Black Hole Mass in Anti-de Sitter Space

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What is the mass of a black hole?

- Consider the $D = 4$ Schwarzschild solution

$$ds^2 = -(1 - \tfrac{2M}{r})\, dt^2 + (1 - \tfrac{2M}{r})^{-1} dr^2 + r^2 d\Omega_2^2$$

- Extract the Newtonian potential

$$-g_{tt} \sim 1 + 2V(r) \qquad \Rightarrow \qquad V = -\frac{M}{r}$$

- This is essentially the ADM procedure

Look for the $1/r$ deviation from asymptotically Minkowski space

What about AdS black holes?

- For the Schwarzschild-AdS black hole

$$ds^2 = -(1 - \tfrac{2M}{r} + g^2r^2)dt^2 + (1 - \tfrac{2M}{r} + g^2r^2)^{-1}dr^2 + r^2 d\Omega_2^2$$

- Focusing on the $1/r$ behavior of g_{tt} again yields mass M

  Is this what we expect?

- Consider that the cosmological term grows as r^2

$$-g_{tt} \sim g^2r^2 \left(1 + \mathcal{O}(\frac{1}{r^2}) + \mathcal{O}(\frac{M}{r^3})\right)$$

  Thus we must extract the subleading $1/r^3$ behavior

The addition of matter

- Consider the four-dimensional $\mathcal{N} = 8$ gauged supergravity

$$\mathcal{N} = 2 \text{ truncation:} \qquad \mathrm{SO}(8)_R \supset \mathrm{U}(1)^4$$

- This admits a four-charge AdS$_4$ black hole

$$ds^2 = -\mathcal{H}(r)^{-1/2} f(r) dt^2 + \mathcal{H}(r)^{1/2}\Big(f(r)^{-1} dr^2 + r^2 d\Omega_2^2\Big)$$

$$\mathcal{H} = \prod_i \left(1 + \frac{q_i}{r}\right), \qquad f = 1 - \frac{\mu}{r} + g^2 r^2 \mathcal{H}$$

- Can we extract the mass from the $1/r^3$ term?

The addition of matter

- Asymptotically:

$$-g_{tt} \sim g^2r^2\bigg(1+\frac{\alpha_1/2}{r}+\frac{1/g^2-\frac{1}{8}(\alpha_1^2-4\alpha_2)}{r^2} - \frac{(\mu+\alpha_1/2)/g^2+\frac{1}{16}(\alpha_1^3-4\alpha_1\alpha_2+8\alpha_3)}{r^3}+\cdots\bigg)$$

where $\alpha_1 = q_1+q_2+q_3+q_4$, $\alpha_2 = q_1q_2+\cdots$, etc.

- How do we unambiguously extract the mass?
- Can we recover a linear BPS-like mass relation?

$$M \sim \tfrac{1}{2}\mu + \tfrac{1}{4}(q_1+q_2+q_3+q_4) \qquad ??$$

Mass in anti-de Sitter space

- The notion of mass can be made more precise

  Abbott and Deser (generalization of ADM)

  Henneaux and Teitelboim (asymptotic SO(3,2) Killing symmetry)

  Ashtekar and Das

- Alternatively, we may follow the holographic renormalization approach in AdS

  In addition to mass and charge, this also yields the thermodynamic potential (regulated on-shell action)

Holographic renormalization

1. Start with the gravitational action

$$S = S_{\rm bulk}(\textsf{Einstein} + \textsf{matter}) + S_{\rm GH} + S_{\rm ct}$$

2. Single out a radial coordinate r

$$ds^2 = N^2\,dr^2 + h_{ab}(dx^a + V^a\,dr)(dx^b + V^b\,dr)$$

3. The boundary stress tensor is given by

$$T^{ab} = \frac{2}{\sqrt{-h}}\frac{\delta S}{\delta h_{ab}}$$

4. Conserved charges are obtained by integrating T^{ab} over a spacelike hypersurface at infinity

Divergences of the action

The action in AdS is divergent and needs to be regulated

- Consider pure gravity

$$S_{\rm bulk} = -\frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\,(R - 2\Lambda)$$

- Use the Einstein equation $R_{\mu\nu} = \Lambda\, g_{\mu\nu}$

$$-I_{\rm bulk} = \frac{\Lambda}{8\pi G}\int d^4x\,\sqrt{-g} = \frac{\Lambda\beta\omega_2}{8\pi G}\int_0^\infty r^2\,dr = \lim_{r\to\infty}\frac{\Lambda\beta\omega_2}{24\pi G}\,r^3 = \infty$$

where $ds^2 = -(1+\Lambda r^2/3)dt^2 + (1+\Lambda r^2/3)^{-1}dr^2 + r^2 d\Omega_2^2$

Divergences of the action

- For the four-charge AdS_4 black hole

$$S_{\rm bulk} + S_{\rm GH} = \mathcal{O}(r^3) + \mathcal{O}(r^2) + \mathcal{O}(r) + \text{finite} + \cdots$$

The boundary stress tensor is likewise divergent

$$T^{ab}_{\rm unreg} = \frac{1}{8\pi G}(\Theta^{ab} - \Theta h^{ab})$$

- This yields a divergent mass

$$M_{\rm unreg} = \lim_{r\to\infty}\left(-g^2r^3 - \tfrac{3}{4}g^2\alpha_1 r^2 - (1+\tfrac{1}{2}g^2\alpha_2)r + (\mu + \tfrac{1}{4}\alpha_1 - \tfrac{1}{4}g^2\alpha_3)\right)$$

The counterterm action

The divergences may be removed by the addition of boundary counterterms

- Cut off the spacetime at the large value $r = r_0$
- Introduce the counterterm action

$$S_{\rm ct} = \frac{1}{8\pi G} \int_{r=r_0} d^3x \sqrt{-h} \Big(W(\phi) + C(\phi)\mathcal{R} + D(\phi)\mathcal{R}^2 + E(\phi)\mathcal{R}_{ab}\mathcal{R}^{ab} + \cdots \Big)$$

- How do we choose the counterterms W, C, D, E, $\ldots$?

Hamilton-Jacobi counterterms

- Consider 'time' evolution in the r direction

$$S[g_{\mu\nu}, \phi^i, A^I_\mu] \qquad \Rightarrow \qquad \mathcal{H}[\pi^{ab}, h_{ab}, \pi_i, \phi^i, \pi^\mu_I, A^I_\mu]$$

- Diffeomorphism invariance implies $\mathcal{H} = 0$
- The momenta may be written as functional derivatives of the on-shell action

$$\mathcal{H}\left[\frac{\delta S}{\delta h_{ab}}, h_{ab}, \frac{\delta S}{\delta \phi^i}, \phi^i, \frac{\delta S}{\delta A^I_\mu}, A^i_\mu\right] = 0$$

This is the Hamilton-Jacobi equation

Hamilton-Jacobi counterterms

- After expanding $\mathcal{H}$ and collecting appropriate terms, one ends up with differential equations for the counterterms W, C, D, $E, \ldots$
- For W one obtains

$$V = 2\mathcal{G}^{ij}\partial_{\phi^i}W\partial_{\phi^j}W - \frac{3}{2}W^2$$

This is just the relation between potential and superpotential in gauged supergravity

Hamilton-Jacobi counterterms

In four dimensions, we obtain

- The unregulated stress tensor

$$T^{ab}_{\rm unreg} = \frac{1}{8\pi G}(\Theta^{ab} - \Theta h^{ab})$$

- The counterterm contribution

$$T^{ab}_{\rm ct} = \frac{1}{8\pi G}\left(h^{ab}W(\phi) - \frac{1}{2g}(2\mathcal{R}^{ab} - \mathcal{R}h^{ab})\right)$$

The regulated black hole mass

Back to the four-charge black hole...

- Three dilatonic scalars parameterized by a constrained set of fields $X_1X_2X_3X_4 = 1$
- The potential and superpotential

$$V = -g^2 \sum_{i<j} X_i X_j, \qquad W = \tfrac{1}{2} g \sum_i X_i$$

- For the black hole solution

$$X_i = \frac{\mathcal{H}^{1/4}}{H_i}, \qquad H_i = 1 + \frac{q_i}{r}$$

The regulated black hole mass

- The unregulated mass

$$M_{\rm unreg} = \lim_{r\to\infty}\left(-g^2r^3 - \tfrac{3}{4}g^2\alpha_1 r^2 - (1+\tfrac{1}{2}g^2\alpha_2)r + (\mu + \tfrac{1}{4}\alpha_1 - \tfrac{1}{4}g^2\alpha_3)\right)$$

- The counterterm contribution

$$M_{\rm ct} = \lim_{r\to\infty}\left(g^2r^3 + \tfrac{3}{4}g^2\alpha_1 r^2 + (1+\tfrac{1}{2}g^2\alpha_2)r + (-\tfrac{1}{2}\mu + \tfrac{1}{4}g^2\alpha_3)\right)$$

- This results in a finite black hole mass

$$M = \tfrac{1}{2}\mu + \tfrac{1}{4}\alpha_1 = \tfrac{1}{2}\mu + \tfrac{1}{4}(q_1+q_2+q_3+q_4)$$

The regulated black hole mass

- The expression for the mass

$$M = \frac{1}{2}\mu + \frac{1}{4}(q_1 + q_2 + q_3 + q_4)$$

  agrees with expectations

  1. Could have been obtained by *careful* expansion of g_{tt}
  2. Yields a linear BPS relation
  3. Dual to energy in the CFT (obeys the first law of black hole thermodynamics)

Advantages of holographic renormalization

- No need for a reference background subtraction

  Intrinsically defined quantities for arbitrary black hole configurations

  Avoids issues of insufficiently rapid falloff of fields in non-asymptotically flat backgrounds

- Easily applied to more complicated objects

  Non-extremal charged rotating black holes. . .

  Thermodynamic investigations of the dual field theories (AdS/CFT)

Advantages of holographic renormalization

- Additional symmetries may be read off from the boundary stress tensor

$$\mathrm{AdS}_5 \times S^5 \quad \Rightarrow \qquad \begin{matrix} SO(2,4) & \times & SO(6) \\ E, S_1, S_2, & & J_1, J_2, J_3 \end{matrix}$$

- Rigorous treatment of black hole thermodynamics

$$\Omega_{\rm reg} = E_{\rm reg} - TS - \Phi^I Q_I$$

. . . but watch out for potential log divergences

Related to the conformal anomaly