Entanglement and Geometry in 2d CFT

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Emergent Geometry

AdS/CFT gives a UV-complete definition of quantum gravity in certain cases.

This is not the end of the story, even in AdS:

- How does the bulk geometry emerge from CFT?
- CFT language is so different from the bulk that basic bulk questions cannot be answered (high-E scattering, information, firewalls, etc.)

Need to find right language to describe physics in the bulk.
- “Build AdS from CFT.”
Emergent Geometry

A first step: Can we derive semiclassical gravity in $\text{AdS}_{d+1}$ from universal features of (special) CFTs in $d$ dimensions?
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A first step: Can we derive semiclassical gravity in AdS\(_{d+1}\) from universal features of (special) CFTs in \(d\) dimensions?

We should be able to find a large class of CFTs with:

- Thermodynamic entropy = area of black hole horizon
- \(\eta/s \sim 1/4\pi\)
- Hawking-Page transition
- Entanglement entropies = Ryu-Takayanagi formula
- etc....

These features are universal in any CFT with an Einstein-gravity dual.

--> towards a language for bulk quantum gravity
Emergent Geometry

In 1+1d CFT / 3d AdS:

- YES we can derive this, probably.
- A natural conjecture is:

  “Virasoro symmetry + large $c$ + gap $\Rightarrow$ 3d gravity”

Symmetry is not enough.

- These assumptions allow a bulk dual which is a perturbative effective field theory with a finite number of light fields.

In such theories, 3d geometries appear directly from CFT.

(Higher dimensions looks much harder.)
Emergent Geometry

There is a rough idea that emergent geometry comes from entanglement:
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ex: Maldacena ’01; van Raamsdonk ’10; Maldacena & Susskind ’13
Emergent Geometry

There is a rough idea that emergent geometry comes from entanglement:

In this talk: study entanglement entropy in 2d CFT.
- Will NOT assume holography
- 3d geometries will pop out of the CFT calculation automatically

ex: Maldacena ’01; van Raamsdonk ’10; Maldacena & Susskind ’13
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**Outline**

I. Motivation

II. Entanglement entropy, 2d CFT, and holography

III. CFT calculation of entanglement entropy

IV. 3d geometries

V. Comments
Entanglement Entropy

Choose a spatial region and trace out the exterior:

\[ \rho_A = \text{Tr}_B \rho_{total} \]

\[ S_A = -\text{Tr} \rho_A \log \rho_A \]

Thermal entropy is a special case with \( A=\text{system}, B=\text{bath} \).
I will always work in the vacuum state, \( \rho_{total} = |0\rangle \langle 0| \)
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In 1+1 dimensions:
Space is a line, so A consists of one or more intervals:
Ryu-Takayanagi Formula

In holographic CFTs, entanglement entropy is computed by a simple geometric formula:
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In holographic CFTs, entanglement entropy is computed by a simple geometric formula:

\[ S_A = \frac{\text{Area(minimal surface)}}{4G_N} \]

This generalizes the Bekenstein-Hawking entropy to other types of surfaces, including Rindler horizons.

Ryu & Takayanagi '06
Lewkowycz & Maldacena '13
2d CFT: The entropy of a single interval is fixed by conformal symmetry:

\[ S_A = \frac{c}{3} \log \left( \frac{L}{\epsilon_{UV}} \right) \]

where \( c \) is the central charge.

CFT: Holzhey, Larsen & Wilczek; Cardy & Calabrese
AdS: Ryu & Takayanagi
Single Interval (2d CFT)

2d CFT: The entropy of a single interval is fixed by conformal symmetry:

\[ S_A = \frac{c}{3} \log \left( \frac{L}{\epsilon_{UV}} \right) \]

\[ c = \text{central charge} \]

This CFT result is reproduced by the length of a geodesic in AdS$_3$.

CFT: Holzhey, Larsen & Wilczek; Cardy & Calabrese
AdS: Ryu & Takayanagi
In a general CFT:

- Not universal: depends on the full operator content of the CFT.
- $S_A$ is not known exactly in any theory, even free fields.
- Encodes the organization of quantum information in the groundstate in a more detailed way than a single interval.
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In holographic CFT:

- From Ryu-Takayanagi, expect a universal answer at leading order in $G_N \sim 1/c$
The Plan

This talk:

- Compute entanglement entropies for multiple intervals in 2d CFT

Assumptions

- Large central charge
- Gap: Not too many low-dimension operators \( O(c^0) \)
- Example: Symmetric orbifold CFTs (AdS/CFT at the free-field point in moduli space)
- These assumptions are motivated by holography, but this is a CFT calculation and does not assume any duality.

Results

- Leading 1/c contribution is universal
- Agrees with the holographic formula
- In the CFT calculation, 3d geometries will appear automatically
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II. Entanglement entropy and 2d CFT

III. CFT calculation of entanglement entropy

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V. Comments
Replica trick

We want to compute the entanglement entropy

\[ S_A = -\text{Tr} \rho_A \log \rho_A \]

First compute the Renyi/replica partition functions for \( n=2,3,\ldots \)

\[ Z^{(n)} = \text{Tr} \rho^n_A \]

and use

\[ S_A = -\partial_n Z^{(n)} \big|_{n=1} \]

This is useful because \( Z^{(n)} \) can be computed by a Euclidean path integral.

Calabrese & Cardy '04
Replica Partition Functions

Example where $A$ is 2 intervals, replica number=3:
Replica Partition Functions

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\[
\text{Tr } \rho_A^3 = Z(\begin{array}{ccc}
\hspace{1cm} \text{---} & \hspace{1cm} \text{---} & \hspace{1cm} \text{---} \\
\end{array})
\]
Replica Partition Functions

Example where $A$ is 2 intervals, replica number=3:

$$\text{Tr} \ 3 \ A = Z(\_\_ \ \_\_ \ \_\_ \ _\_, \_\_, \_\_, \_\_, \_\_, \_\_, \_\_, \_\_, \_\_, \_\_, \_\_ \_\_)$$
Replica Partition Functions

Example where A is 2 intervals, replica number=3:

\[ \text{Tr } \rho_A^3 = Z(\text{--- } \text{--- } \text{--- } ) \]

This is a Riemann surface with nontrivial topology.
This example (2 slits, 3 replicas) has genus 2:

\[ = Z(\text{--- } \text{--- } ) \]
This partition function can be viewed as a correlation function of “twist operators” that glue the sheets together.

\[ \text{Tr } \rho^n_A = \langle \Phi_+ \Phi_- \Phi_+ \Phi_- \rangle_{\text{CFT}^n} \]
2pt functions are fixed by conformal invariance.

4pt functions are not fixed, but are constrained to have the form

\[ \langle \Phi_+ \Phi_- \Phi_+ \Phi_- \rangle = \sum_{\Delta} \Delta \]

\[ = \sum_{\Delta} c_\Delta^2 F(\Delta, z) F(\Delta, \bar{z}) \]

Virasoro Conformal Blocks

OPE coefficient

Applied to this problem in: Headrick ’10
Calculate at large \( c \)

Virasoro blocks have a nice form at large central charge:

\[
\mathcal{F}(\Delta, z) \approx e^{-c f(\frac{\Delta}{c}, z)}
\]

From this we can evaluate the 4pt function of heavy operators to leading order in \( 1/c \):

\[
\text{Tr } \rho^n_A \approx e^{-c f(0,z)} e^{-c f(0,\bar{z})}
\]

Zamolodchikov ’87
Calculate at large $c$

Virasoro blocks have a nice form at large central charge:

$$\mathcal{F}(\Delta, z) \approx e^{-c f(\frac{\Delta}{c}, z)}$$

From this we can evaluate the 4pt function of heavy operators to leading order in $1/c$:

$$\text{Tr} \; \rho^n_A \approx e^{-c f(0, z)} e^{-c f(0, \bar{z})}$$

Comments:

- This contribution is universal (independent of CFT details)
- Valid at leading order in $1/c$ (but all orders in OPE!)
- Also assumed low operator multiplicities
- It is the Virasoro block for the vacuum rep, which includes the operators
  $$1, T, \partial T, T^2, T\partial T, \cdots$$
- Upshot: Heavy correlators are exponentially dominated by exchange of operators built from the stress tensor. (Dual: 3d gravitons)
Channels

This is a saddlepoint evaluation. Different saddles will dominate for different shapes of region $A$:

\[ s\text{-channel:} \quad A \quad A \quad \Rightarrow \quad 1, T, \partial T, \ldots \]
This is a saddlepoint evaluation. Different saddles will dominate for different shapes of region $A$:

**s-channel:**

\[ A \rightarrow 1, T, \partial T, \ldots \]

**t-channel:**

\[ A \rightarrow 1, T, \partial T, \ldots \]
Semiclassical conformal blocks

So far we found the answer is universal:

$$\text{Tr} \rho^n_A \approx e^{-cf(0,z)} e^{-cf(0,\bar{z})}$$

$f$ is called the ‘semiclassical conformal block.’

The goal now: describe $f$ and take $n=1$ to compute the entanglement entropy.

Punchline will be: $f$ is the on-shell Einstein action for a 3-manifold whose boundary is the replica manifold.
Semiclassical blocks

$f$ is defined by the large-$c$ Virasoro block: \( \mathcal{F}(\Delta, z) \approx e^{-c f(\Delta/c, z)} \)

**Zamolodchikov '87:** It can also be computed by finding a stress tensor in the background of heavy operator insertions.

\[
\begin{array}{c}
\begin{tikzpicture}
\node (z) at (0,0) {$z$};
\node (w) at (1,0) {$w$};
\node (0) at (-1,1) {0};
\node (1) at (1,1) {1};
\node (inf) at (2,0) {$\infty$};
\draw (z) -- (0);
\draw (z) -- (w);
\draw (w) -- (1);
\draw (w) -- (inf);
\node at (0.5,0.5) {$\Delta$};
\end{tikzpicture}
\end{array} \Rightarrow \begin{array}{c}
\begin{tikzpicture}
\node (0) at (0,0) {0};
\node (z) at (0.5,0) {$z$};
\node (1) at (1,0) {1};
\node (inf) at (1.5,0) {$\infty$};
\node (w) at (0.5,0.5) {$T(w)$};
\draw (0) -- (w);\draw (z) -- (w);\draw (1) -- (w);\draw (inf) -- (w);
\end{tikzpicture}
\end{array}
\]

The stress tensor determines the semiclassical block via

\[ \partial_z f = \text{res } T(w) \text{ at } w \sim z \]
Holonomy problem

Operator insertions make double poles in $T(w)$. This does not quite determine $T(w)$. Turns out it is fixed by a holonomy condition around points (0,z):

$$\text{Tr } P \exp \left( \int dw \begin{bmatrix} 0 & 1 \\ -T(w) & 0 \end{bmatrix} \right) = -2 \cos \pi \sqrt{1 - \frac{24\Delta}{c}}$$

For the vacuum block,

$$\Delta = 0 \quad \Rightarrow \quad \text{holonomy} = \text{trivial}$$
Recap

Replica partition functions are given by the semiclassical vacuum block:

$$\text{Tr} \, \rho_A^n \approx e^{-cf(0,z)} e^{-cf(0,\bar{z})}$$

The function $f$ is determined by solving a trivial-holonomy condition on a Riemann surface.

In general, this holonomy problem must be solved numerically or in a series expansion.
Entanglement entropy

For the entanglement entropy, take \( n \to 1 \)

\[
S_A = -\partial_n \text{Tr } \rho_A^n|_{n=1}
\]

In this limit the holonomy problem is easy to solve analytically. The result is
Entanglement entropy

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**s-channel OPE:**

$$S_A = \frac{c}{3} \log(L_1) + \frac{c}{3} \log(L_2)$$
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**t-channel OPE:**

$$S_A = \frac{c}{3} \log(L_3) + \frac{c}{3} \log(L_4)$$
For the entanglement entropy, take $n \to 1$

$$S_A = -\partial_n \text{Tr} \rho^n_A|_{n=1}$$

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**s-channel OPE:**

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**t-channel OPE:**

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*Agrees with holographic Ryu-Takayanagi formula*

*(assuming no other non-perturbative contributions, ie non-geometric saddles)*
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III. CFT calculation of entanglement entropy

IV. 3d geometries

V. Comments
What is an on-shell 3d geometry?

3d gravity has no propagating graviton, so all solutions of Einstein equation are \textit{locally} AdS$_3$.

In Euclidean signature: hyperbolic 3-manifold.

To construct hyperbolic 3-manifolds:

1. Draw a genus-g Riemann surface:

2. “Fill in” g cycles
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What is a 3d geometry?

- A “filled in” cycle means a contractible loop in our geometry.
- For a loop to be contractible, the metric must obey some regularity condition.
- This regularity condition can be stated in terms of gauge-invariant data by setting a gravitational Wilson line to zero:

\[ P \exp \left( \oint dw \left[ \begin{array}{cc} 0 & 1 \\ \delta g_{ww} & 0 \end{array} \right] \right) = 1_{2 \times 2} \]

- In the SL(2) Chern-Simons formulation of classical 3d gravity, this is the ordinary holonomy of the SL(2) gauge field.
- This is exactly Zamolodchikov’s construction of the large-c Virasoro block for the vacuum representation!
3d geometry = Replica partition function

Relation to CFT

- Recall the replica partition function:

\[
\text{Tr } \rho^n_A \approx e^{-cf(0,z)} e^{-cf(0,\bar{z})} = Z_{cft}(\ )
\]

where \( f \) is computed by solving a zero-holonomy condition.

- To compute this in CFT, we secretly constructed a 3d geometry.

- The precise relation is “large-c vacuum block = Einstein action”:

\[
f = S_{Einstein}(\ )
\]

- Different ways of filling in the Riemann surface = saddlepoints in different OPE channels
I. Motivation

II. Entanglement entropy and 2d CFT

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V. Two Comments
Comment 1: Results in 2d CFT

2d CFT / 3d Gravity

- Derived holographic entanglement entropies directly from CFT, for an arbitrary region $A$.

- Did not rely on microscopic details of CFT; assumed only large-$c$ and a gap in operator dimensions.

- 3d geometries naturally appear in CFT calculations in this regime, without assuming holography.
Comment 2: Higher dimensions?

This argument is very special to 2d CFT. What about $d>2$?

- 3d gravity comes from large-$c$ limit of the Virasoro algebra
  - In higher dimensions, we only have $SO(d,2)$; operators built from stress tensor live in many different conformal families, and organize into a “gravity” family only at large $c$

- 3d gravity has only one parameter:
  \[ G_N \sim 1/c \]

- Higher-dimension gravity is parameterized by an infinite number of higher-derivative couplings,
  \[ S = \frac{1}{G_N} \int \left[ R + \alpha R^2 + \cdots \right] \]

- This requires (at the very least) a much stronger type of “gap” in operator dimensions for universal Einstein-gravity-like behavior.
- However: 4d higher spin gravity?
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- However:
  \[ G_N \sim \frac{1}{c^{10}} S_{12}^{G_N} \cdot R + R^2 + \cdots \sim c^{12} \text{spectrum} \]
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