \operatorname{tr}_F \sigma^2 + \operatorname{tr}_{\text{Ad}} F_V(\sigma) + \sum_I \operatorname{tr}_{\mathcal{R}_I} F_H(\sigma)\right)} + \text{instantons}\,,$$

where at large $|\sigma|$,

$$F_V(\sigma) \approx \frac{\pi}{6}|\sigma|^3 - \pi|\sigma|\,,$$
$$F_H(\sigma) \approx -\frac{\pi}{6}|\sigma|^3 - \frac{\pi}{8}|\sigma|\,.$$

- Set $g_{\text{YM}} = \infty$ and ignore instantons.

- For Seiberg's field theory, we should extremize

$$F(\lambda_i) = \sum_{i \neq j} \left[F_V(\lambda_i - \lambda_j) + F_V(\lambda_i + \lambda_j) + F_H(\lambda_i - \lambda_j) + F_H(\lambda_i + \lambda_j) \right]$$
$$+ \sum_i \left[F_V(2\lambda_i) + F_V(-2\lambda_i) + N_f F_H(\lambda_i) + N_f F_H(-\lambda_i) \right] .$$

- Assuming $\lambda_i = N^\alpha x_i$ and introducing the density of x_i,

$$F \approx -\frac{9\pi}{8} N^{2+\alpha} \int dx\, dy\, \rho(x)\rho(y) \left(|x - y| + |x + y| \right)$$
$$+ \frac{\pi(8 - N_f)}{3} N^{1+3\alpha} \int dx\, \rho(x) |x|^3 .$$

- To balance the two terms one requires $\alpha = 1/2$ and $F \propto N^{5/2}$.

- Extremizing $F[\rho]$ w.r.t. ρ one obtains

$$\rho(x) = \frac{2|x|}{|x_*|}\,, \qquad x_* = \frac{9}{2(8-N_f)}\,.$$

- The extremum of F is in agreement with holographic entanglement entropy computation!

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- We've managed to extract a property of the UV theory from IR physics!
- This is a test of the Ryu-Takayanagi prescription.
- There exist RG flows for which N_f decreases. Clearly, $(-F)$ decreases too.

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Conclusions

- Disk renormalized entanglement entropy / free energy on S^3 is monotonic under RG flow.
- F can be computed in perturbation theory and via localization in 3d supersymmetric theories.
- Potentially confusing: the interpolating function between F_{UV} and F_{IR} does not appear to always be stationary at fixed points.
- Is there an F theorem in 5d?