Conformal Symmetry of General Black Holes

Mirjam Cvetič
Progress to extract from geometry (mesoscopic approach) an underlying conformal symmetry & promoting it to two-dimensional conformal field theory → governing microscopic structure of four and five dimensional asymptotically flat non-extreme rotating charged black holes

Recent efforts: w/ Finn Larsen 1106.3341 & 1112.4856
w/ Gary Gibbons 1201.0601
w/ Monica Guica & Zain Saleem 1302.7032
[w/ G. Gibbons, Chris Pope & Z. Saleem, to appear]

[Earlier work: w/ Donam Youm ’94–’96: multi-charged rotating asympt. Mink. BH’s
w/ Finn Larsen ’97–’99,’10: greybody factors; special(BPS) microsc.
w/ Chong, Lü & Pope ’06–’08: (AdS) rotating black hole solutions
w/ Chow, Lü & Pope ’09: special (Kerr/CFT) microscopics
w/Gibbons & Pope’11;w/Lü & Pope’13: products of horizon entropies]
Key Issue in Black Hole Physics:

How to relate Bekenstein-Hawking - thermodynamic entropy: $S_{\text{thermo}} = \frac{1}{4} A_{\text{hor}}$

($A_{\text{hor}} = \text{area of the black hole horizon}; c=\hbar=1$)

to

Statistical entropy:

$S_{\text{stat}} = \log N_i$?

Where do black hole microscopic degrees $N_i$ come from?
Microscopic origin of entropy for supersymmetric (BPS) multi-charged black holes with (schematic)

\[ M = Q_i + P_i \]

M-mass, Q\(_i\)-electric charges, P\(_i\)-magnetic charges

Systematic study of microscopic degrees quantified via: AdS/CFT (Gravity/Field Theory) correspondence

[A string theory on a specific Curved Space-Time (in D-dimensions) related to specific Field Theory (in (D-1)-dimensions) on its boundary]

Maldacena’97
Specific microscopic studies of black holes in string theory, in particular relation to 2d-dim CFT via AdS$_3$/CFT$_2$ correspondence extensively explored:

- BPS (supersymmetric) limit ($m \to 0$) [M=Q]  
  Strominger & Vafa’96

- near-BPS limit ($m \ll 1$)  
  . . . Maldacena & Strominger’97

- near-BPS multi-charged rotating black holes  
  M.C. & Larsen’98

Further developments:

- (near-)extreme rotating black holes ($m - l \ll 1$)  
  Kerr/CFT correspondence  
  Guica, Hartman, Song & Strominger 0809.4266…

- extreme AdS charged rotating black holes in diverse dim.  
  . . . . . M.C., Chow, Lü & Pope 0812.2918
Another approach: internal structure of black holes via probes such as scalar wave equation in the black hole background (greybody factors)

If certain terms in the wave equation omitted → SL(2,R)^2 symmetry & radial solution hypergeometric functions

Omission justified for special backgrounds:
- near-BPS limit \(m \ll 1\) Maldacena & Strominger’97
- near-extreme Kerr limit \(m - l \ll 1\) M.C. & Larsen’97
- low-energy probes \(\omega \ll 1\) Das & Mathur’96…

Also super-radiant limit \(\omega - n\Omega \ll 1\)
D=4 Kerr Bredberg, Hartman, Song & Strominger 0907.3477
D=4,5 multi-charged rotating M.C. & Larsen 0908.1136
On the other hand for general (nonextreme) black hole backgrounds there is NO SL(2,R)$^2$ symmetry.

This would seem to doom a CFT interpret. of the general BH’s

Related proposal dubbed “hidden conformal symmetry”

Castro, Maloney & Strominger 1004.0096

asserts conformal symmetry suggested by certain terms of the massless wave equation is there, just that it is spontaneously broken (w/ $\omega \rightarrow 0$ restoring it)

Extensive follow up…
In this talk a different perspective:

Program to quantify "conventional wisdom" that also non-extreme (asymptotically flat) black holes might have microscopic explanation in terms of 2D CFT

M.C. & Larsen ‘97-’99

But such black holes have typically negative specific heat $c_p < 0$ due to the coupling between the internal structure of the black hole and modes that escape to infinity

Should focus on the black hole "by itself" → one must necessarily enclose the black hole in a box, thus creating an equilibrium system

[Must be taken into account in any precise discussion of black hole microscopics]

→ The box leads to a "mildly" modified geometry, dubbed Subtracted Geometry
The rest of the talk:

I. Quantify subtracted geometry of a black hole in a box
   - via conformal symmetry of a wave equation
   - its lift on $S^1 \rightarrow \text{AdS}_3 \times \text{Sphere}$
   M.C. & Larsen 1106.3341 & 1112.4856

II. Identify gauge and scalar field sources supporting subtracted geometry:
   - as a scaling limit of certain BH’s
   - as an “infinite boost” Harrison transf. on the original BH
   M.C. & Gibbons 1201.0601

III. Interpolating geometries between the original black hole and their subtracted geometries:
   - via solution generating technics (reducing on time)
   - in lifted geometry via T-dualities and Melvin twists
   M.C., Guica & Saleem 1302.7032
For the case study we choose: most general black holes of D=5 N=4 (or N=8) un-gauged supergravity, actually its generating solution

N=4 (N=8) supersymmetric ungauged SG in D=5 can be obtained as a toroidal reduction of Heterotic String (Type IIA String) on T^{(10-D)} (D=5). The relevant subsector for generating solutions can also be viewed as D=5 N=2 SG coupled to three vector super-multiplets:

\[ e^{-1} \mathcal{L} = R - \frac{1}{2} \partial \varphi^2 - \frac{1}{4} \sum_{i=1}^{3} X_i^{-2} (F^i)^2 + \frac{1}{24} |\varepsilon_{ijk}| \epsilon^{\mu\nu\rho\sigma\lambda} F_{\mu\nu}^i F_{\rho\sigma}^j A_k^k \]

\[ X_1 = e^{-\frac{1}{\sqrt{6}} \varphi_1 - \frac{1}{\sqrt{2}} \varphi_2} , \quad X_2 = e^{-\frac{1}{\sqrt{6}} \varphi_1 + \frac{1}{\sqrt{2}} \varphi_2} , \quad X_3 = e^{\frac{2}{\sqrt{6}} \varphi_1} \]

Gravity with two scalar fields & three U(1)-gauge fields
[special case: when U(1) gauge fields identified \(\rightarrow\) Maxwell-Einstein Theory in D=5]
Such three charge rotating solutions were obtained by employing solution generating techniques

c.f., Ehlers, … Gibbons, Sen

a) Reduce $D=5$ stationary solution-
Kerr BH (with mass $m$ and two angular momenta $l_1$ and $l_2$) to $D=3$ on time-like and one angular Killing vectors

b) $D=3$ Largrangian has $O(3,3)$ symmetry

c) Acting with an $SO(1,1)^3$ subgroup of $O(3,3)$ transformations [preserving asymptotic Minkowski space-time] on the dimensionally reduced solution to generate new solutions with three parameters $\delta_i$

d) Upon lifting back to $D = 5$, arrive at spinning solutions with two angular momenta & three charges parameterised by the three $\delta_i$

M.C. & Youm hep-th/9603100
D=5 Kerr Solution:

\[ ds^2 = -\frac{r^2 + l_1^2 \cos^2 \theta + l_2^2 \sin^2 \theta - 2m}{r^2 + l_1^2 \cos^2 \theta + l_2^2 \sin^2 \theta} dt^2 + \frac{r^2(r^2 + l_1^2 \cos^2 \theta + l_2^2 \sin^2 \theta)}{(r^2 + l_1^2)(r^2 + l_2^2) - 2mr^2} dr^2 \\
+ (r^2 + l_1^2 \cos^2 \theta + l_2^2 \sin^2 \theta) d\theta^2 + \frac{4ml_1 l_2 \sin^2 \theta \cos^2 \theta}{r^2 + l_1^2 \cos^2 \theta + l_2^2 \sin^2 \theta} d\phi d\psi \\
+ \frac{\sin^2 \theta}{r^2 + l_1^2 \cos^2 \theta + l_2^2 \sin^2 \theta} [(r^2 + l_1^2)(r^2 + l_1^2 \cos^2 \theta + l_2^2 \sin^2 \theta) + 2ml_1^2 \sin^2 \theta] d\phi^2 \\
+ \frac{\cos^2 \theta}{r^2 + l_1^2 \cos^2 \theta + l_2^2 \sin^2 \theta} [(r^2 + l_1^2)(r^2 + l_1^2 \cos^2 \theta + l_2^2 \sin^2 \theta) + 2ml_2^2 \cos^2 \theta] d\psi^2 \\
- \frac{4ml_1 \sin^2 \theta}{r^2 + l_1^2 \cos^2 \theta + l_2^2 \sin^2 \theta} dt d\phi - \frac{4ml_2 \cos^2 \theta}{r^2 + l_1^2 \cos^2 \theta + l_2^2 \sin^2 \theta} dt d\psi.\]

m - mass; \( l_{12} \) - two angular momenta

Myers& Perry'86
\[
\begin{align*}
 ds_E^2 & = \frac{1}{\Delta} \left[ -\frac{(r^2 + l_1^2 \cos^2 \theta + l_2^2 \sin^2 \theta)(r^2 + l_1^2 \cos^2 \theta + l_2^2 \sin^2 \theta - 2m)}{\Delta} \right] dt^2 \\
 & + \frac{r^2}{(r^2 + l_1^2)(r^2 + l_2^2) - 2mr^2} dr^2 + d\theta^2 + \frac{4m \cos^2 \theta \sin^2 \theta}{\Delta} [l_1 l_2 \{(r^2 + l_1^2 \cos^2 \theta + l_2^2 \sin^2 \theta) \\
 - 2m(\sinh^2 \delta_1 \sinh^2 \delta_2 + \sinh^2 \delta_e \sinh^2 \delta_1 + \sinh^2 \delta_e \sinh^2 \delta_2)\}] + 2m\{(l_1^2 + l_2^2) \times \cosh \delta_1 \cosh \delta_2 \cosh \delta_1 \sinh \delta_2 \sinh \delta_e - 2l_1 l_2 \sinh^2 \delta_1 \sinh^2 \delta_2 \sinh^2 \delta_e \}d\phi d\psi \\
 & - \frac{4m \sin^2 \theta}{\Delta} [(r^2 + l_1^2 \cos^2 \theta + l_2^2 \sin^2 \theta)(l_1 \cosh \delta_1 \cosh \delta_2 \cosh \delta_e - l_2 \sinh \delta_1 \sinh \delta_2 \sinh \delta_e) \\
 + 2ml_2 \sinh \delta_1 \sinh \delta_2 \sinh \delta_e]d\phi dt - \frac{4m \cos^2 \theta}{\Delta} [(r^2 + l_1^2 \cos^2 \theta + l_2^2 \sin^2 \theta) \times \sin^2 \theta \\
 \times (l_2 \cosh \delta_1 \cosh \delta_2 \cosh \delta_e - l_1 \sinh \delta_1 \sinh \delta_2 \sinh \delta_e) + 2ml_1 \sinh \delta_1 \sinh \delta_2 \sinh \delta_e]d\phi d\psi \\
 + \frac{1}{\Delta} [(r^2 + 2m \sinh^2 \delta_e + l_1^2)(r^2 + 2m \sinh^2 \delta_1 + l_1^2 \cos^2 \theta + l_2^2 \sin^2 \theta)(r^2 + 2m \sinh^2 \delta_e \\
 + l_1^2 \cos^2 \theta + l_2^2 \sin^2 \theta) + 2m \sin^2 \theta \{(l_1^2 \cosh^2 \delta_m - l_2^2 \sinh^2 \delta_m)(r^2 + l_1^2 \cos^2 \theta + l_2^2 \sin^2 \theta) \\
 + 4ml_1 l_2 \cosh \delta_1 \cosh \delta_2 \cosh \delta_1 \sinh \delta_2 \sinh \delta_e - 2m \sinh^2 \delta_e \sinh^2 \delta_e \\
 \times (l_1^2 \cosh^2 \delta_e + l_2^2 \sinh^2 \delta_e) - 2ml_2 \sinh^2 \delta_e \{(\sinh^2 \delta_1 + \sinh^2 \delta_2)\}]d\phi^2 \\
 + \frac{\cos^2 \theta}{\Delta} [(r^2 + 2m \sinh^2 \delta_e + l_2^2)(r^2 + 2m \sinh^2 \delta_1 + l_1^2 \cos^2 \theta + l_2^2 \sin^2 \theta)(r^2 + 2m \sinh^2 \delta_1 \\
 + l_1^2 \cos^2 \theta + l_2^2 \sin^2 \theta) + 2m \cos^2 \theta \{(l_2^2 \cosh^2 \delta_e - l_1^2 \sinh^2 \delta_e)(r^2 + l_1^2 \cos^2 \theta + l_2^2 \sin^2 \theta) \\
 + 4ml_1 l_2 \cosh \delta_1 \cosh \delta_2 \cosh \delta_1 \sinh \delta_2 \sinh \delta_e - 2m \sinh^2 \delta_e \sinh^2 \delta_e \\
 \times (l_1^2 \sinh^2 \delta_e + l_2^2 \cosh^2 \delta_e) - 2ml_2 \sinh^2 \delta_e \{(\sinh^2 \delta_1 + \sinh^2 \delta_2)\}]d\psi^2 \right],
\end{align*}
\]

where

\[
\Delta \equiv (r^2 + 2m \sinh^2 \delta_1 + l_1^2 \cos^2 \theta + l_2^2 \sin^2 \theta)(r^2 + 2m \sinh^2 \delta_2 + l_1^2 \cos^2 \theta + l_2^2 \sin^2 \theta) \\
\times (r^2 + 2m \sinh^2 \delta_e + l_1^2 \cos^2 \theta + l_2^2 \sin^2 \theta),
\]
Scalar and gauge fields:

\[ g_{11} = \frac{r^2 + 2m\sinh^2\delta_{e_1} + l_1^2\cos^2\theta + l_2^2\sin^2\theta}{r^2 + 2m\sinh^2\delta_{e_2} + l_1^2\cos^2\theta + l_2^2\sin^2\theta}, \]

\[ e^{2\varphi} = \frac{(r^2 + 2m\sinh^2\delta_{e} + l_1^2\cos^2\theta + l_2^2\sin^2\theta)^2}{(r^2 + 2m\sinh^2\delta_{e_1} + l_1^2\cos^2\theta + l_2^2\sin^2\theta)(r^2 + 2m\sinh^2\delta_{e_1} + l_1^2\cos^2\theta + l_2^2\sin^2\theta)}, \]

\[ A_{t_1}^{(1)} = \frac{r^2 + 2m\sinh^2\delta_{e_1} + l_1^2\cos^2\theta + l_2^2\sin^2\theta}{m\cosh\delta_{e_1}\sinh\delta_{e_1}}, \]

\[ A_{\phi_1}^{(1)} = \frac{\sin^2\theta l_1\sinh\delta_{e_1}\sinh\delta_{e_2}\cosh\delta_e - l_2\cosh\delta_{e_1}\cosh\delta_{e_2}\sinh\delta_e}{r^2 + 2m\sinh^2\delta_{e_1} + l_1^2\cos^2\theta + l_2^2\sin^2\theta}, \]

\[ A_{\psi_1}^{(1)} = \frac{\cos^2\theta l_1\cosh\delta_{e_1}\sinh\delta_{e_2}\sinh\delta_e - l_2\sinh\delta_{e_1}\cosh\delta_{e_2}\cosh\delta_e}{r^2 + 2m\sinh^2\delta_{e_1} + l_1^2\cos^2\theta + l_2^2\sin^2\theta}, \]

\[ A_{t_1}^{(2)} = \frac{r^2 + 2m\sinh^2\delta_{e_2} + l_1^2\cos^2\theta + l_2^2\sin^2\theta}{m\cosh\delta_{e_2}\sinh\delta_{e_2}}, \]

\[ A_{\phi_1}^{(2)} = \frac{\sin^2\theta l_1\cosh\delta_{e_1}\sinh\delta_{e_2}\cosh\delta_e - l_2\sinh\delta_{e_1}\cosh\delta_{e_2}\sinh\delta_e}{r^2 + 2m\sinh^2\delta_{e_2} + l_1^2\cos^2\theta + l_2^2\sin^2\theta}, \]

\[ A_{\psi_1}^{(2)} = \frac{\cos^2\theta l_1\sinh\delta_{e_1}\cosh\delta_{e_2}\sinh\delta_e - l_2\cosh\delta_{e_1}\sinh\delta_{e_2}\cosh\delta_e}{r^2 + 2m\sinh^2\delta_{e_2} + l_1^2\cos^2\theta + l_2^2\sin^2\theta}, \]

\[ B_{t\phi} = -2m\sin^2\theta (l_1\sinh\delta_{e_1}\sinh\delta_{e_2}\cosh\delta_e - l_2\cosh\delta_{e_1}\cosh\delta_{e_2}\sinh\delta_e)(r^2 + l_1^2\cos^2\theta + l_2^2\sin^2\theta + m\sinh^2\delta_{e_1}) \times (r^2 + l_1^2\cos^2\theta + l_2^2\sin^2\theta + 2m\sinh^2\delta_{e_1}), \]

\[ B_{t\psi} = -2m\cos^2\theta (l_2\sinh\delta_{e_1}\sinh\delta_{e_2}\cosh\delta_e - l_1\cosh\delta_{e_1}\cosh\delta_{e_2}\sinh\delta_e)(r^2 + l_1^2\cos^2\theta + l_2^2\sin^2\theta + m\sinh^2\delta_{e_1}) \times (r^2 + l_1^2\cos^2\theta + l_2^2\sin^2\theta + 2m\sinh^2\delta_{e_1}), \]

\[ B_{\phi\psi} = \frac{2m\cosh\delta_e\sinh\delta_e\cos^2\theta \sin^2\theta (r^2 + l_1^2\cos^2\theta + l_2^2\sin^2\theta + m\sinh^2\delta_{e_1} + m\sinh^2\delta_{e_2})}{(r^2 + l_1^2\cos^2\theta + l_2^2\sin^2\theta + 2m\sinh^2\delta_{e_1})(r^2 + l_1^2\cos^2\theta + l_2^2\sin^2\theta + 2m\sinh^2\delta_{e_2})}, \]
Solution specified by mass, three charges and two angular momenta:

\[ M = 2m(\cosh^2 \delta_{e1} + \cosh^2 \delta_{e2} + \cosh^2 \delta_e) - 3m, \]
\[ Q_1^{(1)} = 2m \cosh \delta_{e1} \sinh \delta_{e1}, \quad Q_1^{(2)} = 2m \cosh \delta_{e2} \sinh \delta_{e2}, \quad Q = 2m \cosh \delta_e \sinh \delta_e \]
\[ J_\phi = 4m(l_1 \cosh \delta_{e1} \cosh \delta_{e2} \cosh \delta_e - l_2 \sinh \delta_{e1} \sinh \delta_{e2} \sinh \delta_e), \]
\[ J_\psi = 4m(l_2 \cosh \delta_{e1} \cosh \delta_{e2} \cosh \delta_e - l_1 \sinh \delta_{e1} \sinh \delta_{e2} \sinh \delta_e). \]

Inner and outer horizon:
\[ r_\pm^2 = m - \frac{1}{2} l_1^2 - \frac{1}{2} l_2^2 \pm \frac{1}{2} \sqrt{(l_1^2 - l_2^2)^2 + 4m(m - l_1^2 - l_2^2)}, \]

Special cases:
- all \( \delta_i \) equal & \( l_1 = l_2 = 0 \) \quad \text{Reissner-Nordström BH in D=5}
- \( m \to 0 \) \( \delta_i \to \infty \) w/ \( Q_i \) finite \quad \text{Supersymmetric (BPS) limit}
- \( (l_1^2 - l_2^2)^2 + 4m(m - l_1^2 - l_2^2) = 0 \) \quad \text{Extreme -Kerr limit}
We shall employ a bit more compact form w/ a warp factor $\Delta_0$ (as $U(1)$ fibration over 4d base):

M. C., Chong, Lü & Pope: hep-th/06006213

**Metric:**

$$ds_5^2 = -\Delta_0^{-2/3} G(dt + A)^2 + \Delta_0^{1/3} ds_4^2 ,$$

$$ds_4^2 = \frac{dx^2}{4X} + \frac{dy^2}{4Y} + \frac{U}{G} (d\chi - \frac{Z}{U} d\sigma)^2 + \frac{XY}{U} d\sigma^2$$

$$\Delta_0 = (x + y)^3 H_1 H_2 H_3$$

$$H_i = 1 + \frac{\mu \sinh^2 \delta_i}{x + y} , \ (i = 1, 2, 3)$$

Two Horizons ($X=0$)

$$X = (x + a^2)(x + b^2) - \mu x ,$$

$$Y = -(a^2 - y)(b^2 - y) ,$$

$$U = yX - xY ,$$

$$Z = ab(X + Y)$$

$$G = (x + y)(x + y - \mu) ,$$

$$A = \frac{\mu \Pi_c}{x + y - \mu} [(a^2 + b^2 - y)d\sigma - abd\chi] - \frac{\mu \Pi_s}{x + y} (ab d\sigma - y d\chi)$$

Ergosphere $G=0$

$$\Pi_c \equiv \prod_{i=1}^3 \cosh \delta_i, \quad \Pi_s \equiv \prod_{i=1}^3 \sinh \delta_i$$

$l_1 \rightarrow a, l_2 \rightarrow b$

$m \rightarrow \mu$

$x = r^2 ,$

$y = a^2 \cos^2 \theta + b^2 \sin^2 \theta$

$$\sigma = \frac{1}{a^2 - b^2} (a \phi - b \psi) ,$$

$$\chi = \frac{1}{a^2 - b^2} (b \phi - a \psi) .$$
Sources:

two scalars:

\[ X_i = H_i^{-1} (H_1 H_2 H_3)^{1/3} \quad i=1,2,3 \quad \text{w/ } X_1 X_2 X_3 = 1 \]

three gauge potentials:

\[
A^1 = \frac{2m}{(x + y)H_1} \left\{ \sinh \delta_1 \cosh \delta_1 dt \\
+ \sinh \delta_1 \cosh \delta_2 \cosh \delta_3 [abd\chi + (y - a^2 - b^2)d\sigma] \\
+ \cosh \delta_1 \sinh \delta_2 \sinh \delta_3 (abd\sigma - yd\chi) \right\}
\]

\[ A^2, A^3 \text{ via cyclic permutations} \]
Thermodynamics - Suggestive of weakly interacting 2-dim CFT w/ "left-" & "right-moving" excitations [noted already, M.C. & Youm’96]

Area of outer horizon \( S_+ = S_L + S_R \)

[Area of inner horizon \( S_- = S_L - S_R \)]

\[
S_L = \pi \mu \sqrt{\mu - (b-a)^2} (\Pi_c + \Pi_s) \\
S_R = \pi \mu \sqrt{\mu - (b+a)^2} (\Pi_c - \Pi_s)
\]

Surface gravity (inverse temperature) of

outer horizon \( \beta_H = \frac{1}{2} (\beta_L + \beta_R) \)

[inner horizon \( \beta_- = \frac{1}{2} (\beta_L - \beta_R) \)]

\[
\beta_R = \frac{2\pi \mu}{\sqrt{\mu - (b+a)^2}} (\Pi_c + \Pi_s) \\
\beta_L = \frac{2\pi \mu}{\sqrt{\mu - (b-a)^2}} (\Pi_c - \Pi_s)
\]

\[
\beta_H \Omega_R = \frac{2\pi (b+a)}{\sqrt{\mu - (b+a)^2}} \\
\beta_H \Omega_L = \frac{2\pi (b-a)}{\sqrt{\mu - (b-a)^2}}
\]

Two angular velocities:

\[
\Pi_c \equiv \prod_{i=1}^{3} \cosh \delta_i, \; \Pi_s \equiv \prod_{i=1}^{3} \sinh \delta_i
\]

Shown recently, all independent of the warp factor \( \Delta_o \)!

M.C. & Larsen 1106.3341
Subtracted geometry obtained by changing only the warp factor $\Delta_0 \rightarrow \Delta$ such that the scalar wave eq. preserves precisely $SL(2,R)^2$

Wave eq. written for a metric with an implicit warp factor $\Delta$:

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu \nu} \partial_\nu \Phi) = 0$$

Equation separable: $\Phi \sim e^{-i \omega t + i m_R (\phi + \psi) + i m_L (\phi - \psi)} \eta(x) \zeta(y)$

$$\left[ 4 \partial_x X \partial_x + \frac{x_+ - x_-}{x - x_+} \left( \frac{\beta_R \omega}{4 \pi} - m_R \frac{\beta_H \Omega_R}{2 \pi} + \frac{\beta_L \omega}{4 \pi} - m_L \frac{\beta_H \Omega_L}{2 \pi} \right)^2 - \frac{x_+ - x_-}{x - x_-} \left( \frac{\beta_R \omega}{4 \pi} - m_R \frac{\beta_H \Omega_R}{2 \pi} - \frac{\beta_L \omega}{4 \pi} + m_L \frac{\beta_H \Omega_L}{2 \pi} \right)^2 + \mu \omega^2 \left( 1 + \sum_i \sinh^2 \delta_i \right) + x^2 + y^2 + \frac{\Delta - \Delta_0}{G} \omega^2 \right] \Phi$$

$$= j(j + 2) \Phi$$

S$^3$ Laplacian eigenvalues

$$\Delta_0 = \prod_{i=1}^{3} (x + y + \mu \sinh^2 \delta_i)$$

Original warp factor

$$\Delta = \mu^2 \left[(x + y) (\Pi_c^2 - \Pi_s^2) + \mu \Pi_s^2 \right]$$

Subtracted geometry warp factor

$$\Pi_c \equiv \prod_{i=1}^{3} \cosh \delta_i, \quad \Pi_s \equiv \prod_{i=1}^{3} \sinh \delta_i$$
Remarks:

I. Subtracted geometry does not satisfy Einstein’s equation with original sources $\rightarrow$ additional gauge & scalar field sources

II. Subtraction that results in exact conformal symmetry $\leftrightarrow\rightarrow$

Black hole in a box, which has to be supported by additional sources (return to them later)

Asymptotic geometry of a Lifshitz-type w/ a deficit angle

$$ds_5^2 = -\left(\frac{R}{R_0}\right)^8 dt^2 + 12dR^2 + R^2 d\Omega_3^2$$

$\rightarrow$ black hole in an \``asymptotically conical box\'' (w/ deficit angle)

$\rightarrow$ the box is confining (\``softer\'' than AdS)
Lift on a circle to 6-dimensions:

\[
\begin{align*}
    ds_6^2 &= \Delta\left(\frac{1}{\mu}d\alpha + B\right)^2 + \Delta^{-1/3}ds_5^2 \\
    &= \Delta\left(\frac{1}{\mu}d\alpha + B\right)^2 - \Delta^{-1}G(dt + A)^2 + ds_4^2
\end{align*}
\]

where the KK-field along \(\alpha\) is

\[
    B = \frac{1}{\Delta} \left[ \mu((a^2 + b^2 - y)\Pi_s - ab\Pi_c)d\sigma + \mu(y\Pi_c - ab\Pi_s)d\chi - \frac{\Pi_s\Pi_c}{\Pi_c^2 - \Pi_s^2}dt \right]
\]
Geometry factorizes: locally $\text{AdS}_3 \times S^3$

globally $S^3$ (trivially) fibered over BTZ black hole

$$8G_3M_3 = \frac{\ell^2}{\mu^2(\Pi_c^2 - \Pi_s^2)^2} \left[ (\mu - a^2 - b^2)(\Pi_c^2 + \Pi_s^2) + 4ab\Pi_c\Pi_s \right],$$
$$4G_3J_3 = \frac{\ell^3}{\mu^2(\Pi_c^2 - \Pi_s^2)^2} \left[ ab(\Pi_c^2 + \Pi_s^2) - (a^2 + b^2 - \mu)\Pi_s\Pi_c \right].$$

w/ geometry $[\text{SL}(2,R) \times \text{SL}(2,R)]/\mathbb{Z}_2 \times \text{SO}(4)$

→ conformal symmetry of $\text{AdS}_3$ can be promoted to Virasoro algebra & standard microscopic calculation (via $\text{AdS}_3/\text{CFT}_2$) w/ Cardy formula à la Brown-Hennaux

w/ central charge, and conformal weights

$$c = \frac{3\ell_A}{2G_3},$$
$$h_+ = \frac{M_3\ell_A + J_3}{2},$$
$$h_- = \frac{M_3\ell_A - J_3}{2}.$$

guarantees identification of statistical and BH entropy

$$S = 2\pi\left( \sqrt{\frac{c}{6}}h_L + \sqrt{\frac{c}{6}}h_R \right)$$

[long spinning string interpretation]

$$S = \pi\mu\sqrt{\mu - (b - a)^2(\Pi_c + \Pi_s)} + \pi\mu\sqrt{\mu - (b + a)^2(\Pi_c - \Pi_s)}.$$
Sources supporting subtracted geometry were originally obtained as a scaling limit of another black hole (denoted \( \text{w/ ``tilded'' variables} \)) \( \text{w/ two equal infinite boosts } \tilde{\delta}_1 = \tilde{\delta}_2 \equiv \tilde{\delta} \) \& one \( \tilde{\delta}_3 \) finite one (formally near horizon region for BH \( \text{w/ two-charges } \rightarrow \infty \), one \( \rightarrow 0 \)):

\[ \epsilon \rightarrow 0 \]

\[ \tilde{x} = x \epsilon, \quad \tilde{t} = t \epsilon^{-1}, \quad \tilde{y} = y \epsilon, \quad \tilde{\sigma} = \sigma \epsilon^{-1/2}, \quad \tilde{\chi} = \chi \epsilon^{-1/2}, \]

\[ \tilde{m} = m \epsilon, \quad \tilde{a}^2 = a^2 \epsilon, \quad \tilde{b}^2 = b^2 \epsilon, \]

\[ 2\tilde{m} \sinh^2 \tilde{\delta} \equiv Q = 2m \epsilon^{-1/2}(\Pi_c^2 - \Pi_s^2)^{1/2}, \quad \sinh^2 \tilde{\delta}_3 = \frac{\Pi_s^2}{\Pi_c^2 - \Pi_s^2} \]

``Untilded'' variables are those of the subtracted geometry metric of non-extreme black hole with \( \text{w/ three charges } \delta_1, \delta_2, \delta_3 \) and subtracted warp factor

\[ \Delta = (2m)^2(x + y)(\Pi_c^2 - \Pi_s^2) + (2m)^3\Pi_s^2 \]

Fully determined sources: Scalars:

\[ X_1 = X_2 = X_3^{-1/2} = \frac{\Delta^{1/3}}{2m}. \]

Gauge potentials:

\[ A^1 = A^2 = -\frac{x + y}{2m} \, dt + y \Pi_c \, d\sigma - y \Pi_s \, d\chi, \]

\[ A^3 = \frac{(2m)^4 \Pi_s \Pi_c}{(\Pi_c^2 - \Pi_s^2)\Delta} \, dt + \frac{\Pi_s}{\Delta} \left[ ab \, d\chi + (y - a^2 - b^2) d\sigma \right] + \frac{\Pi_c}{\Delta} (ab \, d\sigma - y \, d\chi) \]
Comments:

a) Scaling limit of BH (with ``tilded'' parameters), resulting in subtracted geometry of non-extreme BH (with ``untilded'' parameters) is reminiscent of near-BPS limit (``dilute gas'') in the near horizon limit, w/ two (equal) charges $\rightarrow \infty$ & third one $\rightarrow 0$

b) Infinite charges can be gauged away (by rescaling the scalars). However, the asymptotic metric is of Lifshitz type (``softer'' than AdS)

c) In retrospect the lift to D=6 as AdS$_3 \times$ S$^3$ expected (due to BPS-like nature of the scaling limit for BH’s with tilded parameters)
Further Remarks: D=4 Black Holes

Non-extreme Rotating Asymptotically Minkowski BH’s in D=4, parameterized by mass, angular momentum and (four-)charges
Metric of D=4 rotating four-charge black holes
(generating solutions of N=4,8 ungauged supergravity)
M.C. & Youm '96, Chong, M.C., Lü & Pope’04

\[ ds^2_4 = - \Delta_0^{-1/2} G(dt + \mathcal{A})^2 + \Delta_0^{1/2} \left( \frac{dr^2}{X} + d\theta^2 + \frac{X}{G} \sin^2 \theta d\phi^2 \right) \]

\[ X = r^2 - 2mr + a^2 , \]
\[ G = r^2 - 2mr + a^2 \cos^2 \theta , \]
\[ \mathcal{A} = \frac{2ma \sin^2 \theta}{G} \left[ (\Pi_c - \Pi_s)r + 2m\Pi_s \right] d\phi , \]
\[ \Delta_0 = \prod_{I=0}^{3} (r + 2m \sinh^2 \delta_I) + 2a^2 \cos^2 \theta \left[ r^2 + mr \sum_{I=0}^{3} \sinh^2 \delta_I + 4m^2 (\Pi_c - \Pi_s) \Pi_s \right. \]
\[ \left. - 2m^2 \sum_{I<J<K} \sinh^2 \delta_I \sinh^2 \delta_J \sinh^2 \delta_K \right] + a^4 \cos^4 \theta . \]

\[ G_4M = \frac{1}{4} m \sum_{I=0}^{3} \cosh 2\delta_I , \]
\[ G_4Q_I = \frac{1}{4} m \sinh 2\delta_I , \quad (I = 0, 1, 2, 3) \]
\[ G_4J = ma(\Pi_c - \Pi_s) , \]

Special cases:

- Mass \( \delta_1 = \delta \) Kerr-Newman & \( a = 0 \) Reisner-Nordström
- Four charges \( \delta_1 = 0 \) Kerr & \( a = 0 \) Schwarzschild
Further Remarks:

Rotating Asymptotically Minkowski BH’s in D=4, parameterized by mass, angular momentum and (four-)charges.

Subtracted geometry prescription works in D=4 for general (four-)charge rotating black holes, too!  

M.C. & Larsen 1112.4856
Subtracted geometry for D=4 rotating four-charge rotating black holes:

\[ ds_4^2 = -\Delta_0^{-1/2} G(dt + A)^2 + \Delta_0^{1/2} \left( \frac{dr^2}{X} + d\theta^2 + \frac{X}{G} \sin^2 \theta d\phi^2 \right) \]

\[ X = r^2 - 2mr + a^2 , \]
\[ G = r^2 - 2mr + a^2 \cos^2 \theta , \]
\[ A = \frac{2ma \sin^2 \theta}{G} \left[ (\Pi_c - \Pi_s) r + 2m\Pi_s \right] d\phi , \]
\[ \Delta_0 = \prod_{I=0}^{3} (r + 2m \sinh^2 \delta_I) + 2a^2 \cos^2 \theta [r^2 + mr \sum_{I=0}^{3} \sinh^2 \delta_I + 4m^2 (\Pi_c - \Pi_s) \Pi_s] \]
\[ - 2m^2 \sum_{I<J<K} \sinh^2 \delta_I \sinh^2 \delta_J \sinh^2 \delta_K ] + a^4 \cos^4 \theta . \]

\[ \Delta_0 \rightarrow \Delta = (2m)^3 r (\Pi_c^2 - \Pi_s^2) + (2m)^4 \Pi_s^2 - (2m)^2 (\Pi_c - \Pi_s)^2 a^2 \cos^2 \theta \]
Further Remarks:

Rotating Asymptotically Minkowski BH’s in D=4, parameterized by mass, angular momentum and (four-)charges

Subtracted geometry prescription works in D=4 for general (four-) charge rotating black holes, too! M.C. & Larsen 1112.4856

Metric written with a warp factor; thermodynamics independent of a warp factor

Allows for restoration of $SL(2,R)^2$ in the wave eq.

Lift to D=5: locally $4 \text{AdS}_3 \times S^2$; globally $S^2$ fibered over BTZ

Quantitative microscopics again à la Brown-Henneaux

Conformal Symmetry of the Full D=4, N=4,8 rotating BH w/ 28 el. & 28 mag. charges? Chow & Compere 1310.1925

Work in progress w/ F. Larsen
Further Insights into the origin of subtracted geometry:

These geometries can be obtained as an infinite boost Harrison transformations on the original solution, i.e. SO(1,1) transformations [change asymptotics]:

\[ H \sim \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \quad \beta \to 1 \]

acting on the original solution reduced on a time-like Killing vector.

Explicitly shown for a D=4 Schwarzschild BH & conjectured that it works for general D=4 multi-charged BH’s

M.C. & Gibbons 1201.0601

Subsequently confirmed

for D=4 four-charge BH’s Virmani 1203.5088

for D=5 three-charge BH’s Sahay & Virmani 1305.2800

Interpolating geometries studied for:

Static BH’s; CFT interpretation as (2,2) irrelevant deformations

Baggio, de Boer, Jottar & Mayerson 1210.7695

General rotating BH’s
Highlights for interpolating geometries for general rotating BH’s
M. C., Guica & Saleem 1302.7036

a) Harrison transformations on D=4 BH’s -- specific SO(1,1)^4 boosts of O(4,4) symmetry [change asymptotics] for time-like Killing vector reduced action

D=4 BH \[\mapsto\] D=4 subtracted geometry

\[H \sim \begin{pmatrix} 1 & 0 \\ \beta_i & 1 \end{pmatrix}, \quad \beta_i \to 1 \ (i=1,2,3), \quad \beta_4 \to \text{finite}\]

[\beta_i = 1 \ (i=1,2,3)-garantees AdS_3 lift!]

b) Harrison transformations \[\leftrightarrow\] on S^1 lifted geometry: coordinate transf. \[t + \beta z\] (Melvin) twists [related to spectral flow, in \((t,z)\)]

c) Interpolating solutions obtained in lifted geometry via sets of T-duality transf. followed by Melvin twists

d) Deformations from subtracted geometry (\beta_i-1<<1) in dual CFT via (2,2) & (1,2) (irrelevant) operators
Digression:

How about Harrison transf. on angular Killing vector reduced action $\rightarrow S^1$ lifted geometry should correspond to $\phi + \alpha z$ coordinate transf. (``standard” Melvin twist)

$\rightarrow$ New geometrical insights?

c.f., Maxwell-Einstein Gravity: Gibbons, Mujtaba & Pope 1301.3927

Multi-charged BH’s work in progress M.C., Gibbons, Pope & Saleem
Final comments: Entropy Products

Without cosmological constant, the product of areas associated with two horizons quantised $\rightarrow$ indicative of 2D CFT

$D=4$ \[ A_+A_- = 64\pi^2 \left( \prod_{i=1}^{3} Q_i + J^2 \right) \]

$Larsen’97 (J=0)$, M.C. & Larsen’97 $(J\neq0)$

Generalized to rings & strings

$D=5$ \[ A_+A_- = 64\pi^2 \left( \prod_{i=1}^{3} Q_i + J_R^2 - J_L^2 \right) \]

$Castro & Rodriguez 1204.1284$

$KK$ dyons $M.C., Lü & Pope1306.4522$

Full $D=4$ $N=4,8$ BH $Chow& Compere 1310.1925$

With cosmological constant $(1/g^2)$, more that two horizons, and yet the product of areas associated with all (analytically continued) horizons also quantised:

$D=4$ \[ \prod_{\alpha=1}^{4} A_\alpha = 64\pi^2 \frac{1}{g^4} \left( \prod_{i=1}^{4} Q_i + J^2 \right) \]

$M.C., Gibbons & Pope 1011.008 (PRL)$

Generalized to Wu’s solution:

$D=5$ \[ \prod_{\alpha=1}^{3} A_\alpha = 64\pi^2 \frac{1}{g^3} \left( \prod_{i=1}^{3} Q_i + J_R^2 - J_L^2 \right) \]

$M.C., Lü & Pope1306.4522$

$\rightarrow$ possible underlying (higher dim conformal) symmetry?

Higher derivative gravities $\rightarrow$ evidence for quantisation

$D=4$ Maxwell-Weyl gravity: $M.C., Lü & Pope1306.4522$

product of Wald entropies (over ALL horizons): $\prod_{I} S_I = J^2$

[related: Castro, Dehmami, Giribet & Kastor 1304.1696]

$\rightarrow$ FURTHER STUDY