Entanglement and Geometry in 2d CFT

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U. Michigan Black Holes in String theory
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Emergent Geometry

AdS/CFT gives a UV-complete definition of quantum gravity in certain cases.

This is not the end of the story, even in AdS:

- How does the bulk geometry emerge from CFT?
- CFT language is so different from the bulk that basic bulk questions cannot be answered (high-E scattering, information, firewalls, etc.)

Need to find right language to describe physics in the bulk.

- "Build AdS from CFT."

Emergent Geometry

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We should be able to find a large class of CFTs with:

- Thermodynamic entropy = area of black hole horizon
- $\eta/s \sim 1/4\pi$
- Hawking-Page transition
- Entanglement entropies = Ryu-Takayanagi formula
- etc....

These features are universal in any CFT with an Einstein-gravity dual.

--> towards a language for bulk quantum gravity

Emergent Geometry

In 1+1d CFT / 3d AdS:

- YES we can derive this, probably.
- A natural conjecture is:

$$\text{“Virasoro symmetry} \;+\; \text{large } c \;+\; \text{gap} \;\Rightarrow\; \text{3d gravity”}$$

Symmetry is not enough.

- These assumptions allow a bulk dual which is a perturbative effective field theory with a finite number of light fields.

In such theories, 3d geometries appear directly from CFT.

(Higher dimensions looks much harder.)

Emergent Geometry

There is a rough idea that emergent geometry comes from entanglement:

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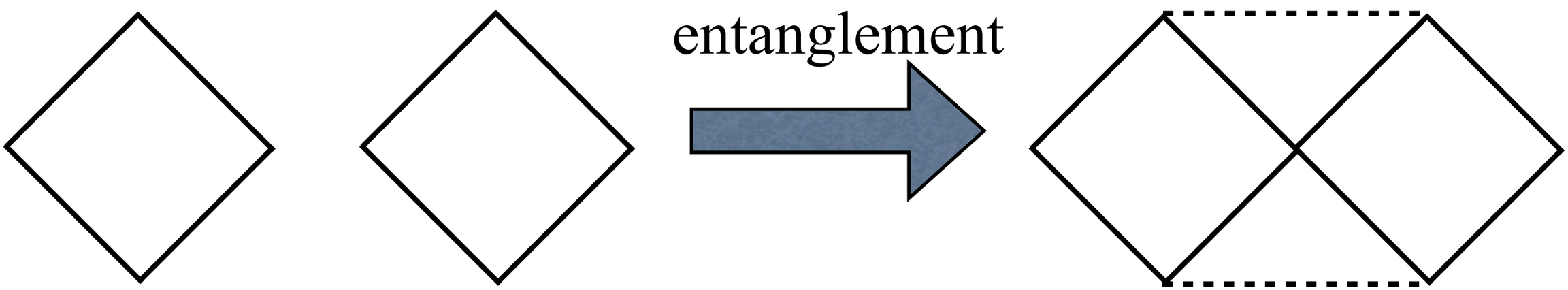

ex: Maldacena '01; van Raamsdonk '10; Maldacena & Susskind '13

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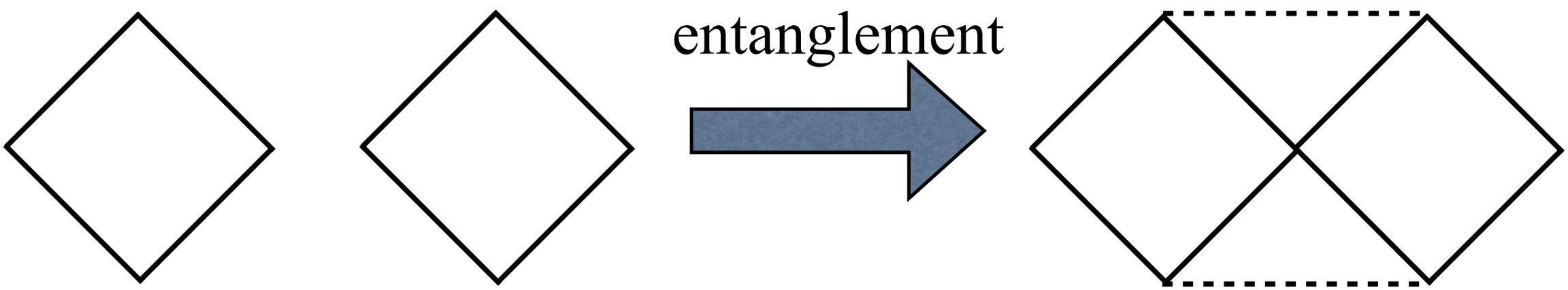

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In this talk: study entanglement entropy in 2d CFT.

- Will NOT assume holography
- 3d geometries will pop out of the CFT calculation automatically

Outline

I. Motivation

II. Entanglement entropy, 2d CFT, and holography

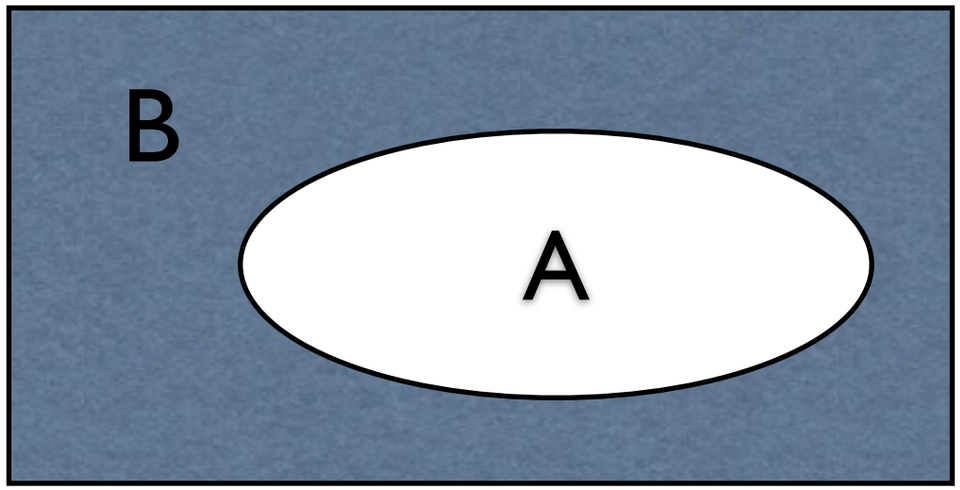
III. CFT calculation of entanglement entropy

IV. 3d geometries

V. Comments

Entanglement Entropy

Choose a spatial region and trace out the exterior:



$$\rho_A = \mathrm{Tr}_B \ \rho_{total}$$

$$S_A = -\mathrm{Tr}\rho_A \log \rho_A$$

Thermal entropy is a special case with A=system, B=bath.

I will always work in the vacuum state, $\rho_{total} = |0\rangle\langle 0|$

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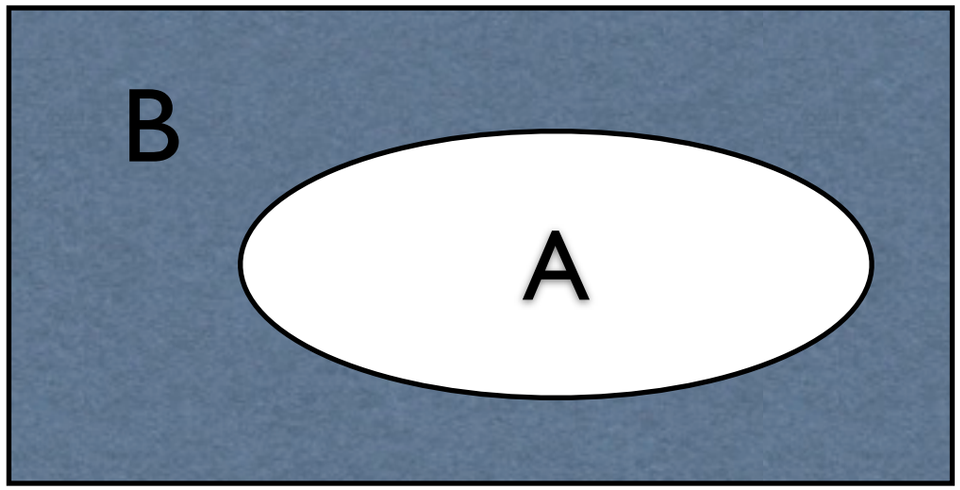

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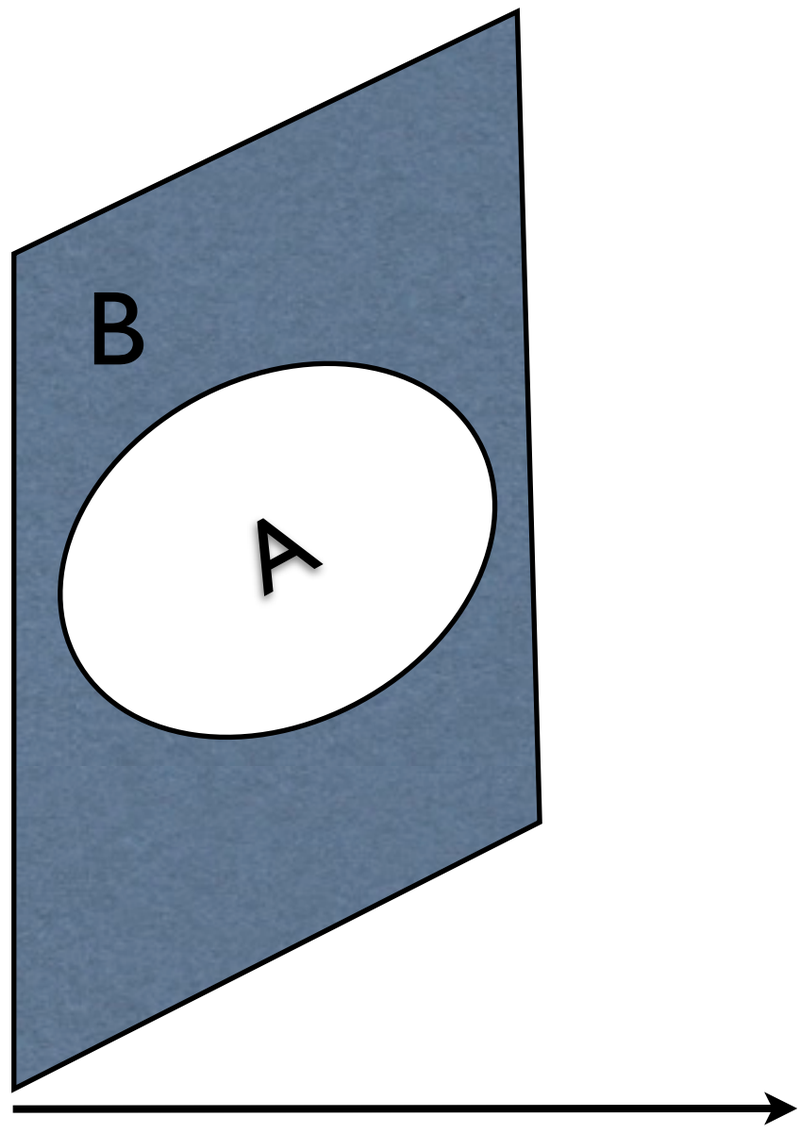
In 1+1 dimensions:

Space is a line, so **A** consists of one or more intervals:

Ryu-Takayanagi Formula

In holographic CFTs, entanglement entropy is computed by a simple geometric formula:



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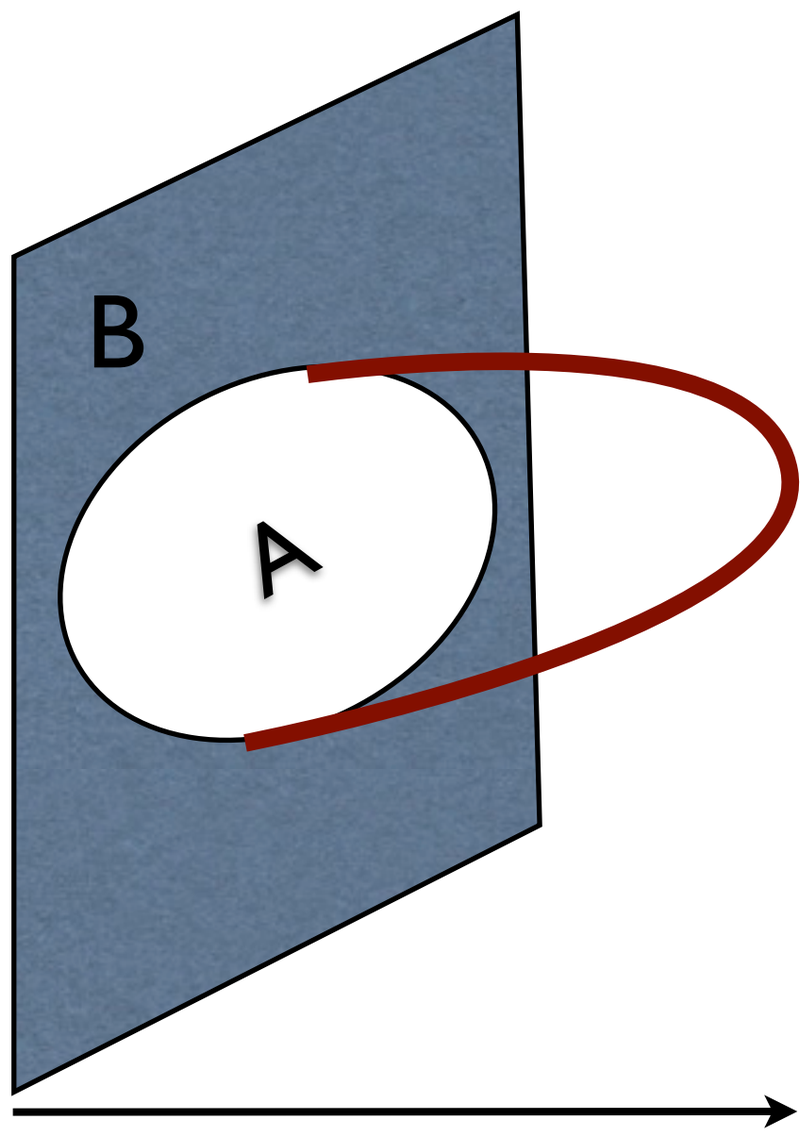

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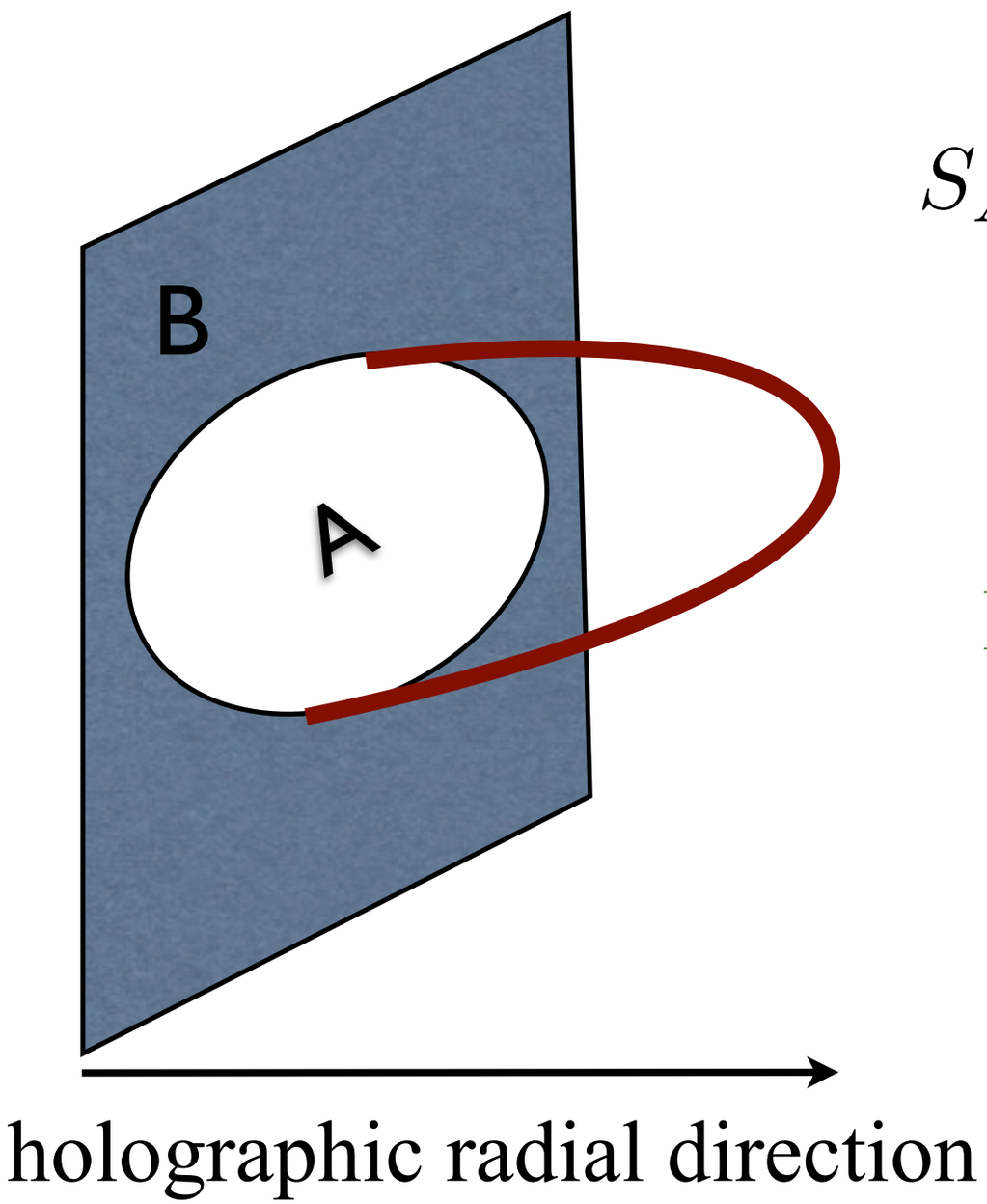


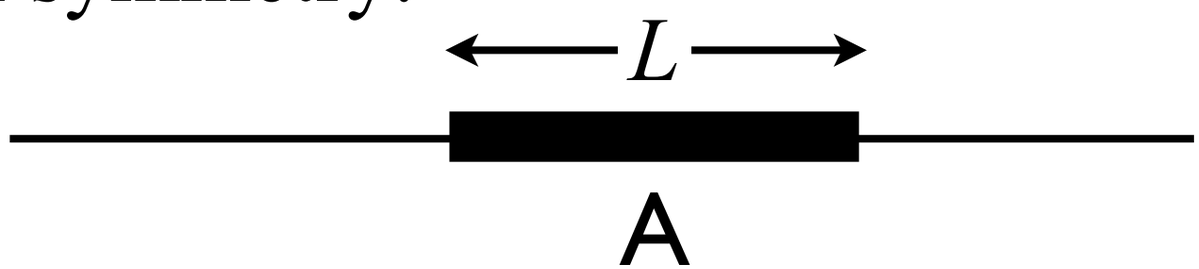
$$S_A = \frac{\text{Area(minimal surface)}}{4G_N}$$

This generalizes the Bekenstein-Hawking entropy to other types of surfaces, including Rindler horizons.

Ryu & Takayanagi '06
Lewkowycz & Maldacena '13

Single Interval (2d CFT)

2d CFT: The entropy of a single interval is fixed by conformal symmetry:



$$S_A = \frac{c}{3} \log \left(\frac{L}{\epsilon_{UV}} \right) \qquad c = \text{central charge}$$

CFT: Holzhey, Larsen & Wilczek; Cardy & Calabrese
AdS: Ryu & Takayanagi

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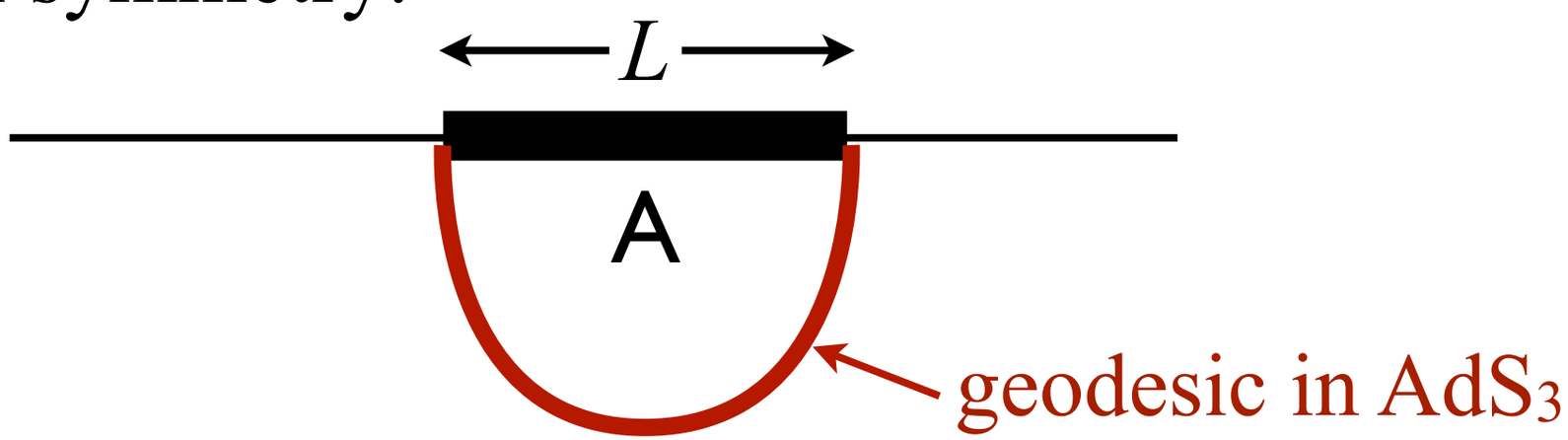

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This CFT result is reproduced by the length of a geodesic in AdS_3.

CFT: Holzhey, Larsen & Wilczek; Cardy & Calabrese
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Multiple Intervals

In a general CFT:

- Not universal: depends on the full operator content of the CFT.
- S_A is not known exactly in any theory, even free fields.
- Encodes the organization of quantum information in the groundstate in a more detailed way than a single interval.

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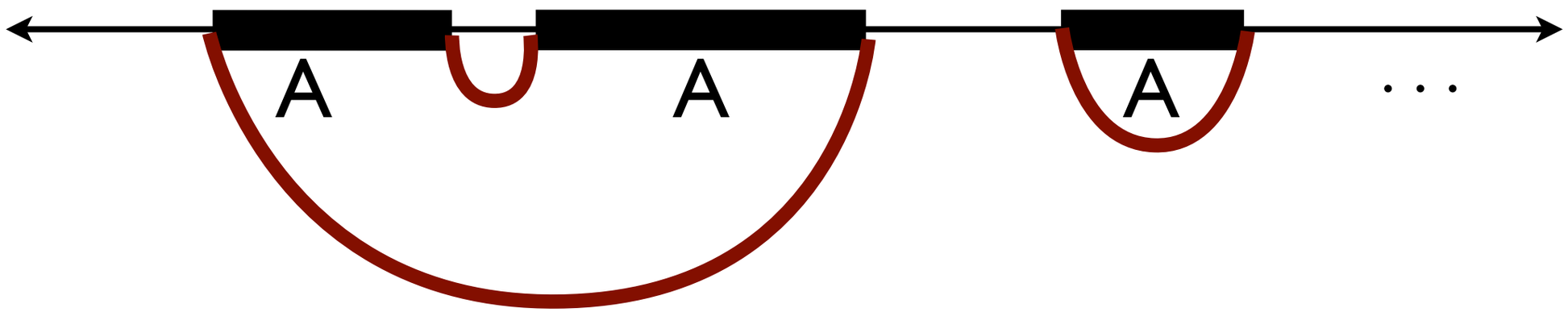

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In holographic CFT:

- From Ryu-Takayanagi, expect a universal answer at leading order in $G_N \sim 1/c$

The Plan

This talk:

- Compute entanglement entropies for multiple intervals in 2d CFT

TH '13

Assumptions

- Large central charge
- Gap: Not too many low-dimension operators $O(c^0)$
- Example: Symmetric orbifold CFTs (AdS/CFT at the free-field point in moduli space)
- These assumptions are motivated by holography, but this is a CFT calculation and does not assume any duality.

Results

- Leading 1/c contribution is universal
- Agrees with the holographic formula
- In the CFT calculation, 3d geometries will appear automatically

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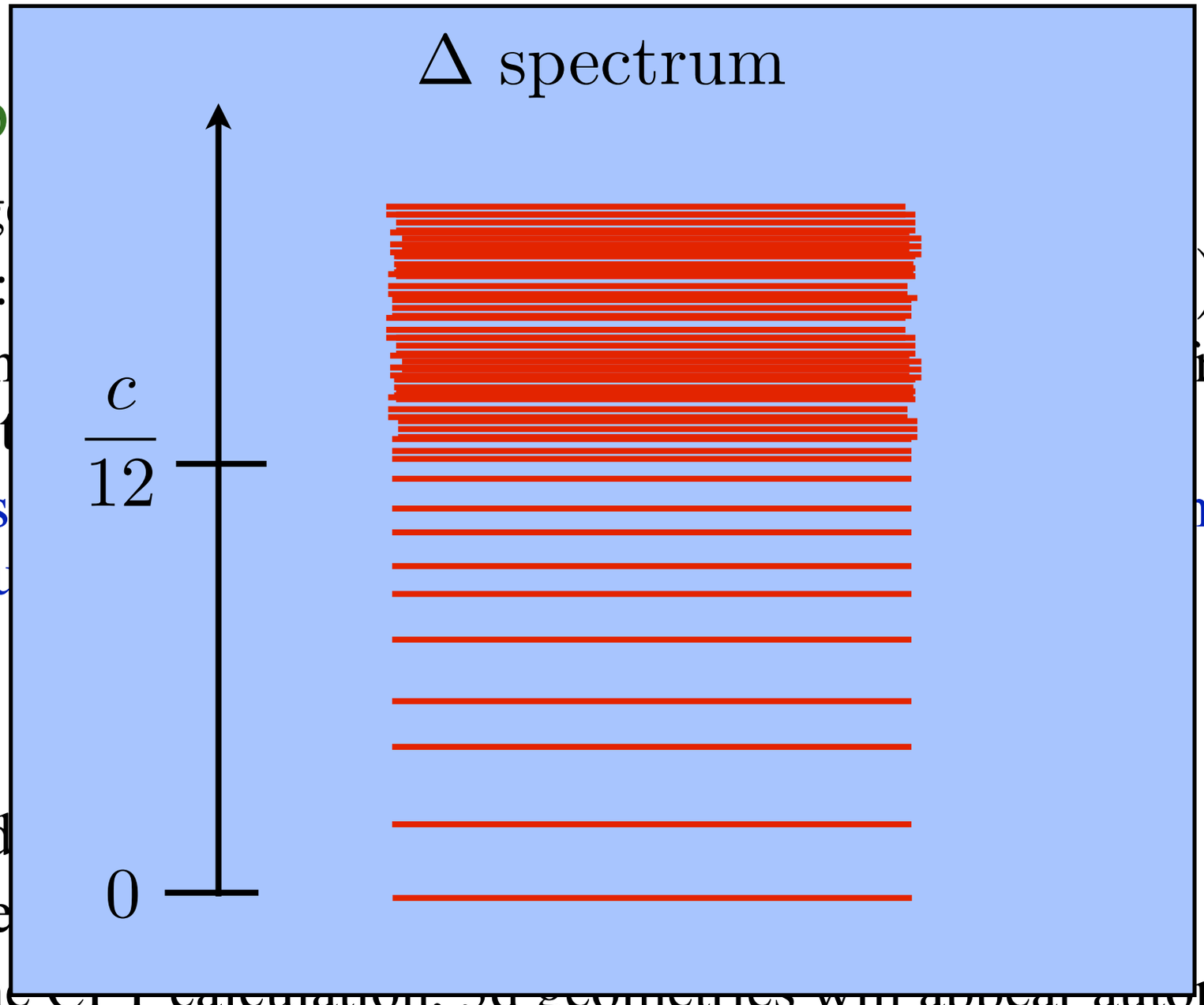

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Replica trick

We want to compute the entanglement entropy

$$S_A = -\mathrm{Tr}\ \rho_A \log \rho_A$$

First compute the Renyi/replica partition functions for n=2,3,...

$$Z^{(n)} = \mathrm{Tr}\ \rho_A^n$$

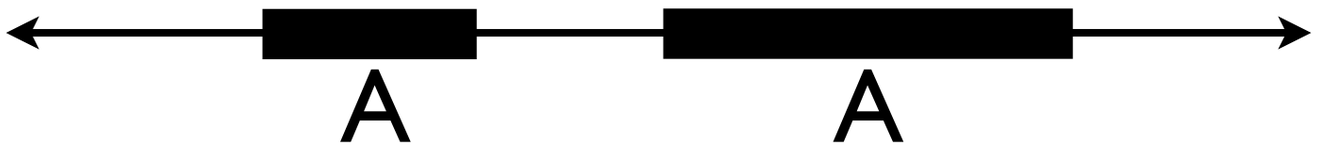
and use

$$S_A = -\partial_n Z^{(n)}|_{n=1}$$

This is useful because $Z^{(n)}$ can be computed by a Euclidean path integral.

Calabrese & Cardy '04

Replica Partition Functions



Example where A is 2 intervals, replica number=3:

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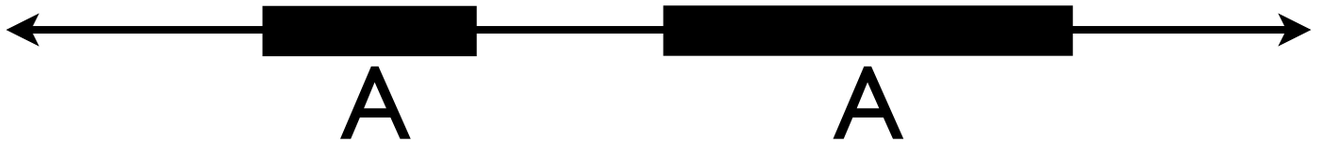

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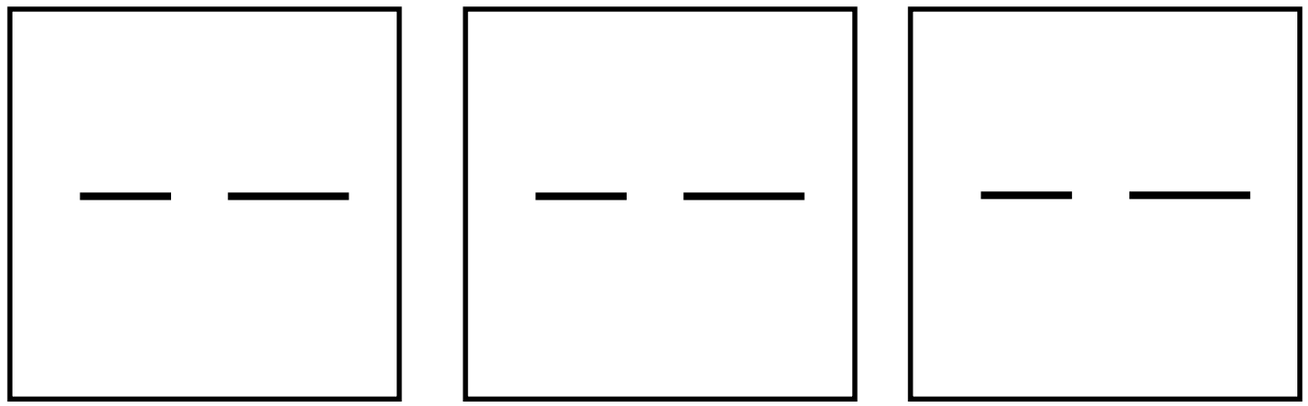

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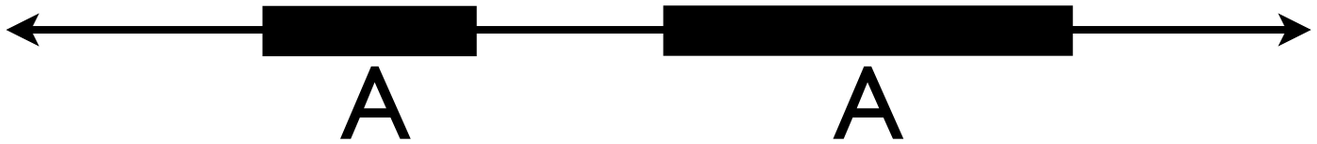

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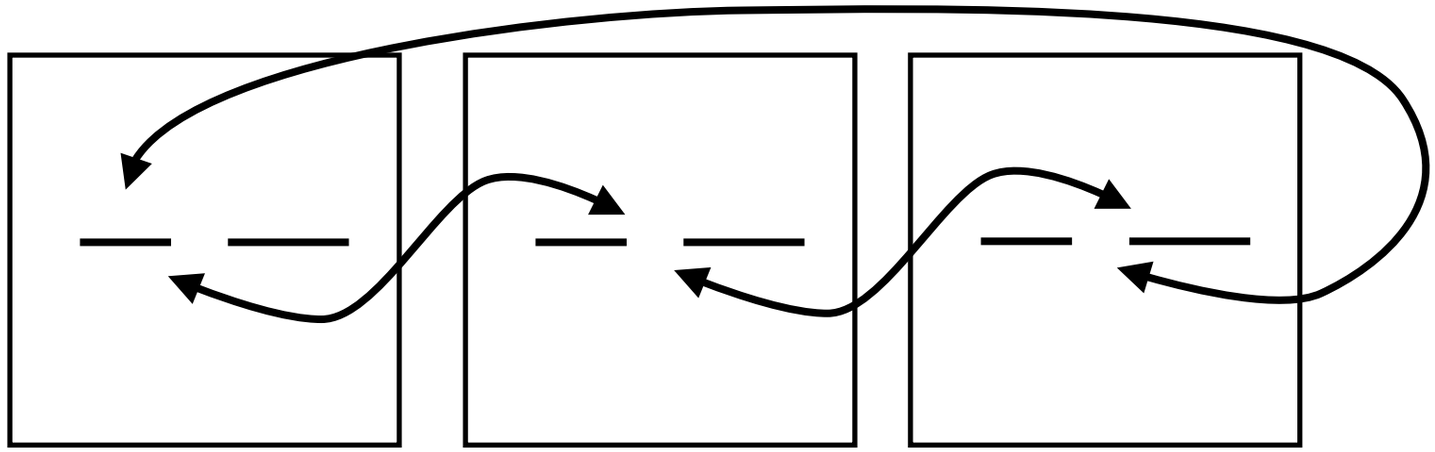
$\mathrm{Tr}\ \rho_A^3 = Z($  $)$

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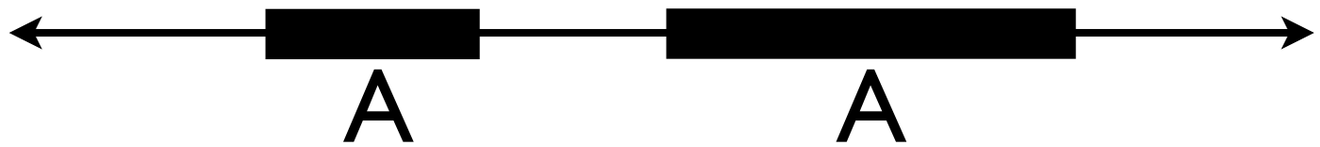

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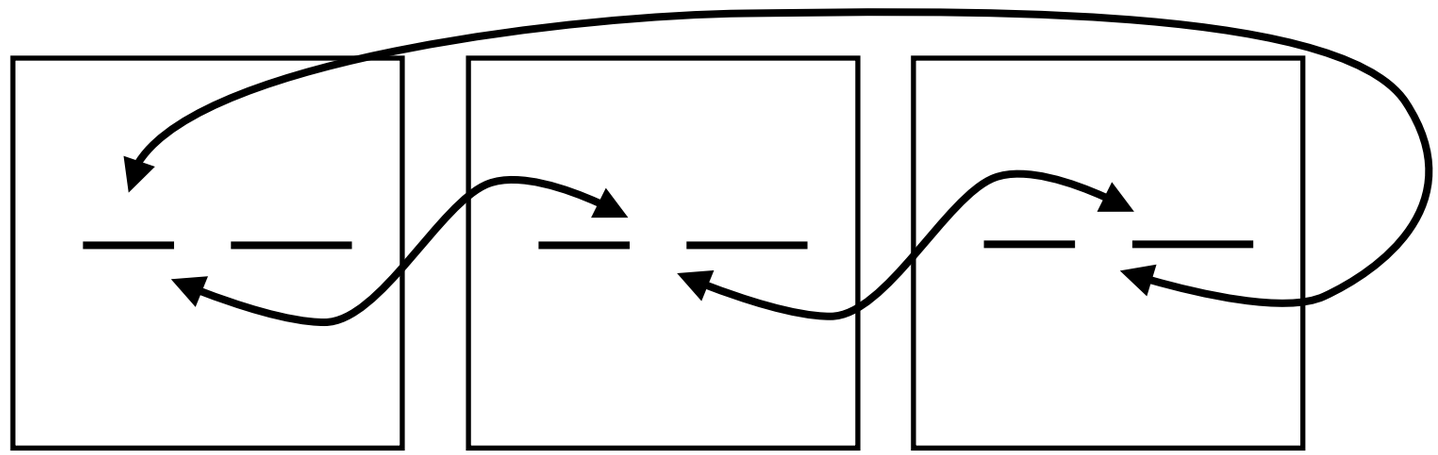
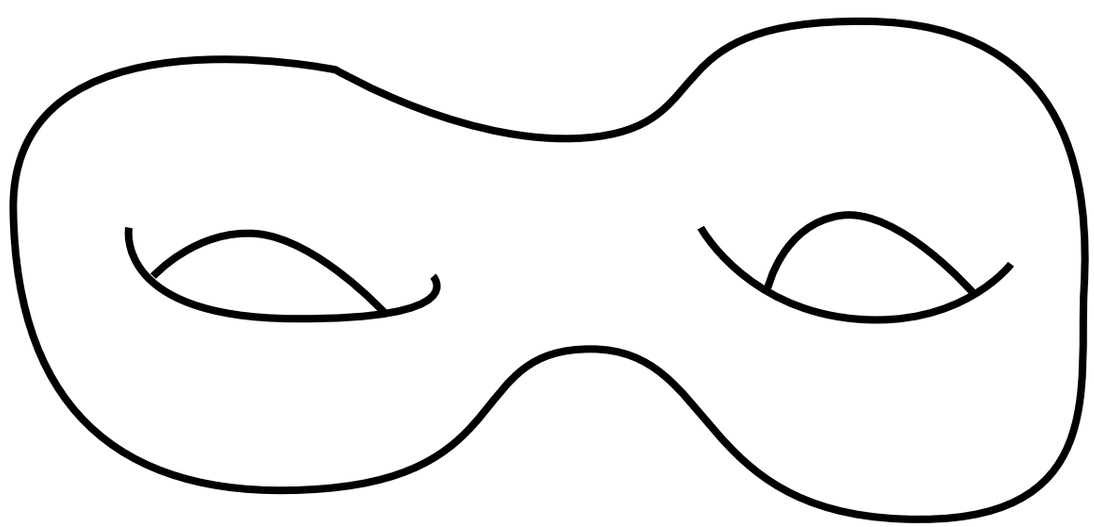


$)$

This is a Riemann surface with nontrivial topology.

This example (2 slits, 3 replicas) has genus 2:

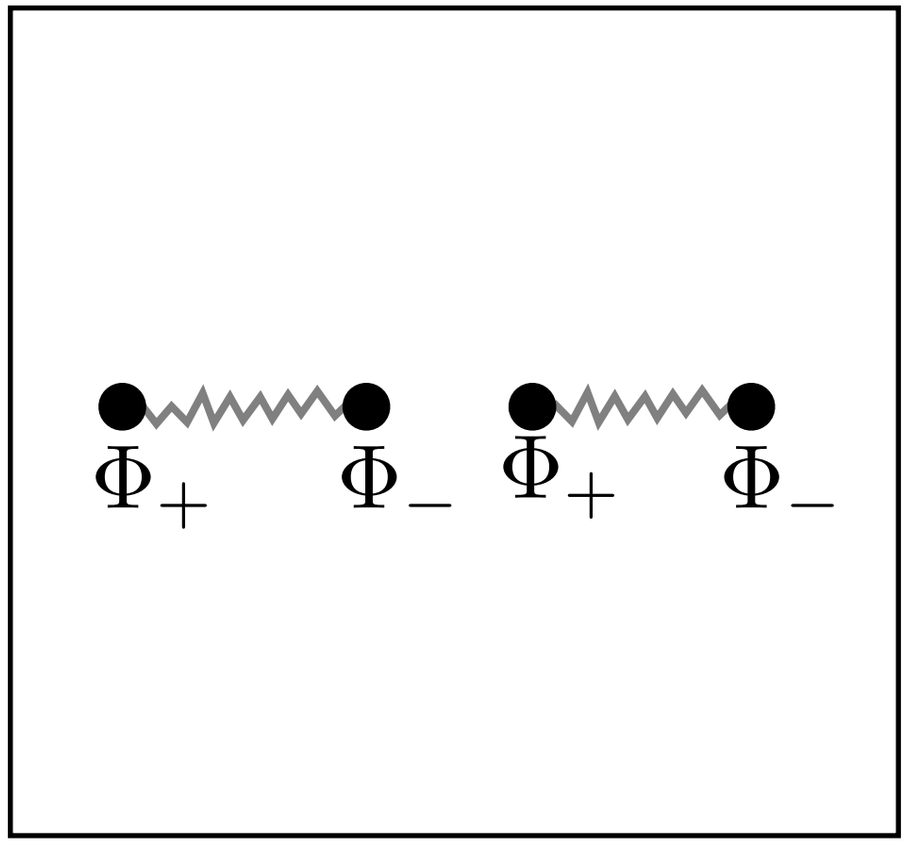
$= Z($



$)$

Twist operators

This partition function can be viewed as a correlation function of "twist operators" that glue the sheets together.



$$\mathrm{Tr}\ \rho_A^n = \langle \Phi_+ \Phi_- \Phi_+ \Phi_- \rangle_{CFT^n}$$

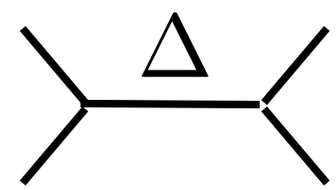
Dixon, Friedan, Martinec & Shenker '87
Calabrese, Cardy, Tonni '04 - '13

OPE Expansion

2pt functions are fixed by conformal invariance.

4pt functions are not fixed, but are constrained to have the form

$$\langle \Phi_+ \Phi_- \Phi_+ \Phi_- \rangle = \sum_\Delta$$



$$= \sum_\Delta c_\Delta^2 \mathcal{F}(\Delta, z)\mathcal{F}(\bar{\Delta}, \bar{z})$$

Virasoro Conformal Blocks

OPE coefficient

Applied to this problem in: Headrick '10

Calculate at large c

Virasoro blocks have a nice form at large central charge:

$$\mathcal{F}(\Delta, z) \approx e^{-cf(\frac{\Delta}{c}, z)}$$

Zamolodchikov '87

From this we can evaluate the 4pt function of heavy operators to leading order in 1/c:

$$\mathrm{Tr}\ \rho_A^n \approx e^{-cf(0,z)} e^{-cf(0,\bar{z})}$$

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Comments:

- This contribution is universal (independent of CFT details)
- Valid at leading order in 1/c (but all orders in OPE!)
- Also assumed low operator multiplicities
- It is the Virasoro block for the vacuum rep, which includes the operators

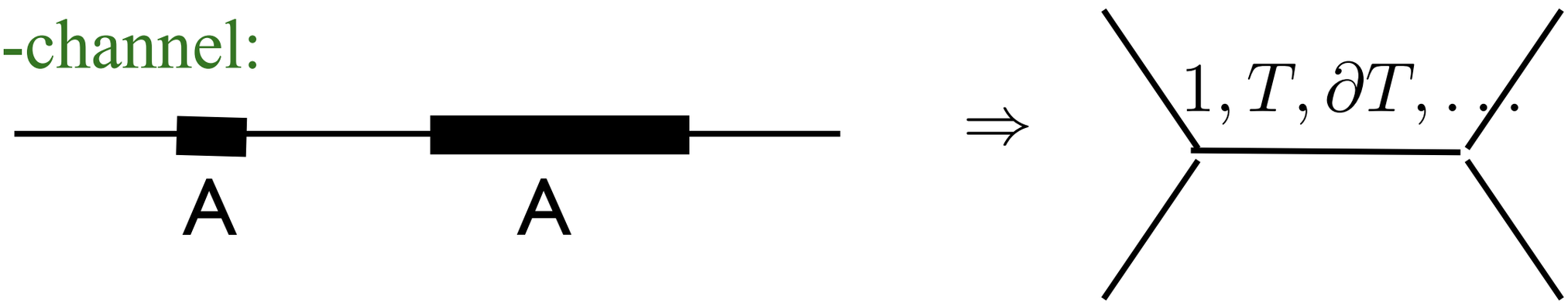
$$1\ , T\ , \partial T\ , T^2\ , T\partial T\ , \cdots$$

- Upshot: Heavy correlators are exponentially dominated by exchange of operators built from the stress tensor. (Dual: 3d gravitons)

Channels

This is a saddlepoint evaluation. Different saddles will dominate for different shapes of region A:

s-channel:



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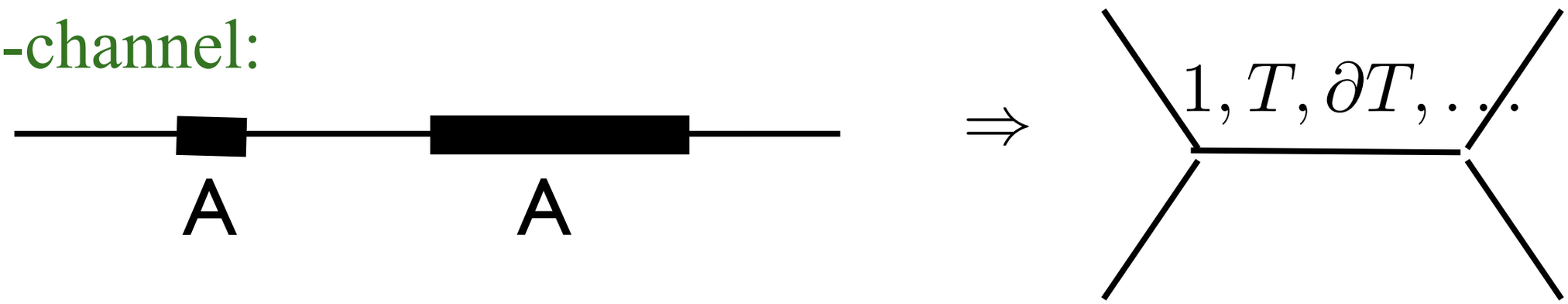
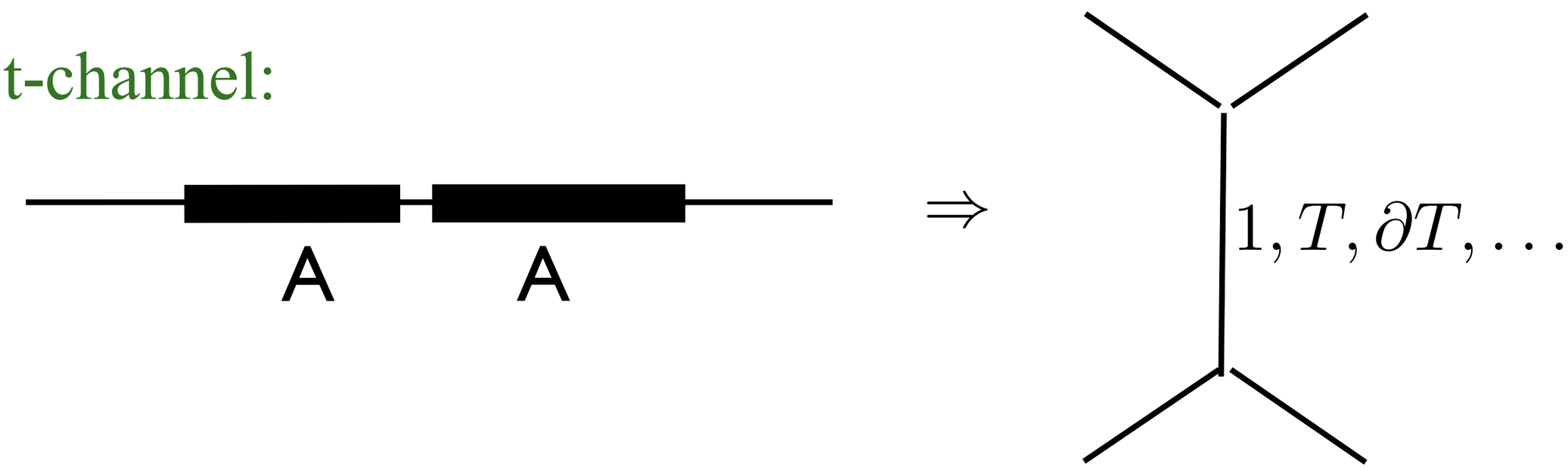


t-channel:



Semiclassical conformal blocks

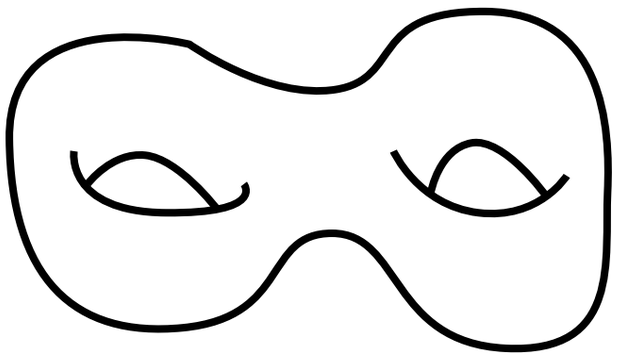
So far we found the answer is universal:

$$\mathrm{Tr}\ \rho_A^n \approx e^{-cf(0,z)} e^{-cf(0,\bar{z})}$$

f is called the 'semiclassical conformal block.'

The goal now: describe *f* and take n=1 to compute the entanglement entropy.

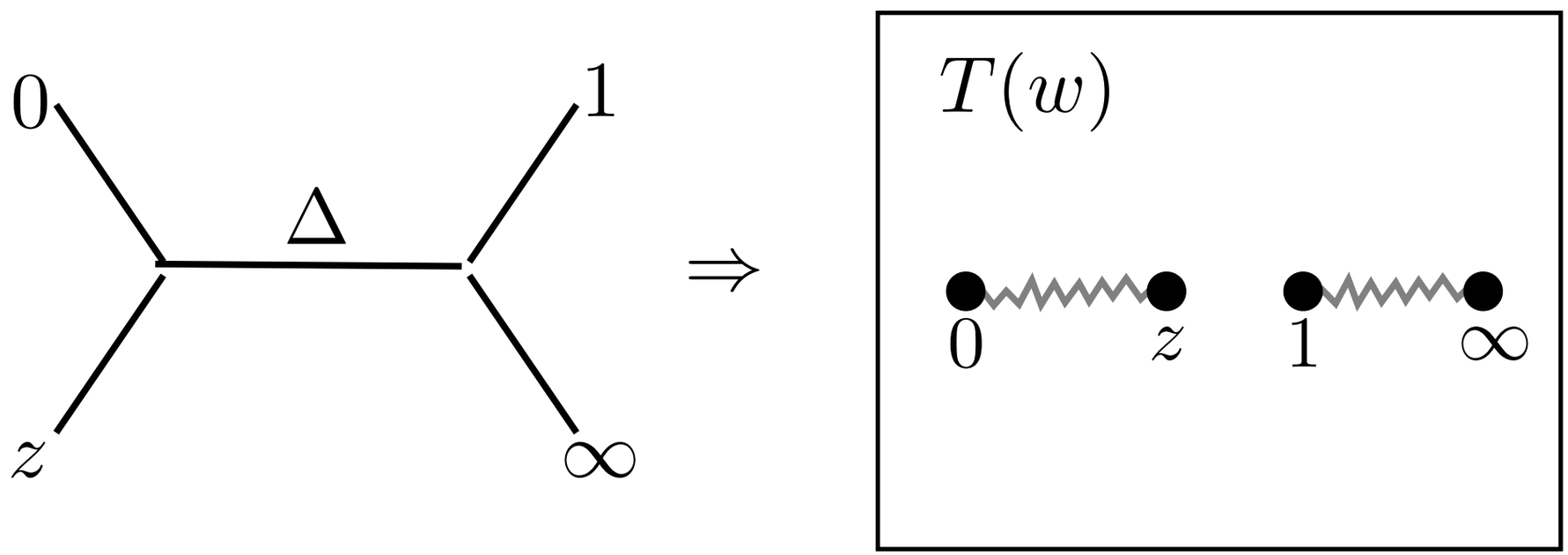
Punchline will be: *f* is the on-shell Einstein action for a 3-manifold whose boundary is the replica manifold



Semiclassical blocks

f is defined by the large-c Virasoro block: $\mathcal{F}(\Delta, z) \approx e^{-cf(\frac{\Delta}{c}, z)}$

Zamolodchikov '87: It can also be computed by finding a stress tensor in the background of heavy operator insertions.



The stress tensor determines the semiclassical block via

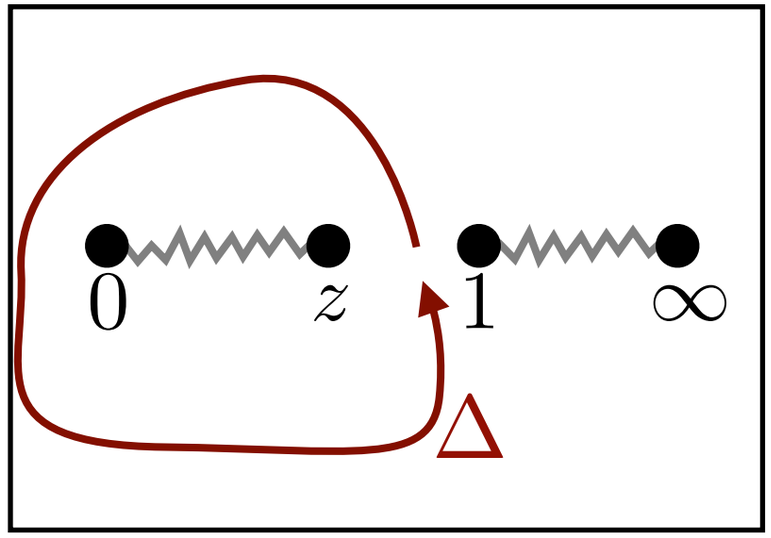
$$\partial_z f = \text{res } T(w) \text{ at } w \sim z$$

Holonomy problem

Operator insertions make double poles in *T(w)*. This does not quite determine *T(w)*. Turns out it is fixed by a holonomy condition around points (0,z):

$$\mathrm{Tr}\ P\exp\left(\oint dw \begin{bmatrix} 0 & 1 \\ -T(w) & 0 \end{bmatrix}\right) = -2\cos\pi\sqrt{1-\frac{24\Delta}{c}}$$

Zamolodchikov '87



For the vacuum block,

$$\Delta = 0 \quad \Rightarrow \quad \text{holonomy} = \text{trivial}$$

Recap

Replica partition functions are given by the semiclassical vacuum block:

$$\mathrm{Tr}\ \rho_A^n \approx e^{-cf(0,z)} e^{-cf(0,\bar{z})}$$

The function *f* is determined by solving a trivial-holonomy condition on a Riemann surface.

In general, this holonomy problem must be solved numerically or in a series expansion.

Entanglement entropy

For the entanglement entropy, take $n \to 1$

$$S_A = -\partial_n \mathrm{Tr}\ \rho_A^n|_{n=1}$$

In this limit the holonomy problem is easy to solve analytically. The result is

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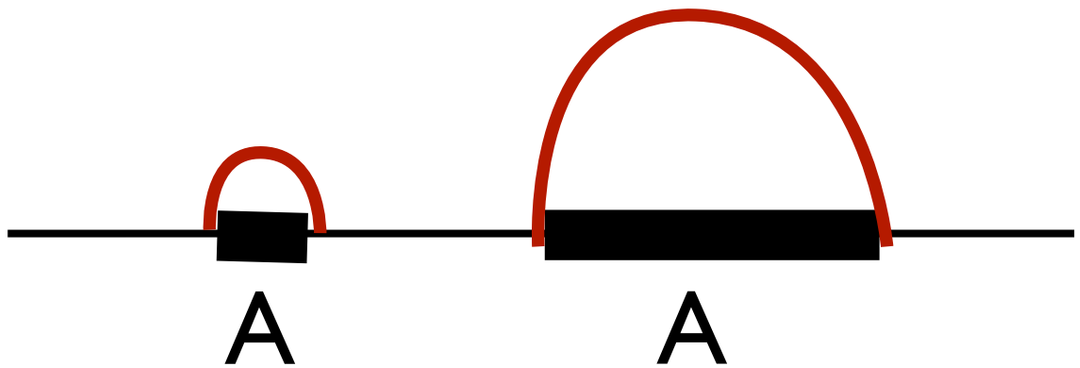

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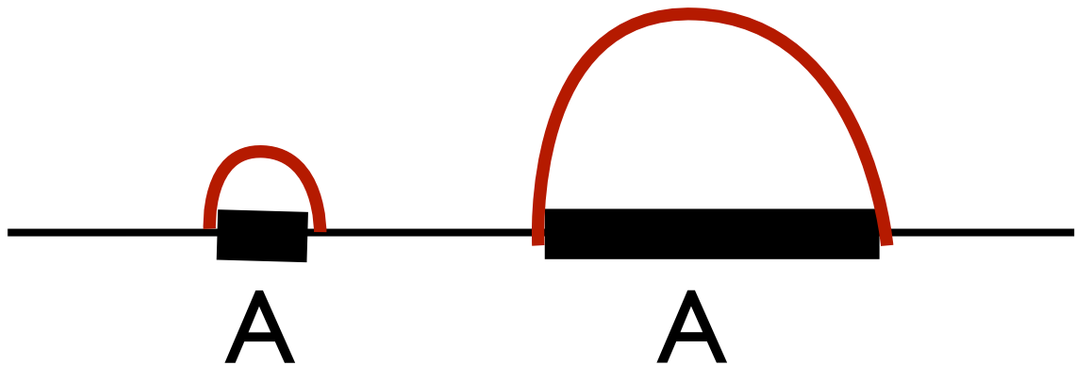

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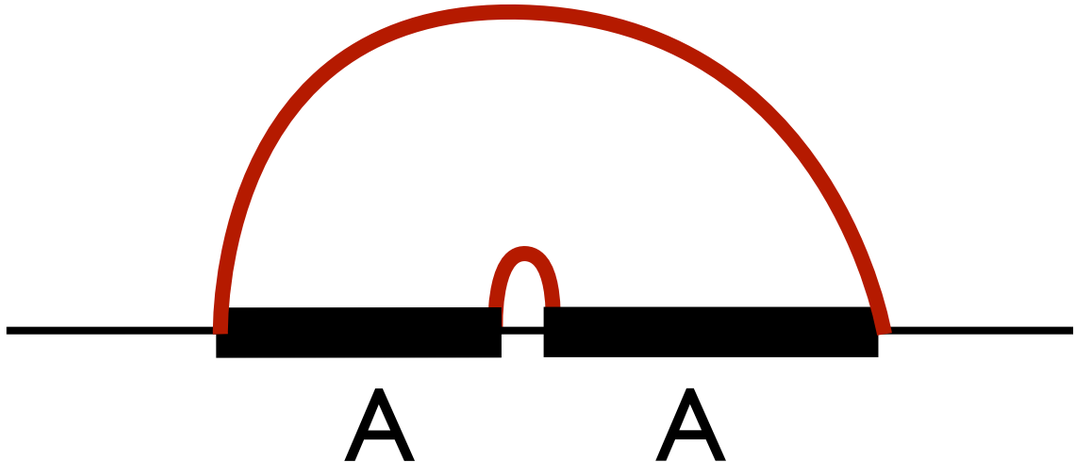

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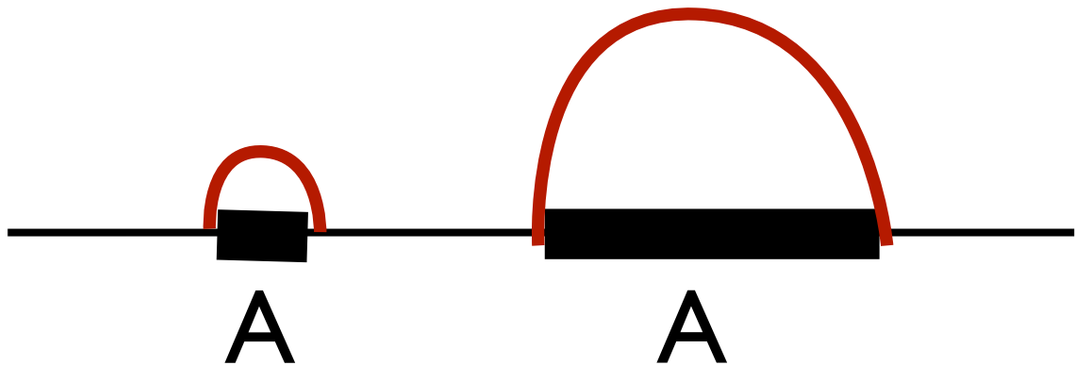

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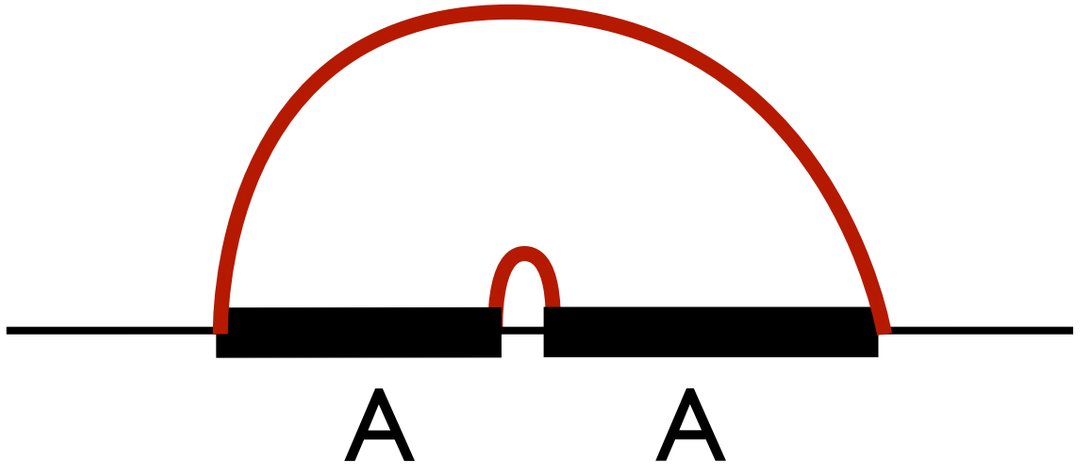

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Agrees with holographic Ryu-Takayanagi formula

(assuming no other non-perturbative contributions, ie non-geometric saddles)

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II. Entanglement entropy and 2d CFT

III. CFT calculation of entanglement entropy

IV. 3d geometries T. Faulkner '13

V. Comments

What is an on-shell 3d geometry?

3d gravity has no propagating graviton, so all solutions of Einstein equation are *locally* AdS_3.

In Euclidean signature: hyperbolic 3-manifold.

To construct hyperbolic 3-manifolds:

1. Draw a genus-g Riemann surface:

2. "Fill in" g cycles

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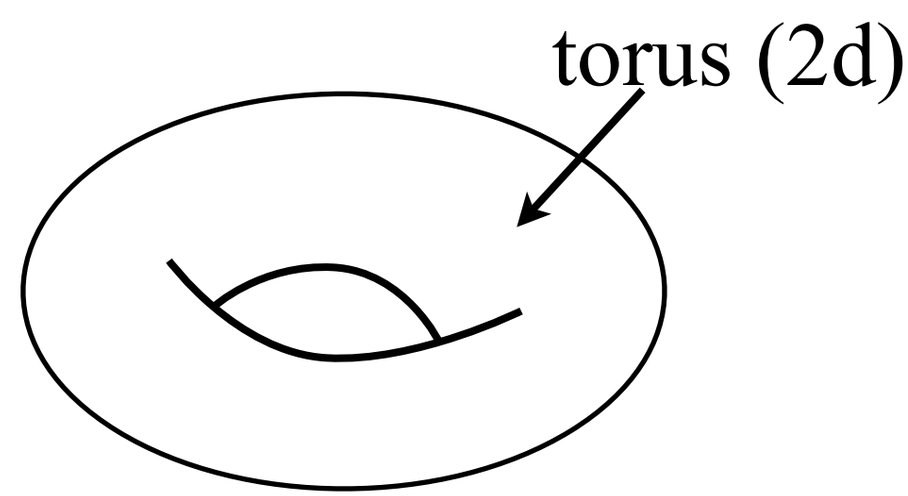

2. "Fill in" g cycles

What is an on-shell 3d geometry?

3d gravity has no propagating graviton, so all solutions of Einstein equation are *locally* AdS_3.

In Euclidean signature: hyperbolic 3-manifold.

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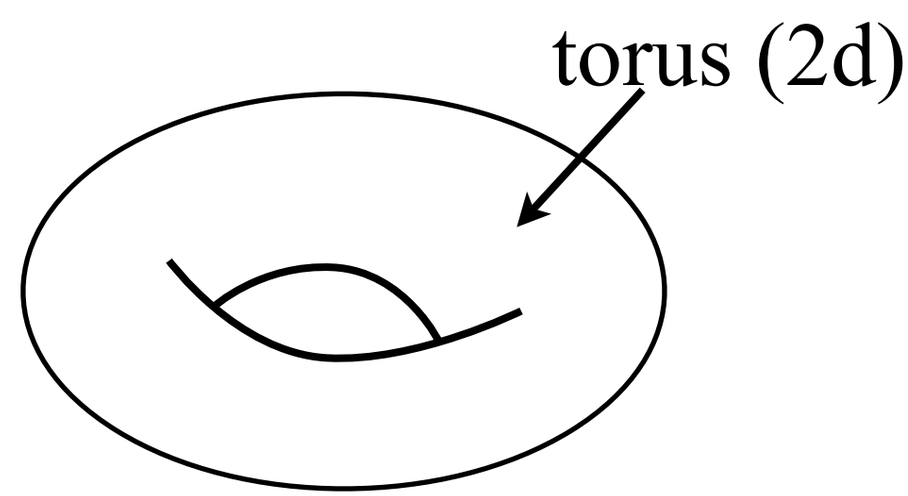

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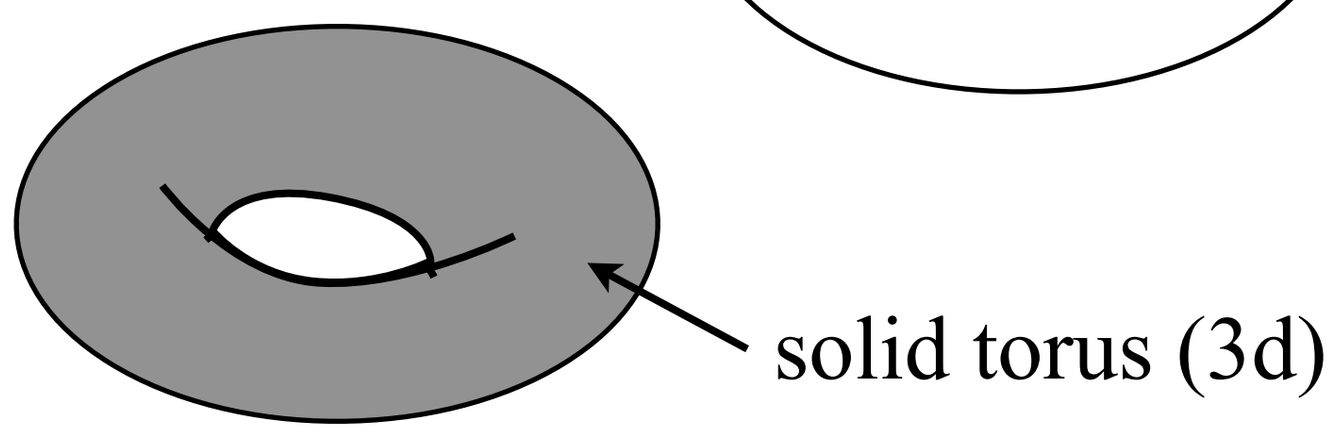
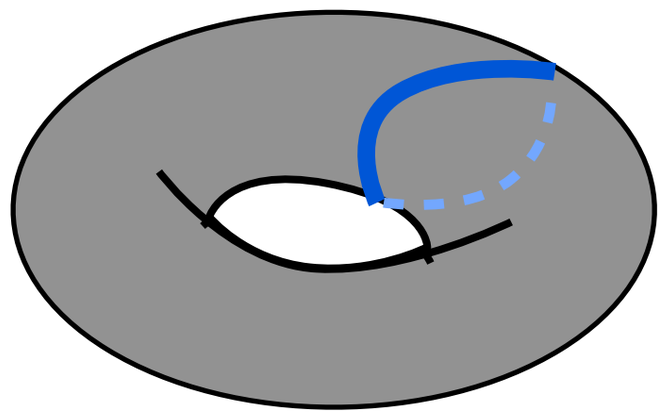


What is a 3d geometry?

- A "filled in" cycle means a contractible loop in our geometry.
- For a loop to be contractible, the metric must obey some regularity condition.
- This regularity condition can be stated in terms of gauge-invariant data by setting a gravitational Wilson line to zero:



$$P\exp\left(\oint dw \begin{bmatrix} 0 & 1 \\ \delta g_{ww} & 0 \end{bmatrix}\right) = \mathbf{1}_{2x2}$$

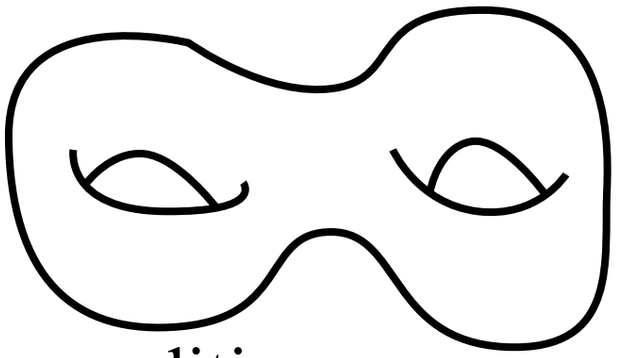
- In the SL(2) Chern-Simons formulation of classical 3d gravity, this is the ordinary holonomy of the SL(2) gauge field.
- This is exactly Zamolodchikov's construction of the large-c Virasoro block for the vacuum representation!

3d geometry = Replica partition function

Relation to CFT

- Recall the replica partition function:

$$\mathrm{Tr}\ \rho_A^n \approx e^{-cf(0,z)} e^{-cf(0,\bar{z})} = Z_{cft}($$



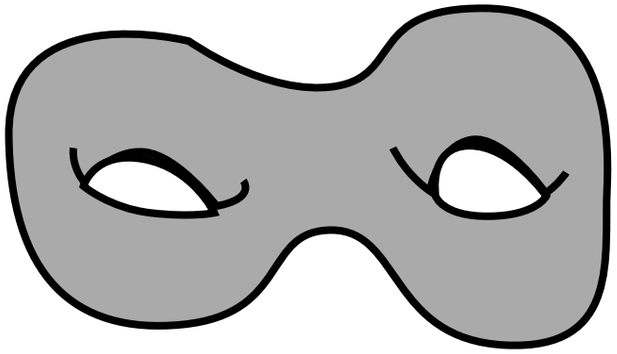
$$)$$

where f is computed by solving a zero-holonomy condition.

- To compute this in CFT, we secretly constructed a 3d geometry.

- The precise relation is "large-c vacuum block = Einstein action":

$$f = S_{Einstein}($$



$$)$$

T. Faulkner '13

- Different ways of filling in the Riemann surface = saddlepoints in different OPE channels

Outline

I. Motivation

II. Entanglement entropy and 2d CFT

III. CFT calculation of entanglement entropy

IV. 3d geometries

V. Two Comments

Comment 1: Results in 2d CFT

2d CFT / 3d Gravity

- Derived holographic entanglement entropies directly from CFT, for an arbitrary region *A*.

- Did not rely on microscopic details of CFT; assumed only large-*c* and a gap in operator dimensions.

- 3d geometries naturally appear in CFT calculations in this regime, without assuming holography.

Comment 2: Higher dimensions?

This argument is very special to 2d CFT. What about d>2?

- 3d gravity comes from large-c limit of the Virasoro algebra
  - In higher dimensions, we only have SO(d,2); operators built from stress tensor live in many different conformal families, and organize into a "gravity" family only at large *c*

- 3d gravity has only one parameter:

$$G_N \sim 1/c$$

- Higher-dimension gravity is parameterized by an infinite number of higher-derivative couplings,

$$S = \frac{1}{G_N} \int \left[R + \alpha R^2 + \cdots \right]$$

- This requires (at the very least) a much stronger type of "gap" in operator dimensions for universal Einstein-gravity-like behavior.
- However: 4d higher spin gravity?

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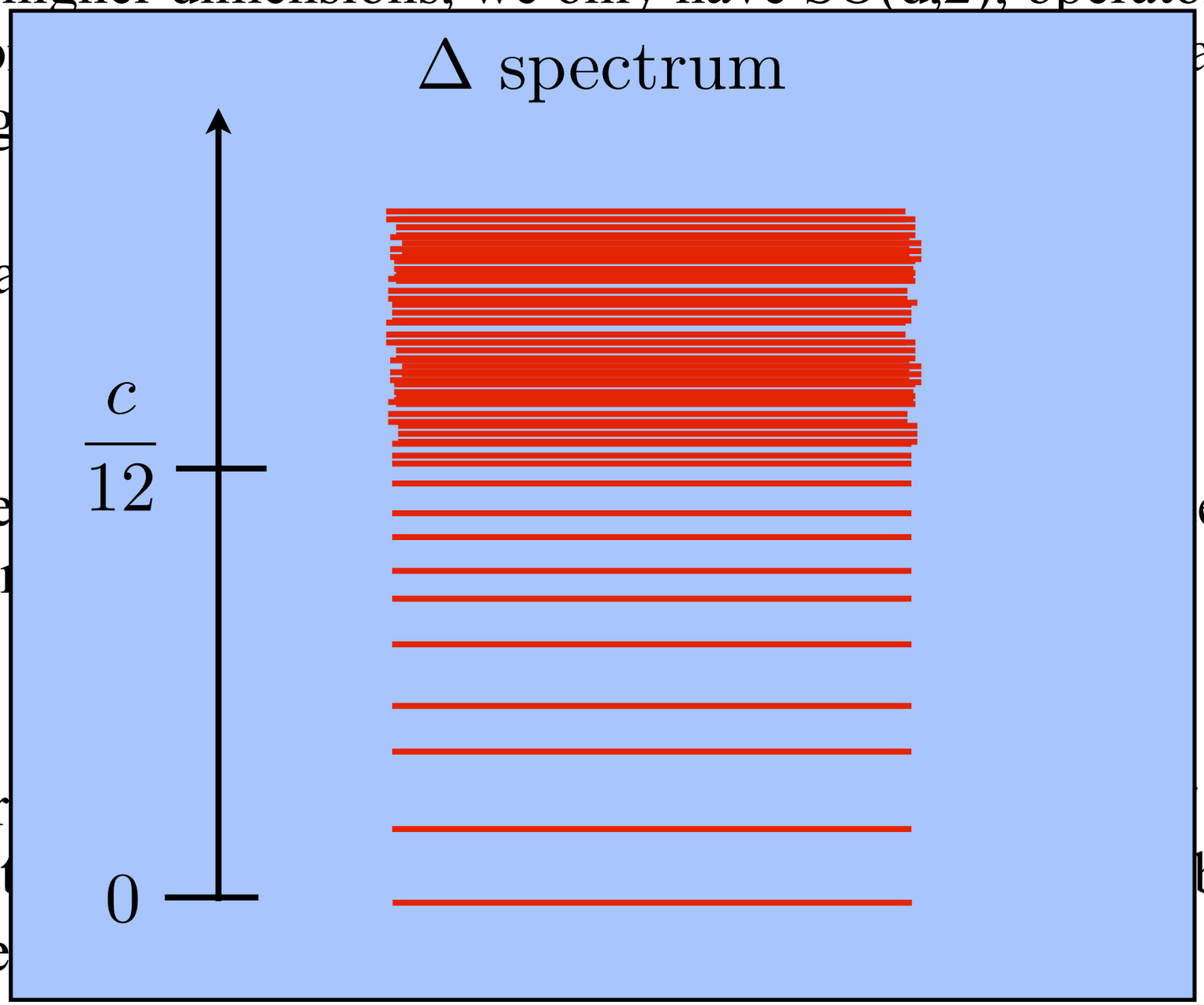

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